

Circular motion: A Level Physics notes and worked problems

These notes cover the Motion in a circle topic of the Cambridge International A Level Physics syllabus (9702). They explain how to measure angles in radians, how angular velocity links to ordinary speed, why an object moving in a circle at constant speed is accelerating, and how to identify which real force provides the centripetal force in horizontal circles, vertical circles and on banked tracks. Take $g = 9.81 \text{ m s}^{-2}$ throughout, as in the data booklet.

1. Radian measure and angular displacement

1.1 The radian

In circular motion, angles are measured in radians, not degrees. All the equations in this topic only work when angles are in radians.

Radian: one radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

If an arc of length s is drawn on a circle of radius r , the angle θ it subtends at the centre is

$$\theta = s / r \text{ (}\theta \text{ in radians, } s \text{ and } r \text{ in metres)}$$

Rearranged: $s = r\theta$. For a complete circle the arc length is the circumference, $2\pi r$, so a full turn is $2\pi r / r = 2\pi$ radians.

Degrees	Radians
360°	$2\pi \text{ rad}$ (about 6.28 rad)
180°	$\pi \text{ rad}$
90°	$\pi/2 \text{ rad}$
1 rad	about 57.3°

To convert degrees to radians multiply by $\pi/180$. To convert radians to degrees multiply by $180/\pi$. The radian is a ratio of two lengths, so it has no dimensions, but you should still write "rad" after an angle.

1.2 Angular displacement

Angular displacement: the angle through which an object has turned about the centre of the circle, measured in radians.

Worked example 1: angles and arc lengths

A point on the rim of a wheel of radius 0.20 m turns through 2.5 revolutions. Calculate its angular displacement and the distance it travels.

Angular displacement: $2.5 \times 2\pi = 5\pi = 15.7 \text{ rad}$.



Distance along the arc: $s = r\theta = 0.20 \times 15.7 = 3.14 \text{ m}$.

2. Angular velocity

2.1 Definition

Angular velocity (ω): the rate of change of angular displacement. $\omega = \Delta\theta / \Delta t$, unit rad s^{-1} .

For an object moving at a steady rate, one full turn ($2\pi \text{ rad}$) takes one period T , so

$$\omega = 2\pi / T = 2\pi f$$

where f is the frequency (number of revolutions per second, unit Hz).

2.2 Linking speed and angular velocity

In a time Δt the object moves through angle $\Delta\theta$ and travels an arc length $\Delta s = r\Delta\theta$. Dividing by Δt :

$$\Delta s / \Delta t = r \times \Delta\theta / \Delta t, \text{ so } \mathbf{v = r\omega}$$

Here v is the linear speed along the circle. The direction of the velocity at any instant is along the tangent to the circle.

All points on a rigid rotating object (a wheel, a turntable, the Earth) have the same angular velocity, but points further from the axis have a greater linear speed, because v is proportional to r .

Worked example 2: a clock hand

The minute hand of a clock is 0.12 m long. Calculate the angular velocity of the hand and the speed of its tip.

Period: $T = 60 \text{ minutes} = 3600 \text{ s}$.

$$\omega = 2\pi / T = 2\pi / 3600 = 1.75 \times 10^{-3} \text{ rad s}^{-1}.$$

$$v = r\omega = 0.12 \times 1.75 \times 10^{-3} = 2.09 \times 10^{-4} \text{ m s}^{-1}.$$

Exam tip: when rotation rate is given in revolutions per minute (rpm), divide by 60 to get revolutions per second (f), then multiply by 2π to get ω in rad s^{-1} . Forgetting the 2π is the most common error in this topic.

3. Centripetal acceleration

3.1 Why constant speed in a circle still means acceleration

Velocity is a vector: it has a size and a direction. An object moving round a circle at constant speed is continually changing direction, so its velocity is changing. A changing velocity means the object is accelerating, even though its speed does not change.

The change in velocity over a short time points towards the centre of the circle. So the acceleration is directed towards the centre. This is called the **centripetal acceleration** ("centripetal" means "centre-seeking").

Centripetal acceleration: the acceleration of an object moving in a circle, directed towards the centre of the circle.

$$a = v^2 / r = r\omega^2 = v\omega$$

The forms are equivalent: substitute $v = r\omega$ into v^2/r to get $r^2\omega^2/r = r\omega^2$. Choose the form that matches the data you are given. If you know v , use v^2/r ; if you know ω or the period, use $r\omega^2$.

3.2 Where the formula comes from

Consider an object moving at speed v round a circle of radius r . In a short time Δt it turns through a small angle $\Delta\theta$. The velocity vectors at the start and end have the same length v , and the angle between them is also $\Delta\theta$. The change in velocity, Δv , is the short side of an isosceles triangle with two sides v and angle $\Delta\theta$ between them. For a small angle, $\Delta v \approx v\Delta\theta$, and Δv points towards the centre.

$$a = \Delta v / \Delta t = v \times \Delta\theta / \Delta t = v\omega. \text{ Using } \omega = v/r, a = v^2/r.$$

You may be asked to explain the direction of the acceleration in words, but the 9702 syllabus does not require this derivation to be reproduced in full.

4. Centripetal force

4.1 Newton's second law applied to circles

By Newton's second law, an acceleration needs a resultant force in the same direction. So an object moving in a circle must have a resultant force acting on it towards the centre.

Centripetal force: the resultant force acting on an object moving in a circle, directed towards the centre of the circle.

$$F = mv^2 / r = mr\omega^2$$

Key points:

- Centripetal force is not a new kind of force. It is the name for the resultant of the real forces acting (tension, friction, gravity, normal reaction, or a combination). Never add "centripetal force" as an extra arrow on a free-body diagram.
- The force is perpendicular to the velocity at every instant. So it changes the direction of motion but not the speed.
- Because the force is perpendicular to the direction of motion, it does **no work** on the object, and the kinetic energy stays constant.
- If the centripetal force is removed (for example, a string breaks), the object moves off in a straight line along the tangent at that point, not outwards along the radius.

Worked example 3: a car on a bend

A car of mass 900 kg travels round a flat, circular bend of radius 40 m at a constant speed of 15 m s⁻¹. Calculate the centripetal acceleration and the centripetal force, and state what provides this force.

$$a = v^2/r = 15^2/40 = 225/40 = 5.6 \text{ m s}^{-2}.$$

$$F = ma = 900 \times 5.625 = 5.1 \times 10^3 \text{ N.}$$

The force is provided by friction between the tyres and the road, acting sideways towards the centre of the bend.

4.2 Identifying the force that provides the centripetal force

Situation	Force providing the centripetal force
Ball whirled on a string in a horizontal circle on a smooth table	Tension in the string
Car going round a flat bend	Friction between tyres and road
Satellite or the Moon in orbit	Gravitational force of the planet
Electron in a magnetic field (moving at right angles to it)	Magnetic force on the moving charge
Person on a rotating drum (fairground rotor) pressed against the wall	Normal reaction force from the wall
Conical pendulum	Horizontal component of the tension
Car on a banked track at the design speed	Horizontal component of the normal reaction
Car over the top of a hump-backed bridge	Resultant of weight (down) and normal reaction (up)

Exam tip: a reliable method

1. Draw a free-body diagram showing only the real forces. 2. Find the centre of the circle and the direction towards it. 3. Resolve forces along that direction (and perpendicular to it if needed). 4. Write: resultant force towards the centre = mv^2/r (or $m\omega^2 r$). 5. If the object does not move up or down, the vertical forces balance.

5. Horizontal circles

5.1 Object on a string on a smooth surface

The weight is balanced by the normal reaction from the surface, so the only horizontal force is the tension, which provides all of the centripetal force: $T = mv^2/r$. A faster speed needs a greater tension; if the tension needed exceeds the breaking strength, the string snaps and the object leaves along the tangent.

5.2 Car on a flat bend

The car's weight is balanced by the normal reaction. Friction from the road provides the centripetal force. There is a maximum friction force the tyres can supply, so there is a maximum safe speed: from $F_{\max} = mv^2/r$, $v_{\max} = \sqrt{(F_{\max} r / m)}$. On a wet or icy road the maximum friction is smaller, so the safe speed is lower, and a car that is too fast skids outwards towards a wider path because there is not enough inward force.

5.3 The conical pendulum

A mass on a string moves in a horizontal circle while the string sweeps out a cone. The string makes

an angle θ with the vertical. Only two forces act: weight mg downwards and tension T along the string.

- Vertically (no vertical acceleration): $T \cos \theta = mg$
- Horizontally (towards the centre): $T \sin \theta = mv^2/r$

Dividing the second equation by the first: $\tan \theta = v^2/(rg)$. The faster the mass moves, the larger the angle. The string can never be exactly horizontal, because then the tension would have no vertical component to support the weight.

Worked example 4: conical pendulum

A mass of 0.40 kg moves in a horizontal circle of radius 0.50 m on a string that makes 40° with the vertical. Calculate the tension and the speed.

$$T = mg / \cos 40^\circ = (0.40 \times 9.81) / 0.766 = 5.1 \text{ N.}$$

$$v^2 = rg \tan 40^\circ = 0.50 \times 9.81 \times 0.839 = 4.12, \text{ so } v = 2.0 \text{ m s}^{-1}.$$

6. Vertical circles

6.1 Why speed changes in a vertical circle

In a vertical circle, gravity does work as the object rises and falls, so the speed is not constant (unless a motor keeps it constant). The object is fastest at the bottom and slowest at the top. Link the speeds at the top and bottom using conservation of energy. For an object on a string of radius r , the top is $2r$ higher than the bottom:

$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2r), \text{ so } v_{\text{bottom}}^2 = v_{\text{top}}^2 + 4gr$$

6.2 Tension at the top and the bottom

At the bottom: tension acts up (towards the centre), weight acts down (away from the centre). The resultant towards the centre is $T - mg$, so

$$T_{\text{bottom}} - mg = mv^2/r, \text{ giving } T_{\text{bottom}} = mv^2/r + mg$$

At the top: both tension and weight act down, which is towards the centre. So

$$T_{\text{top}} + mg = mv^2/r, \text{ giving } T_{\text{top}} = mv^2/r - mg$$

The tension is greatest at the bottom (the string is most likely to break there) and least at the top.

6.3 Minimum speed at the top

The string goes slack if the tension at the top would be negative. At the minimum speed, $T_{\text{top}} = 0$ and weight alone provides the centripetal force:

$$mg = mv_{\text{min}}^2/r, \text{ so } v_{\text{min}} = \sqrt{gr}$$

This is why water stays in a bucket swung quickly over your head: at the top, the weight of the water plus the push from the bucket base provides the centripetal force. If the speed is less than $\sqrt{gr}$, gravity provides more force than is needed for the circle, so the water falls out of the circular path.

Worked example 5: ball on a string in a vertical circle

A ball of mass 0.25 kg on a string of length 0.60 m moves in a vertical circle. Its speed at the top is 3.5 m s^{-1} . Calculate the tension at the top, the speed at the bottom and the tension at the bottom.

Top: $T = mv^2/r - mg = (0.25 \times 3.5^2)/0.60 - 0.25 \times 9.81 = 5.10 - 2.45 = 2.65 \text{ N}$.

Bottom speed: $v^2 = 3.5^2 + 4 \times 9.81 \times 0.60 = 12.25 + 23.54 = 35.79$, so $v = 5.98 \text{ m s}^{-1}$.

Bottom: $T = mv^2/r + mg = (0.25 \times 35.79)/0.60 + 2.45 = 14.91 + 2.45 = 17.4 \text{ N}$.

6.4 Over the top of a hump: the hump-backed bridge

A car of mass m crosses the top of a curved bridge of radius r at speed v . The centre of the circle is below the car. Weight acts down (towards the centre) and the normal reaction N acts up (away from the centre):

$$mg - N = mv^2/r, \text{ so } N = mg - mv^2/r$$

The faster the car, the smaller the normal reaction, so passengers feel lighter. When $v = \sqrt{gr}$, N becomes zero and the car loses contact with the road. A similar idea explains why riders feel lighter at the top of a roller-coaster hill and heavier at the bottom of a dip, where $N = mg + mv^2/r$.

7. Banked tracks (qualitative)

Roads, cycle tracks and railway curves are often banked (tilted) towards the centre of the bend. On a banked surface, the normal reaction is perpendicular to the surface, so it is tilted towards the centre of the circle.

- The **vertical component** of the normal reaction balances the weight.
- The **horizontal component** of the normal reaction points towards the centre and can provide the centripetal force.

At one particular speed, the design speed, the horizontal component of the normal reaction alone provides exactly the centripetal force needed, so no sideways friction is required. This makes the bend safer, especially in wet or icy conditions, and reduces tyre wear. Resolving as for the conical pendulum gives $\tan \theta = v^2/(rg)$ at this speed, where θ is the angle of the bank, so steeper banking suits higher speeds.

- If the vehicle goes faster than the design speed, more centripetal force is needed, so friction acts down the slope to supply the extra force.
- If it goes slower, less is needed, so friction acts up the slope to stop it sliding down.

An aircraft turns in a similar way: it banks so that the lift force, which is perpendicular to the wings, has a horizontal component towards the centre of the turn, while the vertical component balances the weight. A cyclist leans into a bend for the same reason.

8. Summary of equations

Quantity	Equation	Unit
Angle in radians	$\theta = s / r$	rad
Angular velocity	$\omega = \Delta\theta/\Delta t = 2\pi/T = 2\pi f$	rad s^{-1}

Linear speed	$v = r\omega$	m s^{-1}
Centripetal acceleration	$a = v^2/r = r\omega^2$	m s^{-2}
Centripetal force	$F = mv^2/r = mr\omega^2$	N
Minimum speed at top of vertical circle	$v = \sqrt{gr}$	m s^{-1}

Quick check

- Define the radian. [1]
 - Convert 75° to radians, and calculate the arc length subtended by this angle on a circle of radius 0.40 m. [2]
- A wheel turns at 1200 revolutions per minute. Calculate (a) its angular velocity in rad s^{-1} and (b) the speed of a point 0.30 m from the axle. [3]
- The Earth orbits the Sun once every 365 days in an approximately circular orbit of radius 1.5×10^{11} m. Calculate (a) its angular velocity, (b) its orbital speed and (c) its centripetal acceleration. State which force provides the centripetal force. [5]
- A body moves in a circle at constant speed. Explain why it is accelerating, state the direction of the resultant force on it, and explain why this force does no work. [4]
- A car of mass 1200 kg drives round a flat, horizontal bend of radius 50 m. The maximum sideways friction force the road can provide is 7.2 kN. Calculate the maximum speed at which the car can take the bend without skidding, and describe what happens if it goes faster. [3]
- A stone of mass 0.20 kg is attached to a string and moves in a horizontal circle of radius 0.80 m on a smooth table at 3.0 revolutions per second.
 - Calculate the tension in the string. [2]
 - The string breaks when the tension reaches 80 N. Calculate the greatest number of revolutions per second possible. [2]
 - Describe the path of the stone just after the string breaks. [1]
- A conical pendulum consists of a mass of 0.50 kg on a string of length 1.2 m. The string makes an angle of 30° with the vertical. Calculate (a) the tension, (b) the radius of the circle, (c) the speed of the mass and (d) the period of the motion. [6]
- A bucket of water is swung in a vertical circle of radius 0.90 m. Calculate the minimum speed at the top for the water to stay in the bucket, and explain your reasoning. [3]
- A mass of 0.30 kg on a string of length 0.50 m moves in a vertical circle. Its speed at the lowest point is 6.0 m s^{-1} . Calculate (a) the tension at the lowest point, (b) the speed at the highest point, assuming no energy losses, and (c) the tension at the highest point. [6]
- A car of mass 800 kg drives over a hump-backed bridge whose top is part of a circle of radius 20 m.
 - Calculate the normal reaction on the car at the top when its speed is 10 m s^{-1} . [2]
 - Calculate the speed at which the car would just lose contact with the road. [2]
- Explain how banking a track allows a car to travel round a bend without relying on friction, and state the direction of the friction force if the car travels faster than the design speed. [4]
- In a fairground rotor, riders stand against the inside wall of a vertical cylinder of radius 2.0 m that rotates once every 2.0 s. The floor is then lowered and the riders stay in place.

- (a) Identify the force that provides the centripetal force on a rider. [1]
(b) Calculate this force for a rider of mass 60 kg. [2]
(c) Explain why the rider does not slide down the wall. [2]

1. (a) One radian is the angle subtended at the centre of a circle by an arc equal in length to the radius.

(b) $\theta = 75 \times \pi/180 = 1.31 \text{ rad}$. $s = r\theta = 0.40 \times 1.309 = 0.52 \text{ m}$.

2. (a) $f = 1200/60 = 20 \text{ rev s}^{-1}$. $\omega = 2\pi f = 2\pi \times 20 = 126 \text{ rad s}^{-1}$ (125.7).

(b) $v = r\omega = 0.30 \times 125.7 = 37.7 \text{ m s}^{-1}$.

3. (a) $T = 365 \times 24 \times 3600 = 3.15 \times 10^7 \text{ s}$. $\omega = 2\pi/T = 6.283 / 3.154 \times 10^7 = 1.99 \times 10^{-7} \text{ rad s}^{-1}$.

(b) $v = r\omega = 1.5 \times 10^{11} \times 1.99 \times 10^{-7} = 3.0 \times 10^4 \text{ m s}^{-1}$.

(c) $a = r\omega^2 = 1.5 \times 10^{11} \times (1.99 \times 10^{-7})^2 = 5.9 \times 10^{-3} \text{ m s}^{-2}$. (Check: $v^2/r = (2.99 \times 10^4)^2 / 1.5 \times 10^{11} = 5.9 \times 10^{-3} \text{ m s}^{-2}$.) The centripetal force is the gravitational force of the Sun on the Earth.

4. Velocity is a vector. The direction of motion is always changing (it is along the tangent), so the velocity is changing even though the speed is constant, and a changing velocity is an acceleration. The resultant force acts towards the centre of the circle. This force is always perpendicular to the velocity, so there is no displacement in the direction of the force; therefore no work is done and the kinetic energy (speed) is unchanged.

5. Friction provides the centripetal force: $F = mv^2/r$, so $v^2 = Fr/m = (7200 \times 50)/1200 = 300$, $v = 17 \text{ m s}^{-1}$ (17.3). At a higher speed, the friction cannot provide the larger centripetal force needed, so the resultant inward force is too small for that radius; the car skids outwards into a path of larger radius.

6. (a) $\omega = 2\pi f = 2\pi \times 3.0 = 18.85 \text{ rad s}^{-1}$. $T = m\omega^2 r = 0.20 \times 0.80 \times 18.85^2 = 0.16 \times 355.3 = 57 \text{ N}$.

(b) $\omega_{\max}^2 = T/(mr) = 80/0.16 = 500$, $\omega_{\max} = 22.4 \text{ rad s}^{-1}$. $f = \omega/2\pi = 22.36/6.283 = 3.6 \text{ revolutions per second}$.

(c) It moves off in a straight line along the tangent to the circle at the point where the string broke (at constant speed, since the table is smooth).

7. (a) Vertically: $T \cos 30^\circ = mg$, so $T = (0.50 \times 9.81)/0.866 = 5.7 \text{ N}$ (5.66).

(b) $r = L \sin 30^\circ = 1.2 \times 0.50 = 0.60 \text{ m}$.

(c) Horizontally: $T \sin 30^\circ = mv^2/r$. Dividing by the vertical equation: $\tan 30^\circ = v^2/(rg)$, so $v^2 = 0.60 \times 9.81 \times 0.577 = 3.40$, $v = 1.8 \text{ m s}^{-1}$ (1.84).

(d) Period $= 2\pi r/v = (2\pi \times 0.60)/1.84 = 3.77/1.84 = 2.0 \text{ s}$ (2.05).

8. At the top, the weight of the water and the reaction from the bucket base both act downwards, towards the centre. At the minimum speed the reaction is zero, and the weight alone provides the centripetal force: $mg = mv^2/r$, so $v = \sqrt{gr} = \sqrt{(9.81 \times 0.90)} = \sqrt{8.83} = 3.0 \text{ m s}^{-1}$ (2.97). Below this speed the weight is greater than the centripetal force needed, so the water falls inside the circular path and leaves the bucket.

9. (a) At the bottom: $T - mg = mv^2/r$, so $T = (0.30 \times 6.0^2)/0.50 + 0.30 \times 9.81 = 21.6 + 2.94 = 24.5 \text{ N}$.

(b) The top is $2r = 1.0 \text{ m}$ higher. Energy: $v_{\text{top}}^2 = v_{\text{bottom}}^2 - 4gr = 36 - 4 \times 9.81 \times 0.50 = 36 - 19.62 = 16.38$, so $v_{\text{top}} = 4.0 \text{ m s}^{-1}$ (4.05).

(c) At the top: $T + mg = mv^2/r$, so $T = (0.30 \times 16.38)/0.50 - 2.94 = 9.83 - 2.94 = 6.9 \text{ N}$.

10. (a) At the top, weight acts towards the centre and the normal reaction away from it: $mg - N =$

$$mv^2/r. N = m(g - v^2/r) = 800 \times (9.81 - 100/20) = 800 \times 4.81 = 3.8 \times 10^3 \text{ N (3850 N)}.$$

(b) Contact is lost when $N = 0$: $mg = mv^2/r$, $v = \sqrt{gr} = \sqrt{9.81 \times 20} = \sqrt{196.2} = 14 \text{ m s}^{-1}$.

11. On a banked track the normal reaction acts perpendicular to the tilted surface, so it has a horizontal component towards the centre of the bend and a vertical component. The vertical component balances the weight. At the design speed the horizontal component is exactly equal to the centripetal force mv^2/r needed, so no sideways friction is required. If the car travels faster than the design speed, a larger centripetal force is needed, so friction acts down the slope (towards the centre) to provide the extra force.

12. (a) The normal reaction force from the wall, acting horizontally towards the axis.

(b) $\omega = 2\pi/T = 2\pi/2.0 = 3.14 \text{ rad s}^{-1}$. $N = mr\omega^2 = 60 \times 2.0 \times 3.14^2 = 60 \times 2.0 \times 9.87 = 1.2 \times 10^3 \text{ N (1180 N)}$.

(c) Friction between the rider and the wall acts upwards. The friction force can be as large as the weight ($60 \times 9.81 = 589 \text{ N}$) because the large normal reaction presses the rider firmly against the wall. With friction equal to the weight, the vertical forces balance, so the rider does not slide down.