

Pure Mathematics 1 (9709): revision notes with worked examples

These notes cover the whole of Paper 1 (Pure Mathematics 1) of Cambridge International AS & A Level Mathematics 9709, in syllabus order: quadratics, functions, coordinate geometry, circular measure, trigonometry, series, differentiation and integration. Each section explains the idea briefly, gives the results you must learn, and then works through a typical exam question. The Quick check at the end has exam-style questions on every chapter with full worked answers.

Paper 1 is the foundation for Paper 3 (Pure Mathematics 3) and for Paper 2 if you take it. Our P2 and P3 notes follow the same layout and assume everything here, so time spent making these methods automatic pays off twice. You may use a calculator in Paper 1, but many questions ask for exact answers (surds, fractions and multiples of π), so show the algebra and only round at the very end. Unless a question says otherwise, give non-exact answers to 3 significant figures and angles in degrees to 1 decimal place.

1. Quadratics

A quadratic expression has the form $ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$. Its graph is a parabola: a "U" shape when $a > 0$ and an upside-down "U" when $a < 0$. Almost every question on quadratics comes down to one of three tools: completing the square, factorising or the formula, and the discriminant.

1.1 Completing the square

Completing the square rewrites $ax^2 + bx + c$ in the form $a(x + p)^2 + q$. Because $(x + p)^2$ is never negative, this form shows the vertex (turning point) of the graph straight away: the vertex is at $(-p, q)$. If $a > 0$ the vertex is a minimum and the least value of the expression is q ; if $a < 0$ it is a maximum and the greatest value is q .

Method. Take out the factor a from the x^2 and x terms only. Inside the bracket, halve the coefficient of x to get p , write $(x + p)^2$, and subtract p^2 to compensate. Multiply back out and collect the constants.

$$x^2 + bx = (x + b/2)^2 - (b/2)^2$$

Worked example 1. Express $2x^2 - 12x + 5$ in the form $a(x + p)^2 + q$ and state the coordinates of the vertex of $y = 2x^2 - 12x + 5$.

$$\begin{aligned} 2x^2 - 12x + 5 &= 2(x^2 - 6x) + 5 \\ &= 2[(x - 3)^2 - 9] + 5 \\ &= 2(x - 3)^2 - 18 + 5 \\ &= 2(x - 3)^2 - 13 \end{aligned}$$

So $a = 2$, $p = -3$, $q = -13$. Since $a > 0$ the vertex is a minimum point at $(3, -13)$, and the least value of the expression is -13 .



Completing the square is also how you find the range of a quadratic function (section 2) and the centre of a circle (section 3), so practise it until it takes seconds.

1.2 Solving quadratic equations

There are three methods. Factorise when the numbers are friendly. Use the quadratic formula when they are not. Complete the square when the question has just asked you to, because the answer then follows almost immediately.

The quadratic formula. The roots of $ax^2 + bx + c = 0$ are
 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked example 2. Solve $3x^2 - 5x - 2 = 0$.

We need two numbers with product $3 \times (-2) = -6$ and sum -5 : these are -6 and 1 .

$$3x^2 - 6x + x - 2 = 0$$

$$3x(x - 2) + 1(x - 2) = 0$$

$$(3x + 1)(x - 2) = 0$$

$$x = -1/3 \text{ or } x = 2.$$

Worked example 3. Using your answer to Worked example 1, solve $2x^2 - 12x + 5 = 0$, giving exact answers.

$$2(x - 3)^2 - 13 = 0$$

$$(x - 3)^2 = 13/2$$

$$x - 3 = \pm\sqrt{13/2}$$

$$x = 3 \pm \sqrt{13/2}, \text{ which can also be written } 3 \pm (\sqrt{26})/2.$$

1.3 The discriminant

The expression under the square root in the formula, $b^2 - 4ac$, is called the discriminant. Its sign tells you how many real roots the equation has, and so how many times the graph meets the x-axis.

Learn. For $ax^2 + bx + c = 0$:

$b^2 - 4ac > 0$: two distinct real roots (the graph crosses the x-axis twice).

$b^2 - 4ac = 0$: two equal real roots, a repeated root (the graph touches the x-axis).

$b^2 - 4ac < 0$: no real roots (the graph does not meet the x-axis).

"Real roots" (with no other word) means $b^2 - 4ac \geq 0$.

Worked example 4. Find the values of k for which $x^2 + kx + 9 = 0$ has equal roots.

Here $a = 1$, $b = k$, $c = 9$. Equal roots require $b^2 - 4ac = 0$:

$$k^2 - 4 \times 1 \times 9 = 0$$

$$k^2 = 36$$

$$k = 6 \text{ or } k = -6.$$

A quadratic that is positive for all x (for example $x^2 + 2x + 7$) has $a > 0$ and $b^2 - 4ac < 0$. You can

prove it either way: the discriminant is $4 - 28 = -24 < 0$, or completing the square gives $(x + 1)^2 + 6$, which is at least 6.

1.4 Where a line meets a curve

To find where a straight line meets a curve, substitute the line into the curve and form one equation in x . If that equation is a quadratic, its discriminant tells you whether the line cuts the curve at two points, is a tangent (one point, repeated root) or misses it entirely. This is one of the most frequently examined ideas in Paper 1.

Worked example 5. The line $y = mx$ passes through the origin. Find the values of m for which the line is a tangent to the curve $y = x^2 + 3$.

At intersection: $x^2 + 3 = mx$, so $x^2 - mx + 3 = 0$.

For a tangent the roots are equal: $b^2 - 4ac = 0$

$$(-m)^2 - 4 \times 1 \times 3 = 0$$

$$m^2 = 12$$

$$m = \pm\sqrt{12} = \pm 2\sqrt{3}.$$

Exam tip. When a question says "the line does not meet the curve", you need $b^2 - 4ac < 0$, which usually leads to a quadratic inequality in k . Solve it properly (section 1.5): the answer is normally a range such as $-2 < k < 6$, not two separate values.

1.5 Quadratic inequalities

To solve a quadratic inequality, first solve the corresponding equation to find the critical values, then sketch the parabola and read off where it is above or below the x -axis. For a positive x^2 coefficient, the graph is below the axis between the roots and above it outside them.

Worked example 6. Solve $2x^2 - x - 6 > 0$.

Critical values: $2x^2 - x - 6 = 0$, so $(2x + 3)(x - 2) = 0$, giving $x = -3/2$ and $x = 2$.

The graph is a "U" crossing the x -axis at $-3/2$ and 2 . It is above the axis (positive) outside the roots.

Solution: $x < -3/2$ or $x > 2$.

For " < 0 " you would take the region between the roots, $-3/2 < x < 2$, written as one inequality. Keep "or" for two separate regions; never write $-3/2 > x > 2$, which is meaningless.

1.6 Equations that reduce to quadratics

Some equations are quadratics in disguise. Look for a pattern of powers where one is double the other, such as x^4 and x^2 , or x and $\sqrt{x}$. Substitute a new letter, solve the quadratic, then go back to the original variable and check each answer is allowed.

Worked example 7. Solve (a) $x^4 - 5x^2 + 4 = 0$ and (b) $x - 5\sqrt{x} + 6 = 0$.

(a) Let $u = x^2$: $u^2 - 5u + 4 = 0$, so $(u - 1)(u - 4) = 0$ and $u = 1$ or 4 .

$x^2 = 1$ gives $x = \pm 1$; $x^2 = 4$ gives $x = \pm 2$. Four solutions: $x = -2, -1, 1, 2$.

(b) Let $u = \sqrt{x}$, so $x = u^2$: $u^2 - 5u + 6 = 0$, so $(u - 2)(u - 3) = 0$ and $u = 2$ or 3 .
 $\sqrt{x} = 2$ gives $x = 4$; $\sqrt{x} = 3$ gives $x = 9$. (A negative value of u would have been rejected, since $\sqrt{x}$ cannot be negative.)

1.7 Simultaneous equations: one linear, one quadratic

Rearrange the linear equation to make one variable the subject, substitute into the quadratic equation, solve, and then find the matching value of the other variable for each solution. Always give answers as pairs.

Worked example 8. Solve the simultaneous equations $x + y = 5$ and $x^2 + y^2 = 13$.

From the first equation, $y = 5 - x$. Substitute:

$$x^2 + (5 - x)^2 = 13$$

$$x^2 + 25 - 10x + x^2 = 13$$

$$2x^2 - 10x + 12 = 0$$

$$x^2 - 5x + 6 = 0, \text{ so } (x - 2)(x - 3) = 0.$$

$x = 2$ gives $y = 3$; $x = 3$ gives $y = 2$. Solutions: $(2, 3)$ and $(3, 2)$.

2. Functions

This chapter is about the language of functions: what a function is, which inputs it accepts, which outputs it produces, and how functions can be combined, reversed and transformed. Exam questions are usually multi-part, building from a range to a composite to an inverse, so precise vocabulary matters.

2.1 Domain, range and one-one functions

Definitions to learn.

A **function** is a rule that maps each input in its domain to exactly one output.

The **domain** is the set of input values for which the function is defined.

The **range** is the set of output values the function actually takes for its domain.

A function is **one-one** if each output comes from exactly one input; otherwise it is **many-one**.

Notation: $f(x) = x^2$, $x \in \mathbb{R}$, or $f : x \mapsto x^2$, $x \in \mathbb{R}$. Here $\mathbb{R}$ is the set of real numbers. The range is described using $f(x)$, for example $f(x) \geq 0$, not using x .

A graph shows a one-one function if every horizontal line meets it at most once. A quadratic on all of $\mathbb{R}$ is many-one, because it is symmetrical about its vertex; restricting the domain to one side of the vertex makes it one-one.

Worked example 9. The function f is defined by $f(x) = x^2 - 4x + 5$ for $x \in \mathbb{R}$.

(a) Find the range of f . (b) Explain why f is not one-one. (c) The domain is restricted to $x \geq k$. Find the least value of k for which f is one-one.

(a) Completing the square: $f(x) = (x - 2)^2 + 1$. The least value is 1 (at $x = 2$), so the range is $f(x) \geq 1$.

- (b) $f(1) = 2$ and $f(3) = 2$, so two different inputs give the same output.
(c) The vertex is at $x = 2$. To the right of it f is increasing, so f is one-one for $x \geq 2$. The least value of k is 2.

Exam tip. To find the range of a function on a restricted domain, sketch it over that domain and look at the end-points and any turning point inside it. For example, $f(x) = (x - 2)^2 + 1$ for $0 \leq x \leq 5$ has least value 1 (at $x = 2$) and greatest value 10 (at $x = 5$), so the range is $1 \leq f(x) \leq 10$. The value at $x = 0$ is 5, which is not the greatest.

2.2 Composite functions

The composite function fg means "do g first, then f ": $fg(x) = f(g(x))$. The order matters, and in general fg is not the same as gf . The composite fg only exists if every output of g is an allowed input of f , that is, the range of g lies within the domain of f . The notation $f^2(x)$ means $ff(x)$, not $[f(x)]^2$.

Worked example 10. $f(x) = 2x + 1$ and $g(x) = x^2 - 3$, both for $x \in \mathbb{R}$.

(a) Find $fg(x)$ and $gf(x)$. (b) Show that the equation $fg(x) = gf(x)$ has no real solutions.

(a) $fg(x) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$.

$gf(x) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.

(b) $2x^2 - 5 = 4x^2 + 4x - 2$ gives $2x^2 + 4x + 3 = 0$.

Discriminant: $4^2 - 4 \times 2 \times 3 = 16 - 24 = -8 < 0$, so there are no real solutions.

2.3 Inverse functions

The inverse function f^{-1} reverses f : if $f(a) = b$ then $f^{-1}(b) = a$. Only a one-one function has an inverse. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

Learn. The domain of f^{-1} is the range of f , and the range of f^{-1} is the domain of f . Also $ff^{-1}(x) = f^{-1}f(x) = x$.

Method to find $f^{-1}(x)$. Write $y = f(x)$. Rearrange to make x the subject. Swap x and y (or simply rename) to give $f^{-1}(x)$. State its domain.

Worked example 11. $f(x) = (x - 2)^2 + 1$ for $x \geq 2$. Find an expression for $f^{-1}(x)$ and state its domain.

$$y = (x - 2)^2 + 1$$

$$(x - 2)^2 = y - 1$$

$$x - 2 = \sqrt{y - 1} \text{ (positive root, because } x \geq 2 \text{ means } x - 2 \geq 0)$$

$$x = 2 + \sqrt{y - 1}$$

So $f^{-1}(x) = 2 + \sqrt{x - 1}$. The range of f is $f(x) \geq 1$, so the domain of f^{-1} is $x \geq 1$.

Worked example 12. $g(x) = (3x + 1)/(x - 2)$ for $x > 2$. Find $g^{-1}(x)$.

$$y = (3x + 1)/(x - 2)$$

$$y(x - 2) = 3x + 1$$

$$xy - 2y = 3x + 1$$

$$xy - 3x = 2y + 1$$

$$x(y - 3) = 2y + 1$$

$$x = (2y + 1)/(y - 3)$$

So $g^{-1}(x) = (2x + 1)/(x - 3)$. (For $x > 2$, $g(x) = 3 + 7/(x - 2)$ takes every value greater than 3, so the domain of g^{-1} is $x > 3$.)

When a question asks you to sketch f and f^{-1} on one diagram, draw the line $y = x$, use the same scale on both axes, and make the two curves mirror images. If they meet, they meet on $y = x$, so solving $f(x) = x$ is often quicker than solving $f(x) = f^{-1}(x)$.

2.4 Transformations of graphs

You need to recognise and describe these transformations of $y = f(x)$, using the correct technical words: translation (with a vector), stretch (with a scale factor and direction), reflection (with the mirror line).

New graph	Transformation of $y = f(x)$
$y = f(x) + a$	Translation by vector $(0, a)$: up by a
$y = f(x + a)$	Translation by vector $(-a, 0)$: left by a
$y = af(x)$	Stretch parallel to the y -axis, scale factor a
$y = f(ax)$	Stretch parallel to the x -axis, scale factor $1/a$
$y = -f(x)$	Reflection in the x -axis
$y = f(-x)$	Reflection in the y -axis

Changes inside the bracket act on x and behave "the opposite way" to what you might expect: $f(x + 3)$ moves the graph 3 units to the left, and $f(2x)$ squashes it to half its width. Changes outside the bracket act on y and behave as expected.

Worked example 13. Describe a sequence of transformations that maps $y = x^2$ onto $y = 2(x + 1)^2 - 3$.

Build the new equation one step at a time:

$$x^2 \rightarrow (x + 1)^2: \text{translation by vector } (-1, 0).$$

$$(x + 1)^2 \rightarrow 2(x + 1)^2: \text{stretch parallel to the } y\text{-axis, scale factor } 2.$$

$$2(x + 1)^2 \rightarrow 2(x + 1)^2 - 3: \text{translation by vector } (0, -3).$$

The order of the last two matters: if you translated down by 3 first and then stretched, you would get $2(x + 1)^2 - 6$.

Exam tip. When asked for the equation after a given sequence of transformations, apply each one in turn to the current equation, replacing x by $(x - a)$ for a translation of a to the right, and multiplying the whole right-hand side for a stretch in the y -direction.

3. Coordinate geometry

Coordinate geometry links algebra to straight lines and circles. Most marks come from setting up the right equation quickly: midpoints, gradients, perpendicular lines, and the equation of a circle.

3.1 Lines: length, midpoint, gradient and equation

Results to learn. For points $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\text{Length } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Midpoint of } AB = ((x_1 + x_2)/2, (y_1 + y_2)/2)$$

$$\text{Gradient of } AB, m = (y_2 - y_1)/(x_2 - x_1)$$

$$\text{Line through } (x_1, y_1) \text{ with gradient } m: y - y_1 = m(x - x_1)$$

Parallel lines have equal gradients. Perpendicular lines have gradients with product -1 : $m_1 \times m_2 = -1$.

The perpendicular bisector of AB is the line through the midpoint of AB that is perpendicular to AB . Every point on it is the same distance from A as from B , which is why it appears in questions about circles.

Worked example 14. A is $(1, 2)$ and B is $(5, -4)$. Find (a) the length AB in exact form and (b) the equation of the perpendicular bisector of AB in the form $ax + by + c = 0$.

$$(a) \text{ } AB = \sqrt{(5 - 1)^2 + (-4 - 2)^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}.$$

$$(b) \text{ Midpoint} = ((1 + 5)/2, (2 - 4)/2) = (3, -1).$$

$$\text{Gradient of } AB = (-4 - 2)/(5 - 1) = -6/4 = -3/2, \text{ so the perpendicular gradient is } 2/3.$$

$$y + 1 = (2/3)(x - 3)$$

$$3y + 3 = 2x - 6$$

$$2x - 3y - 9 = 0.$$

To find where two lines meet, solve their equations simultaneously. To find the foot of the perpendicular from a point to a line, write the equation of the perpendicular line through the point and find where it meets the original line.

3.2 The equation of a circle

Learn. The circle with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Expanded, an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

If the equation is given in expanded form, the safest method is to complete the square in x and in y separately rather than relying on the formula.

Worked example 15. Find the centre and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

$$(x^2 - 6x) + (y^2 + 4y) = 12$$

$$(x - 3)^2 - 9 + (y + 2)^2 - 4 = 12$$

$$(x - 3)^2 + (y + 2)^2 = 25$$

Centre (3, -2), radius 5.

Useful geometric facts: the angle in a semicircle is a right angle, so if AB is a diameter the centre is the midpoint of AB; the perpendicular from the centre to a chord bisects the chord; and the centre of a circle through three points is where the perpendicular bisectors of any two chords meet.

3.3 Tangents to a circle

Learn. A tangent to a circle is perpendicular to the radius at the point of contact.

Worked example 16. The point P(6, 2) lies on the circle $(x - 3)^2 + (y + 2)^2 = 25$. Find the equation of the tangent at P.

Check: $(6 - 3)^2 + (2 + 2)^2 = 9 + 16 = 25$, so P is on the circle.

Gradient of radius CP, from C(3, -2) to P(6, 2) = $(2 - (-2))/(6 - 3) = 4/3$.

Gradient of tangent = $-3/4$.

$$y - 2 = -(3/4)(x - 6)$$

$$4y - 8 = -3x + 18$$

$$3x + 4y - 26 = 0.$$

3.4 Where a line meets a circle

Substitute the line into the circle to get a quadratic in one variable. Two distinct roots means the line cuts the circle at two points (it is a chord), equal roots means the line is a tangent, and no real roots means the line misses the circle. This is the same discriminant idea as in section 1.4.

Worked example 17. Find the values of k for which the line $y = x + k$ is a tangent to the circle $(x - 3)^2 + (y + 2)^2 = 25$. Give exact answers.

Substitute $y = x + k$:

$$(x - 3)^2 + (x + k + 2)^2 = 25$$

$$x^2 - 6x + 9 + x^2 + 2(k + 2)x + (k + 2)^2 = 25$$

$$2x^2 + (2k - 2)x + (k + 2)^2 - 16 = 0$$

For a tangent, $b^2 - 4ac = 0$:

$$(2k - 2)^2 - 8[(k + 2)^2 - 16] = 0$$

$$4k^2 - 8k + 4 - 8k^2 - 32k + 96 = 0$$

$$-4k^2 - 40k + 100 = 0$$

$$k^2 + 10k - 25 = 0$$

$$k = [-10 \pm \sqrt{(100 + 100)}]/2 = -5 \pm 5\sqrt{2}.$$

Exam tip. A quick check for line and circle questions: the line is a tangent exactly when the perpendicular distance from the centre to the line equals the radius. In Worked example 17, the distance from (3, -2) to $x - y + k = 0$ is $|3 + 2 + k|/\sqrt{2}$, and setting this equal to 5 gives the same values of k. In the exam, use whichever method the question leads you towards, but show full working.

4. Circular measure

In pure mathematics, angles are normally measured in radians rather than degrees. Radians make the formulas for arc length and sector area very simple, and they are essential for calculus with trigonometric functions in Paper 3.

4.1 Radians

Definition to learn. One **radian** is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

π radians = 180° , so 1 radian $\approx 57.3^\circ$.

To convert degrees to radians multiply by $\pi/180$; to convert radians to degrees multiply by $180/\pi$.

Degrees	30°	45°	60°	90°
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$

Degrees	120°	180°	270°	360°
Radians	$2\pi/3$	π	$3\pi/2$	2π

Make sure your calculator is in the correct mode. A very common lost mark is calculating $\sin 1.2$ in degree mode when 1.2 is in radians.

4.2 Arc length and sector area

Learn. For a sector of a circle of radius r with angle θ radians at the centre:

Arc length $s = r\theta$

Sector area $A = \frac{1}{2}r^2\theta$

These formulas only work with θ in radians.

Worked example 18. A sector has radius 8 cm and angle 0.9 radians. Find its perimeter and area.

Arc length = $8 \times 0.9 = 7.2$ cm.

Perimeter = arc + two radii = $7.2 + 8 + 8 = 23.2$ cm.

Area = $\frac{1}{2} \times 8^2 \times 0.9 = 32 \times 0.9 = 28.8$ cm².

4.3 Segments, triangles and composite shapes

A segment is the region between a chord and an arc. Its area is the sector area minus the area of the triangle formed by the two radii and the chord. The triangle area uses $\frac{1}{2}ab \sin C$ with $a = b = r$.

Learn. Area of the triangle formed by two radii with angle θ between them = $\frac{1}{2}r^2 \sin \theta$.

Area of segment = $\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2}r^2(\theta - \sin \theta)$.

Length of the chord = $2r \sin(\theta/2)$, or use the cosine rule.

Worked example 19. A chord AB subtends an angle of $\pi/3$ radians at the centre O of a circle of radius 6 cm. Find (a) the exact area of the minor segment cut off by AB and (b) the perimeter of that segment, correct to 3 significant figures.

(a) Area = $\frac{1}{2} \times 36 \times (\pi/3 - \sin(\pi/3)) = 18(\pi/3 - \sqrt{3}/2) = 6\pi - 9\sqrt{3} \text{ cm}^2$ (about 3.26 cm²).

(b) Arc AB = $6 \times \pi/3 = 2\pi$ cm.

Chord AB = $2 \times 6 \times \sin(\pi/6) = 12 \times \frac{1}{2} = 6$ cm (the triangle is equilateral, since the angle at O is 60° and OA = OB).

Perimeter = $2\pi + 6 = 12.3$ cm (3 s.f.).

Exam questions often combine sectors, triangles and right angles in one diagram. Before calculating, mark every length and angle you know, look for right angles (tangent and radius) and use basic trigonometry (for example $\tan \theta = \text{opposite/adjacent}$) to find missing lengths. Exact values are frequently required, so keep π and surds in your working.

Exact values to learn.

$\sin(\pi/6) = \frac{1}{2}$, $\cos(\pi/6) = \sqrt{3}/2$, $\tan(\pi/6) = 1/\sqrt{3}$

$\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/4) = \sqrt{2}/2$, $\tan(\pi/4) = 1$

$\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/3) = \frac{1}{2}$, $\tan(\pi/3) = \sqrt{3}$

5. Trigonometry

Paper 1 trigonometry covers the graphs of sine, cosine and tangent, exact values, the two basic identities, inverse trigonometric functions and solving equations in a given interval. Paper 3 adds sec, cosec, cot and compound angles, so a secure foundation here is important.

5.1 Graphs of sine, cosine and tangent

$y = \sin x$ and $y = \cos x$ are waves with period 360° (2π) and values between -1 and 1 . The cosine graph is the sine graph translated 90° to the left. $y = \tan x$ has period 180° (π) and vertical asymptotes at $x = 90^\circ$, 270° and so on, where $\cos x = 0$.

Graph	Period	Range	Key points (0 to 360°)
$y = \sin x$	360° (2π)	$-1 \leq y \leq 1$	0 at 0° , 180° , 360° ; max 1 at 90° ; min -1 at 270°
$y = \cos x$	360° (2π)	$-1 \leq y \leq 1$	1 at 0° and 360° ; 0 at 90° , 270° ; min -1 at 180°
$y = \tan x$	180° (π)	all real values	0 at 0° , 180° , 360° ; asymptotes at 90° , 270°

The transformations from section 2.4 apply. For $y = a \sin bx + c$: the amplitude is $|a|$, the period is $360^\circ/b$ (or $2\pi/b$), and the graph is moved up by c , so it oscillates between $c - |a|$ and $c + |a|$.

Worked example 20. $f(x) = 3 - 2 \cos x$ for $0 \leq x \leq 2\pi$. State the range of f and the value of x at which f has its greatest value.

$\cos x$ takes every value from -1 to 1 , so $-2 \cos x$ takes every value from -2 to 2 , and $f(x)$ takes

every value from 1 to 5.

Range: $1 \leq f(x) \leq 5$.

The greatest value 5 occurs when $\cos x = -1$, that is $x = \pi$.

5.2 Identities

Learn.

$$\tan \theta \equiv \sin \theta / \cos \theta$$

$$\sin^2 \theta + \cos^2 \theta \equiv 1, \text{ so also } \sin^2 \theta \equiv 1 - \cos^2 \theta \text{ and } \cos^2 \theta \equiv 1 - \sin^2 \theta.$$

To prove an identity, start from one side (usually the more complicated one) and work towards the other. Do not treat the identity as an equation by doing the same thing to both sides. Common moves are: writing $\tan$ as $\sin/\cos$, putting fractions over a common denominator, and replacing $1 - \cos^2 \theta$ by $\sin^2 \theta$.

Worked example 21. Prove the identity $1/\cos \theta - \cos \theta \equiv \sin \theta \tan \theta$.

$$\text{LHS} = 1/\cos \theta - \cos \theta$$

$$= (1 - \cos^2 \theta)/\cos \theta$$

$$= \sin^2 \theta / \cos \theta$$

$$= \sin \theta \times (\sin \theta / \cos \theta)$$

$$= \sin \theta \tan \theta = \text{RHS}.$$

5.3 Inverse trigonometric functions

Because $\sin$, $\cos$ and $\tan$ are many-one, their inverses are defined on restricted domains. The calculator gives the value in the principal range.

Learn: principal values.

$$\sin^{-1} x: -\pi/2 \leq \sin^{-1} x \leq \pi/2 \text{ } (-90^\circ \text{ to } 90^\circ), \text{ for } -1 \leq x \leq 1$$

$$\cos^{-1} x: 0 \leq \cos^{-1} x \leq \pi \text{ } (0^\circ \text{ to } 180^\circ), \text{ for } -1 \leq x \leq 1$$

$$\tan^{-1} x: -\pi/2 < \tan^{-1} x < \pi/2 \text{ } (-90^\circ \text{ to } 90^\circ), \text{ for all } x$$

5.4 Solving trigonometric equations

Find the principal value with the calculator, then use the symmetry of the graph to find every solution in the required interval.

Rules for finding other solutions (degrees).

$$\sin x = k: x = \alpha \text{ and } 180^\circ - \alpha, \text{ then add or subtract } 360^\circ.$$

$$\cos x = k: x = \alpha \text{ and } 360^\circ - \alpha \text{ (or } -\alpha), \text{ then add or subtract } 360^\circ.$$

$$\tan x = k: x = \alpha, \text{ then add or subtract } 180^\circ \text{ repeatedly.}$$

Here α is the principal value from the calculator. In radians, replace 180° by π and 360° by 2π .

Worked example 22. Solve $2 \sin x = -1$ for $0^\circ \leq x \leq 360^\circ$.

$\sin x = -1/2$. The principal value is -30° , which is outside the interval.

Using the symmetry: $x = 180^\circ - (-30^\circ) = 210^\circ$, and $x = -30^\circ + 360^\circ = 330^\circ$.
Solutions: $x = 210^\circ, 330^\circ$.

Worked example 23 (multiple angle). Solve $\sin 2\theta = 0.6$ for $0^\circ \leq \theta \leq 180^\circ$.

If $0^\circ \leq \theta \leq 180^\circ$ then $0^\circ \leq 2\theta \leq 360^\circ$. Solve for 2θ in this wider interval first.
 $\sin^{-1} 0.6 = 36.87^\circ$, so $2\theta = 36.87^\circ$ or $180^\circ - 36.87^\circ = 143.13^\circ$.
 $\theta = 18.4^\circ$ or 71.6° (1 d.p.).

Worked example 24 (quadratic in $\sin x$). Solve $2 \cos^2 x + 3 \sin x = 3$ for $0^\circ \leq x \leq 360^\circ$.

Use $\cos^2 x = 1 - \sin^2 x$ so that only $\sin x$ appears:

$$2(1 - \sin^2 x) + 3 \sin x - 3 = 0$$

$$-2 \sin^2 x + 3 \sin x - 1 = 0$$

$$2 \sin^2 x - 3 \sin x + 1 = 0$$

$$(2 \sin x - 1)(\sin x - 1) = 0$$

$\sin x = \frac{1}{2}$ gives $x = 30^\circ, 150^\circ$. $\sin x = 1$ gives $x = 90^\circ$.

Solutions: $x = 30^\circ, 90^\circ, 150^\circ$.

Worked example 25 (using $\tan$). Solve $\sin x = 2 \cos x$ for $0^\circ \leq x \leq 360^\circ$.

Divide by $\cos x$ ($\cos x = 0$ is not a solution, because then $\sin x$ would be ± 1 , not 0):
 $\tan x = 2$, so $x = 63.4^\circ$ or $63.4^\circ + 180^\circ = 243.4^\circ$.

Exam tip. Never divide both sides by $\sin x$ or $\cos x$ when it could be zero: you lose solutions. In $\sin x \cos x = \sin x$, factorise instead: $\sin x(\cos x - 1) = 0$, so $\sin x = 0$ or $\cos x = 1$. Always check the interval at the end and give the number of solutions the graph suggests.

6. Series

This chapter has three parts: the binomial expansion of $(a + b)^n$ for a positive integer n , arithmetic progressions and geometric progressions. The expansion for other values of n comes in Paper 3.

6.1 The binomial expansion

Learn. For a positive integer n ,

$$(a + b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_r a^{n-r}b^r + \dots + b^n$$

where ${}^nC_r = \frac{n!}{r!(n-r)!}$, also written as the binomial coefficient " n choose r ".

The general term (the term containing b^r) is ${}^nC_r a^{n-r}b^r$.

The coefficients for small n can be read from Pascal's triangle: 1; 1 1; 1 2 1; 1 3 3 1; 1 4 6 4 1; 1 5 10 10 5 1; 1 6 15 20 15 6 1. When b is negative, put it in brackets so that the signs alternate correctly, and remember that powers of a coefficient apply to the number as well as to x : $(2x)^3 = 8x^3$.

Worked example 26. Find the first three terms in the expansion of $(2 - x)^5$ in ascending powers of

x.

$$\begin{aligned}(2-x)^5 &= 2^5 + {}^5C_1(2^4)(-x) + {}^5C_2(2^3)(-x)^2 + \dots \\ &= 32 + 5(16)(-x) + 10(8)(x^2) + \dots \\ &= 32 - 80x + 80x^2 + \dots\end{aligned}$$

Worked example 27 (term independent of x). Find the term independent of x in the expansion of $(x^2 + 2/x)^6$.

$$\text{General term: } {}^6C_r(x^2)^{6-r}(2/x)^r = {}^6C_r \times 2^r \times x^{12-2r} \times x^{-r} = {}^6C_r \times 2^r \times x^{12-3r}.$$

Independent of x means the power is 0: $12 - 3r = 0$, so $r = 4$.

$$\text{Term} = {}^6C_4 \times 2^4 = 15 \times 16 = 240.$$

Worked example 28 (product of two brackets). Find the coefficient of x^2 in the expansion of $(1 + 2x)(1 - x)^6$.

$$(1 - x)^6 = 1 - 6x + 15x^2 - \dots$$

The x^2 terms in the product come from $1 \times 15x^2$ and $2x \times (-6x)$.

$$\text{Coefficient} = 15 + 2 \times (-6) = 3.$$

6.2 Arithmetic progressions

An arithmetic progression (AP) is a sequence in which each term is obtained from the previous one by adding a fixed number, the common difference d . For example 5, 8, 11, 14, ... has first term $a = 5$ and $d = 3$.

Learn. For an AP with first term a and common difference d :

$$\text{nth term: } u_n = a + (n - 1)d$$

$$\text{Sum of the first } n \text{ terms: } S_n = (n/2)[2a + (n - 1)d] = (n/2)(a + l), \text{ where } l \text{ is the last term.}$$

Worked example 29. The fifth term of an AP is 17 and the sum of the first 10 terms is 185. Find the first term and the common difference.

$$a + 4d = 17 \dots (1)$$

$$(10/2)(2a + 9d) = 185, \text{ so } 2a + 9d = 37 \dots (2)$$

$$\text{Doubling (1): } 2a + 8d = 34. \text{ Subtract from (2): } d = 3. \text{ Then } a = 17 - 12 = 5.$$

$$\text{Check: } 5, 8, 11, 14, 17 \text{ and } S_{10} = 5(10 + 27) = 185.$$

The sum of the first n positive integers is a special case: $1 + 2 + \dots + n = n(n + 1)/2$. "Find the sum of all the multiples of 7 between 100 and 300" is an AP question: the first multiple is 105, the last is 294, and there are $(294 - 105)/7 + 1 = 28$ terms, so the sum is $14(105 + 294) = 5586$.

6.3 Geometric progressions

A geometric progression (GP) is a sequence in which each term is obtained from the previous one by multiplying by a fixed number, the common ratio r . For example 3, 6, 12, 24, ... has $a = 3$ and $r = 2$.

Learn. For a GP with first term a and common ratio r :

n th term: $u_n = ar^{n-1}$

Sum of the first n terms: $S_n = a(1 - r^n)/(1 - r) = a(r^n - 1)/(r - 1)$, for $r \neq 1$

Sum to infinity: $S_\infty = a/(1 - r)$, which exists only when $|r| < 1$, that is $-1 < r < 1$.

A GP is called convergent when $|r| < 1$: the terms get smaller and smaller and their sum approaches a limit. You should be able to state this condition when asked why a sum to infinity exists.

Worked example 30. The second term of a GP is 12 and its sum to infinity is 64. Find the two possible values of the common ratio and the corresponding first terms.

$ar = 12 \dots (1)$ and $a/(1 - r) = 64$, so $a = 64(1 - r) \dots (2)$

Substitute (2) into (1): $64r(1 - r) = 12$

$64r - 64r^2 = 12$

$16r^2 - 16r + 3 = 0$

$(4r - 1)(4r - 3) = 0$, so $r = 1/4$ or $r = 3/4$.

$r = 1/4$ gives $a = 64 \times 3/4 = 48$; $r = 3/4$ gives $a = 64 \times 1/4 = 16$. Both values satisfy $|r| < 1$, so both are valid.

Worked example 31. For the GP 3, 6, 12, ..., find the least number of terms for which the sum exceeds 1000.

$S_n = 3(2^n - 1)/(2 - 1) = 3(2^n - 1)$.

We need $3(2^n - 1) > 1000$, so $2^n > 334.3\dots$

$2^8 = 256$ (too small) and $2^9 = 512$, so $n = 9$.

Check: $S_8 = 765$ and $S_9 = 1533$.

Exam tip. Questions that link an AP and a GP (for example "the first, third and ninth terms of an AP form a GP") are common. Write each term in terms of a and d , then use the GP property that the ratio of consecutive terms is constant: if x, y, z are consecutive terms of a GP then $y/x = z/y$, so $y^2 = xz$.

7. Differentiation

Differentiation measures how fast one quantity changes compared with another. Geometrically, the derivative dy/dx gives the gradient of the curve $y = f(x)$ at each point. In Paper 1 you differentiate powers of x (including negative and fractional powers) and use the chain rule; Paper 3 extends this to exponentials, logarithms, trigonometric functions, products and quotients.

7.1 The derivative as a gradient

The gradient of a curve at a point is the gradient of the tangent there. It is found as a limit: take a chord from $(x, f(x))$ to a nearby point $(x + h, f(x + h))$, work out its gradient $[f(x + h) - f(x)]/h$, and let h tend to 0. You should understand this idea, but in the exam you use the rules below.

Learn.

If $y = x^n$, then $dy/dx = nx^{n-1}$, for any rational n .

If $y = kx^n$, then $dy/dx = knx^{n-1}$. The derivative of a constant is 0.

Differentiate a sum term by term. $f'(x)$ is another notation for dy/dx ; the second derivative d^2y/dx^2 (or $f''(x)$) is the derivative of dy/dx .

Before differentiating, rewrite every term as a power of x : $1/x^2 = x^{-2}$, $\sqrt{x} = x^{1/2}$, $1/\sqrt{x} = x^{-1/2}$. Products and fractions must be multiplied out or divided through first (in Paper 1 there is no product or quotient rule).

Worked example 32. Find dy/dx when $y = 4x^3 - 6/x + 2\sqrt{x}$.

Rewrite: $y = 4x^3 - 6x^{-1} + 2x^{1/2}$

$$dy/dx = 12x^2 + 6x^{-2} + x^{-1/2}$$

$$= 12x^2 + 6/x^2 + 1/\sqrt{x}.$$

7.2 The chain rule

The chain rule differentiates a function of a function. If y is a function of u , and u is a function of x , then the rate of change of y with x is the product of the two rates.

Learn.

$$dy/dx = dy/du \times du/dx$$

In particular, if $y = (ax + b)^n$ then $dy/dx = an(ax + b)^{n-1}$, and if $y = [f(x)]^n$ then $dy/dx = n[f(x)]^{n-1} \times f'(x)$.

Worked example 33. Differentiate (a) $y = (3x^2 - 1)^5$ and (b) $y = 1/\sqrt{2x + 5}$.

(a) Let $u = 3x^2 - 1$, so $y = u^5$. $dy/du = 5u^4$ and $du/dx = 6x$.

$$dy/dx = 5(3x^2 - 1)^4 \times 6x = 30x(3x^2 - 1)^4.$$

(b) $y = (2x + 5)^{-1/2}$

$$dy/dx = -\frac{1}{2}(2x + 5)^{-3/2} \times 2 = -(2x + 5)^{-3/2} = -1/(2x + 5)^{3/2}.$$

7.3 Tangents and normals

The tangent at a point has the same gradient as the curve there. The normal is the line through the same point perpendicular to the tangent, so its gradient is $-1/m$ when the tangent gradient is m .

Worked example 34. Find the equations of the tangent and the normal to $y = x^3 - 4x + 1$ at the point where $x = 2$.

When $x = 2$, $y = 8 - 8 + 1 = 1$. The point is $(2, 1)$.

$$dy/dx = 3x^2 - 4; \text{ at } x = 2, dy/dx = 12 - 4 = 8.$$

Tangent: $y - 1 = 8(x - 2)$, so $y = 8x - 15$.

Normal gradient = $-1/8$: $y - 1 = -(1/8)(x - 2)$, so $8y - 8 = -x + 2$, giving $x + 8y - 10 = 0$.

7.4 Increasing and decreasing functions

Learn. A function f is **increasing** on an interval if $f'(x) > 0$ there, and **decreasing** if $f'(x) < 0$ there.

Worked example 35. (a) Show that $f(x) = x^3 + 3x - 7$ is an increasing function. (b) Find the set of values of x for which $g(x) = x^3 - 12x$ is decreasing.

(a) $f'(x) = 3x^2 + 3$. Since $x^2 \geq 0$, $f'(x) \geq 3 > 0$ for all x , so f is increasing.

(b) $g'(x) = 3x^2 - 12$. g is decreasing when $3x^2 - 12 < 0$, so $x^2 < 4$, giving $-2 < x < 2$.

This links back to functions: a function that is increasing (or decreasing) throughout its domain is one-one, so it has an inverse. Exam questions sometimes ask you to use the derivative to justify that an inverse exists.

7.5 Stationary points

Learn. A **stationary point** is a point where $dy/dx = 0$.

If $d^2y/dx^2 < 0$ there, it is a maximum point.

If $d^2y/dx^2 > 0$ there, it is a minimum point.

If $d^2y/dx^2 = 0$, the test fails: check the sign of dy/dx just either side of the point instead.

Worked example 36. Find the stationary points of $y = 2x^3 - 9x^2 + 12x - 1$ and determine their nature.

$$dy/dx = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2).$$

$$dy/dx = 0 \text{ gives } x = 1 \text{ or } x = 2.$$

$$x = 1: y = 2 - 9 + 12 - 1 = 4. \quad x = 2: y = 16 - 36 + 24 - 1 = 3.$$

$$d^2y/dx^2 = 12x - 18.$$

At $x = 1$: $12 - 18 = -6 < 0$, so $(1, 4)$ is a maximum point.

At $x = 2$: $24 - 18 = 6 > 0$, so $(2, 3)$ is a minimum point.

Practical optimisation problems use the same method. Write the quantity to be maximised or minimised in terms of one variable (use the given constraint to eliminate the other), differentiate, set the derivative to zero, and confirm the nature with the second derivative.

Worked example 37. A farmer has 100 m of fencing to make a rectangular pen against a long straight wall, so fencing is needed on three sides only. The two sides perpendicular to the wall are each x m long. Find the maximum area.

The side parallel to the wall is $100 - 2x$, so the area is $A = x(100 - 2x) = 100x - 2x^2$.

$$dA/dx = 100 - 4x = 0 \text{ gives } x = 25.$$

$$d^2A/dx^2 = -4 < 0, \text{ so this is a maximum.}$$

$$\text{Maximum area} = 25 \times 50 = 1250 \text{ m}^2.$$

7.6 Rates of change

When two or more quantities change with time, the chain rule links their rates of change. Identify the

rate you are given, the rate you want, and the formula that connects the two variables.

Learn. Connected rates of change: $dy/dt = dy/dx \times dx/dt$.

Worked example 38. A spherical balloon is being inflated so that its volume increases at a constant rate of $50 \text{ cm}^3 \text{ s}^{-1}$. Find the rate at which the radius is increasing when the radius is 5 cm. (The volume of a sphere is $V = (4/3)\pi r^3$.)

$$\begin{aligned} dV/dr &= 4\pi r^2; \text{ when } r = 5, dV/dr = 100\pi. \\ dV/dt &= dV/dr \times dr/dt, \text{ so } 50 = 100\pi \times dr/dt. \\ dr/dt &= 50/(100\pi) = 1/(2\pi) \approx 0.159 \text{ cm s}^{-1}. \end{aligned}$$

Exam tip. Write the chain rule out in letters first (for example $dr/dt = dr/dV \times dV/dt$) and check that the units cancel sensibly. A negative rate means the quantity is decreasing; say so in words if the question asks for "the rate of decrease".

8. Integration

Integration is the reverse of differentiation. It is used to find a function from its derivative, to calculate areas under curves and between curves, and to find volumes of revolution.

8.1 Integration as the reverse of differentiation

Learn.

$$\int x^n dx = x^{n+1}/(n+1) + c, \text{ for } n \neq -1.$$

$$\int (ax + b)^n dx = (ax + b)^{n+1}/[a(n+1)] + c, \text{ for } n \neq -1.$$

Every indefinite integral needs the constant of integration, + c.

As with differentiation, rewrite terms as powers of x first. The case $n = -1$ ($\int 1/x dx$) involves logarithms and is in Paper 3, not Paper 1.

Worked example 39. Find (a) $\int (6x^2 - 4/x^3) dx$ and (b) $\int 1/\sqrt{4x+1} dx$.

$$(a) \int (6x^2 - 4x^{-3}) dx = 6x^3/3 - 4x^{-2}/(-2) + c = 2x^3 + 2x^{-2} + c = 2x^3 + 2/x^2 + c.$$

$$(b) \int (4x+1)^{-1/2} dx = (4x+1)^{1/2}/(4 \times 1/2) + c = \frac{1}{2}\sqrt{4x+1} + c.$$

$$\text{Check by differentiating: } d/dx[\frac{1}{2}(4x+1)^{1/2}] = \frac{1}{2} \times \frac{1}{2} \times 4 \times (4x+1)^{-1/2} = (4x+1)^{-1/2}.$$

8.2 Finding the equation of a curve

If you know dy/dx and one point on the curve, integrate to get y with a constant c, then substitute the point to find c.

Worked example 40. A curve has gradient $dy/dx = 3x^2 - 2x$ and passes through (2, 5). Find its equation.

$$y = x^3 - x^2 + c.$$

$$\text{At } (2, 5): 5 = 8 - 4 + c, \text{ so } c = 1.$$

$$y = x^3 - x^2 + 1.$$

8.3 Definite integrals and area

Learn. $\int_a^b f(x) dx = F(b) - F(a)$, where F is any integral of f .

The area between the curve $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$ is $\int_a^b y dx$ when the curve is above the axis.

If the curve is below the axis the integral is negative: the area is its magnitude. If the curve crosses the axis, integrate each part separately.

The area between the curve $x = g(y)$, the y -axis and the lines $y = c$ and $y = d$ is $\int_c^d x dy$.

Worked example 41. Evaluate $\int_1^4 (3\sqrt{x} - 1) dx$.

$$\int (3x^{1/2} - 1) dx = 3x^{3/2}/(3/2) - x = 2x^{3/2} - x.$$

$$[2x^{3/2} - x] \text{ from } 1 \text{ to } 4 = (2 \times 8 - 4) - (2 \times 1 - 1) = 12 - 1 = 11.$$

Worked example 42. Find the area enclosed by the curve $y = x^2 - 4x$ and the x -axis.

The curve meets the axis where $x(x - 4) = 0$, so $x = 0$ and $x = 4$. Between these, the curve is below the axis.

$$\int_0^4 (x^2 - 4x) dx = [x^3/3 - 2x^2] \text{ from } 0 \text{ to } 4 = 64/3 - 32 = -32/3.$$

Area = $32/3$ square units.

8.4 Area between a curve and a line

Find where they meet, then integrate (upper graph – lower graph) between those x -values. This gives the area directly, even if part of the region is below the x -axis.

Worked example 43. Find the area of the region enclosed by $y = 6x - x^2$ and $y = 2x$.

Intersection: $6x - x^2 = 2x$ gives $x^2 - 4x = 0$, so $x = 0$ or $x = 4$.

For $0 < x < 4$ the curve is above the line (for example at $x = 1$: $5 > 2$).

$$\text{Area} = \int_0^4 (6x - x^2 - 2x) dx = \int_0^4 (4x - x^2) dx = [2x^2 - x^3/3] \text{ from } 0 \text{ to } 4 = 32 - 64/3 = 32/3 \text{ square units.}$$

8.5 Improper integrals

An integral is improper when a limit is infinite, or when the function is undefined at a limit. Evaluate it by replacing the problem limit with a letter and seeing what happens as that letter tends to the problem value. If the result tends to a finite number, that is the value of the integral.

Worked example 44. Evaluate (a) $\int_1^\infty 1/x^2 dx$ and (b) $\int_0^1 1/\sqrt{x} dx$.

$$(a) \int_1^k x^{-2} dx = [-1/x] \text{ from } 1 \text{ to } k = -1/k + 1. \text{ As } k \rightarrow \infty, -1/k \rightarrow 0, \text{ so the integral is } 1.$$

(b) $1/\sqrt{x}$ is undefined at $x = 0$. $\int_k^1 x^{-1/2} dx = [2\sqrt{x}]$ from k to $1 = 2 - 2\sqrt{k}$. As $k \rightarrow 0$, this tends to 2, so the integral is 2.

8.6 Volumes of revolution

Learn. When the region under $y = f(x)$ between $x = a$ and $x = b$ is rotated through 360° about the x -axis, the volume of the solid formed is

$$V = \pi \int_a^b y^2 dx$$

For rotation about the y -axis, between $y = c$ and $y = d$: $V = \pi \int_c^d x^2 dy$.

Worked example 45. The region bounded by $y = x^2 + 1$, the axes and $x = 1$ is rotated through 360° about the x -axis. Find the exact volume.

$$y^2 = (x^2 + 1)^2 = x^4 + 2x^2 + 1.$$

$$\begin{aligned} V &= \pi \int_0^1 (x^4 + 2x^2 + 1) dx = \pi [x^5/5 + 2x^3/3 + x] \text{ from } 0 \text{ to } 1 \\ &= \pi(1/5 + 2/3 + 1) = \pi(3/15 + 10/15 + 15/15) = 28\pi/15. \end{aligned}$$

Worked example 46. The region between the curve $y = x^2$ (for $x \geq 0$), the y -axis and the lines $y = 1$ and $y = 4$ is rotated through 360° about the y -axis. Find the exact volume.

$$x^2 = y, \text{ so } V = \pi \int_1^4 y dy = \pi[y^2/2] \text{ from } 1 \text{ to } 4 = \pi(8 - 1/2) = 15\pi/2.$$

Exam tip. For a volume between a curve and a line (a "washer"), subtract volumes, not heights: $V = \pi \int (y_1^2 - y_2^2) dx$, where y_1 is the outer boundary. Squaring $(y_1 - y_2)$ is a common error. Often it is easiest to find the volume under the curve and subtract a cone or cylinder whose volume you know.

Quick check

Answer these exam-style questions without looking back, then mark your work against the full worked answers below. Give non-exact answers to 3 significant figures and angles to 1 decimal place.

- Quadratics.** (a) Express $3x^2 + 12x + 5$ in the form $a(x + b)^2 + c$. (b) State the coordinates of the minimum point of the curve $y = 3x^2 + 12x + 5$.
- Quadratics.** Find the set of values of k for which the line $y = kx - 2$ does not meet the curve $y = x^2 + 2x + 2$.
- Functions.** The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 4x - 1$ and $g(x) = x^2 + 2$. Find $gf(x)$ and solve the equation $gf(x) = 11$.
- Functions.** The function h is defined by $h(x) = x^2 - 8x + 10$ for $x \geq 4$. (a) Find the range of h . (b) Find an expression for $h^{-1}(x)$ and state the domain of h^{-1} .
- Coordinate geometry.** A is the point $(-2, 3)$ and B is the point $(4, 7)$. Find the equation of the perpendicular bisector of AB, giving your answer in the form $ax + by + c = 0$.
- Coordinate geometry.** A circle has centre $C(2, -1)$ and passes through the point $P(5, 3)$. (a) Find the equation of the circle. (b) Find the equation of the tangent to the circle at P.

7. **Circular measure.** OAB is a sector of a circle with centre O, radius 10 cm and angle AOB = 1.2 radians. Calculate (a) the perimeter of the sector and (b) the area of the segment bounded by the chord AB and the arc AB.
8. **Trigonometry.** Solve the equation $3 \sin^2 \theta - 2 \cos \theta - 2 = 0$ for $0^\circ \leq \theta \leq 360^\circ$.
9. **Trigonometry.** Prove the identity $(1 + \sin \theta)/\cos \theta + \cos \theta/(1 + \sin \theta) \equiv 2/\cos \theta$.
10. **Series.** Find the coefficient of x^3 in the expansion of $(3 - 2x)^5$.
11. **Series.** An arithmetic progression has first term 7 and common difference 4. Find the least number of terms needed for the sum to exceed 1000.
12. **Series.** A geometric progression has first term 24 and sum to infinity 96. Find the common ratio and the sum of the first 5 terms.
13. **Differentiation.** Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 - 9x + 2$ and determine their nature.
14. **Differentiation.** A point moves along the curve $y = 2/(3x - 1)$ in such a way that its x-coordinate increases at a constant rate of 0.4 units per second. Find the rate of change of the y-coordinate when $x = 1$.
15. **Integration.** The region R is enclosed by the curve $y = 4 - x^2$ and the x-axis. (a) Find the area of R. (b) Find the exact volume of the solid formed when R is rotated through 360° about the x-axis.

1. (a) $3x^2 + 12x + 5 = 3(x^2 + 4x) + 5 = 3[(x + 2)^2 - 4] + 5 = 3(x + 2)^2 - 7$. So $a = 3$, $b = 2$, $c = -7$.
(b) The minimum point is $(-2, -7)$.

2. At intersection: $x^2 + 2x + 2 = kx - 2$, so $x^2 + (2 - k)x + 4 = 0$.
No intersection requires $b^2 - 4ac < 0$: $(2 - k)^2 - 16 < 0$.
 $(2 - k)^2 < 16$, so $-4 < 2 - k < 4$, giving $-2 < k < 6$.
(Alternatively expand: $k^2 - 4k - 12 < 0$, $(k - 6)(k + 2) < 0$, so $-2 < k < 6$.)

3. $gf(x) = g(4x - 1) = (4x - 1)^2 + 2$.
 $(4x - 1)^2 + 2 = 11$, so $(4x - 1)^2 = 9$ and $4x - 1 = \pm 3$.
 $4x - 1 = 3$ gives $x = 1$; $4x - 1 = -3$ gives $x = -\frac{1}{2}$.

4. (a) $h(x) = (x - 4)^2 - 16 + 10 = (x - 4)^2 - 6$. For $x \geq 4$ the least value is -6 at $x = 4$, so the range is $h(x) \geq -6$.
(b) $y = (x - 4)^2 - 6$, so $(x - 4)^2 = y + 6$ and $x - 4 = \sqrt{y + 6}$ (positive root since $x \geq 4$).
 $h^{-1}(x) = 4 + \sqrt{y + 6}$, with domain $x \geq -6$.

5. Midpoint of AB = $((-2 + 4)/2, (3 + 7)/2) = (1, 5)$.
Gradient of AB = $(7 - 3)/(4 - (-2)) = 4/6 = 2/3$, so the perpendicular gradient is $-3/2$.
 $y - 5 = -(3/2)(x - 1)$, so $2y - 10 = -3x + 3$, giving $3x + 2y - 13 = 0$.

6. (a) $r^2 = (5 - 2)^2 + (3 - (-1))^2 = 9 + 16 = 25$. Equation: $(x - 2)^2 + (y + 1)^2 = 25$.
(b) Gradient of CP = $(3 - (-1))/(5 - 2) = 4/3$, so the tangent gradient is $-3/4$.
 $y - 3 = -(3/4)(x - 5)$, so $4y - 12 = -3x + 15$, giving $3x + 4y - 27 = 0$.

7. (a) Arc AB = $10 \times 1.2 = 12$ cm. Perimeter = $12 + 10 + 10 = 32$ cm.
(b) Sector area = $\frac{1}{2} \times 100 \times 1.2 = 60$ cm². Triangle area = $\frac{1}{2} \times 100 \times \sin 1.2 = 50 \times 0.93204 = 46.602$ cm² (calculator in radians).
Segment area = $60 - 46.602 = 13.4$ cm² (3 s.f.).

8. Replace $\sin^2 \theta$ by $1 - \cos^2 \theta$: $3(1 - \cos^2 \theta) - 2 \cos \theta - 2 = 0$.

$1 - 3 \cos^2 \theta - 2 \cos \theta = 0$, so $3 \cos^2 \theta + 2 \cos \theta - 1 = 0$.

$(3 \cos \theta - 1)(\cos \theta + 1) = 0$, so $\cos \theta = 1/3$ or $\cos \theta = -1$.

$\cos \theta = 1/3$: $\theta = 70.5^\circ$ or $360^\circ - 70.5^\circ = 289.5^\circ$.

$\cos \theta = -1$: $\theta = 180^\circ$.

Solutions: $\theta = 70.5^\circ, 180^\circ, 289.5^\circ$.

9. LHS = $[(1 + \sin \theta)^2 + \cos^2 \theta] / [\cos \theta(1 + \sin \theta)]$

= $[1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta] / [\cos \theta(1 + \sin \theta)]$

= $[2 + 2 \sin \theta] / [\cos \theta(1 + \sin \theta)]$ (using $\sin^2 \theta + \cos^2 \theta = 1$)

= $2(1 + \sin \theta) / [\cos \theta(1 + \sin \theta)]$

= $2/\cos \theta = \text{RHS}$.

10. The x^3 term is ${}^5C_3(3)^2(-2x)^3 = 10 \times 9 \times (-8x^3) = -720x^3$.

Coefficient = -720 .

11. $S_n = (n/2)[14 + 4(n - 1)] = (n/2)(4n + 10) = n(2n + 5)$.

We need $n(2n + 5) > 1000$, so $2n^2 + 5n - 1000 > 0$.

The positive root of $2n^2 + 5n - 1000 = 0$ is $n = [-5 + \sqrt{(25 + 8000)}]/4 = (-5 + 89.58)/4 = 21.1$ (3 s.f.).

So n must be at least 22. Check: $S_{21} = 21 \times 47 = 987$ and $S_{22} = 22 \times 49 = 1078$. Least number of terms = 22.

12. $24/(1 - r) = 96$, so $1 - r = 1/4$ and $r = 3/4$.

$S_5 = 24(1 - (3/4)^5)/(1 - 3/4) = 96(1 - 243/1024) = 96 \times 781/1024 = 73.21875 = 73.2$ (3 s.f.).

13. $dy/dx = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$.

$dy/dx = 0$ gives $x = -1$ or $x = 3$.

$x = -1$: $y = -1 - 3 + 9 + 2 = 7$. $x = 3$: $y = 27 - 27 - 27 + 2 = -25$.

$d^2y/dx^2 = 6x - 6$. At $x = -1$ it is $-12 < 0$, so $(-1, 7)$ is a maximum. At $x = 3$ it is $12 > 0$, so $(3, -25)$ is a minimum.

14. $y = 2(3x - 1)^{-1}$, so $dy/dx = -2(3x - 1)^{-2} \times 3 = -6/(3x - 1)^2$.

At $x = 1$: $dy/dx = -6/4 = -1.5$.

$dy/dt = dy/dx \times dx/dt = -1.5 \times 0.4 = -0.6$.

The y -coordinate is decreasing at 0.6 units per second.

15. (a) The curve meets the x -axis where $4 - x^2 = 0$, $x = \pm 2$.

Area = $\int_{-2}^2 (4 - x^2) dx = [4x - x^3/3]$ from -2 to $2 = (8 - 8/3) - (-8 + 8/3) = 16 - 16/3 = 32/3$ square units.

(b) $y^2 = (4 - x^2)^2 = 16 - 8x^2 + x^4$.

$V = \pi \int_{-2}^2 (16 - 8x^2 + x^4) dx = \pi [16x - 8x^3/3 + x^5/5]$ from -2 to 2 .

The function is even, so this is $2\pi [16x - 8x^3/3 + x^5/5]$ from 0 to $2 = 2\pi (32 - 64/3 + 32/5)$.

= $2\pi (480/15 - 320/15 + 96/15) = 2\pi \times 256/15 = 512\pi/15$.