

Mechanics: revision notes

These notes cover all the Mechanics content for Cambridge International AS & A Level Mathematics 9709, Paper 4. The topics are in syllabus order and use the syllabus numbers: 4.1 Forces and equilibrium, 4.2 Kinematics of motion in a straight line, 4.3 Momentum, 4.4 Newton's laws of motion, and 4.5 Energy, work and power. Each section has the key ideas, the definitions you need to learn, and at least one worked example with every step shown. Some worked examples also show how marks are usually awarded. At the end you will find a glossary and a set of quick-check questions that get harder as you go, with full answers. We use $g = 10 \text{ m s}^{-2}$ throughout, as 9709 does. Give non-exact answers to 3 significant figures, and angles in degrees to 1 decimal place, unless a question says otherwise.

Modelling words you must understand

- **Particle**: an object whose size can be ignored, so all its mass acts at a single point and it does not rotate.
- **Light** (string, rod, pulley): its mass can be ignored.
- **Inextensible** string: it does not stretch, so **while the string is taut** the particles joined by it have the same speed and the same magnitude of acceleration. If the string goes slack (for example when one particle hits the floor), the tension becomes zero and the particles move independently.
- **Smooth** surface or pulley: there is no friction, so the tension is the same on both sides of a smooth pulley.
- **Rough** surface: friction may act.

How marks are shown in the worked examples

Some worked examples show mark labels in the style used by Cambridge mark schemes. The allocations are a guide only; a real mark scheme may differ.

- **M1** method mark: a correct method applied, for example Newton's second law with the right forces.
- **A1** accuracy mark: a correct answer or equation that depends on the method mark before it.
- **B1** an independent mark for a correct value or statement.
- **AG** (answer given): in a "Show that" question, the given answer must appear at the end of correct working.

4.1 Forces and equilibrium

The forces you will meet

Force	Symbol and size	Direction
Weight	$W = mg$	Vertically downwards

Normal contact force (normal reaction)	R , found by resolving	Perpendicular to the surface, away from it
Tension	T	Along the string or rod, pulling away from the object
Thrust (compression)	T	Along a rod, pushing towards the object
Friction	F , with $F \leq \mu R$	Along the surface, opposing motion or the tendency to move
Driving force or resistance	Given, or found from power	Forwards (driving), backwards (resistance)

A string can only pull (tension). A rod can pull (tension) or push (thrust).

Newton's third law in equilibrium problems: if the floor pushes up on a box with a normal contact force R , the box pushes down on the floor with a force of the same size R . These two forces act on *different* bodies, so they never appear on the same force diagram. Draw only the forces acting on the body you are studying.

Resolving forces

A force of magnitude F acting at an angle θ to a chosen direction has:

- a component $F \cos \theta$ **along** that direction
- a component $F \sin \theta$ **perpendicular** to that direction.

To find the **resultant** of several forces, add up the components in two perpendicular directions (say X and Y). Then:

- magnitude of resultant $= \sqrt{X^2 + Y^2}$
- angle β with the X -direction: $\tan \beta = Y \div X$ (draw a quick sketch so you get the quadrant right).

On an inclined plane at angle α to the horizontal, resolve **parallel** and **perpendicular** to the plane. The weight mg then has components $mg \sin \alpha$ down the slope and $mg \cos \alpha$ into the slope.

Learn this: equilibrium

A particle is in **equilibrium** when the resultant force on it is zero. So the sum of the components in any direction is zero. In practice you resolve in two perpendicular directions and set each sum to zero.

Friction

Learn this: the friction law

For two surfaces in contact, the frictional force F satisfies $F \leq \mu R$, where R is the normal contact force and μ is the **coefficient of friction**.

- If the object is at rest and not about to move, F is only as large as it needs to be to stop motion ($F < \mu R$).
- When the object is on the point of moving (**limiting equilibrium**) or is actually sliding, friction takes its maximum value: $F = \mu R$.

μ has no units. A smooth surface has $\mu = 0$. For a particle about to slip on a rough plane at angle α with no other forces along the plane, resolving gives $mg \sin \alpha = \mu mg \cos \alpha$, so $\mu = \tan \alpha$.

Common mistakes

- Writing $R = mg$ when another force has a vertical component, or when the surface is sloped. Always find R by resolving.
- Using $F = \mu R$ when the object is not moving and not about to move.
- Putting friction in the wrong direction. Ask yourself which way the object would move if there were no friction; friction acts the opposite way.

Worked example 1: limiting equilibrium on a horizontal plane

A box of mass 4 kg rests on rough horizontal ground. The coefficient of friction between the box and the ground is 0.25. A force of P N acts on the box at 30° above the horizontal. The box is about to slide. Find P .

Diagram in words: the box sits on a horizontal line. Weight 40 N acts straight down. R acts straight up. P points up and to the right at 30° to the ground. Friction F acts to the left along the ground.

Step 1: resolve vertically. $R + P \sin 30^\circ = 40$, so $R = 40 - 0.5P$.

Step 2: limiting friction. The box is about to slide, so $F = \mu R = 0.25(40 - 0.5P) = 10 - 0.125P$.

Step 3: resolve horizontally. $P \cos 30^\circ = F$

$$0.8660P = 10 - 0.125P$$

$$0.9910P = 10$$

$$P = 10.09\dots, \text{ so } \mathbf{P = 10.1 \text{ N (3 s.f.)}}$$

Check: $R = 40 - 5.05 = 34.95$ N. This is positive, so the box stays in contact with the ground.

Worked example 2: three forces in equilibrium

A particle is in equilibrium under three forces. The first is 9 N in the positive x -direction. The second is 5 N acting at 120° anticlockwise from the positive x -direction. Find the magnitude and direction of the third force.

Step 1: components of the known forces.

- 9 N force: $(9, 0)$
- 5 N force: $(5 \cos 120^\circ, 5 \sin 120^\circ) = (-2.5, 4.330)$

Step 2: add them. The sum of the two known forces is $(6.5, 4.330)$.

Step 3: the third force balances this sum. It is $(-6.5, -4.330)$.

$$\text{Magnitude} = \sqrt{(6.5)^2 + (4.330)^2} = \sqrt{42.25 + 18.75} = \sqrt{61} = \mathbf{7.81 \text{ N}}$$

Direction: $\tan \beta = 4.330 \div 6.5$, so $\beta = 33.7^\circ$. Both components are negative, so the force acts at **33.7° below the negative x -direction** (equivalently, 213.7° anticlockwise from the positive x -direction).

Worked example 3: does it slide?

A block of mass 2 kg is placed at rest on a rough plane inclined at 25° to the horizontal. The coefficient of friction is 0.6. Determine whether the block slides, and find the frictional force acting on it.

Step 1: perpendicular to the plane. $R = 20 \cos 25^\circ = 18.13 \text{ N}$.

Step 2: maximum friction available. $\mu R = 0.6 \times 18.13 = 10.88 \text{ N}$.

Step 3: component of weight down the plane. $20 \sin 25^\circ = 8.452 \text{ N}$.

Step 4: compare. $8.452 < 10.88$, so friction can hold the block. **The block does not slide.**

The block is in equilibrium, so friction exactly balances the weight component: **$F = 8.45 \text{ N up the plane}$** . It is *not* 10.9 N, because the block is not in limiting equilibrium.

4.2 Kinematics of motion in a straight line

Vocabulary

Scalar	Vector partner	Note
Distance (m)	Displacement (m)	Displacement is position relative to a fixed point and can be negative.
Speed (m s ⁻¹)	Velocity (m s ⁻¹)	Speed is the magnitude of velocity.
—	Acceleration (m s ⁻²)	Rate of change of velocity. A negative acceleration while moving forwards is a deceleration (retardation).

Constant acceleration equations

These equations only apply when the acceleration is **constant**. Here u is the initial velocity, v the final velocity, a the acceleration, s the displacement and t the time.

- $v = u + at$
- $s = ut + \frac{1}{2}at^2$
- $s = \frac{1}{2}(u + v)t$
- $v^2 = u^2 + 2as$
- $s = vt - \frac{1}{2}at^2$

These five equations are given in the formula list (MF19), but you should know them well enough to choose one quickly.

Method: write down the three values you know and the one you want, then pick the equation that uses those four. Choose a positive direction first and stick to it. For vertical motion under gravity alone, $a = -10 \text{ m s}^{-2}$ if up is positive.

Graphs

Graph	Gradient gives	Area under the graph gives
Displacement–time (s–t)	Velocity	No useful meaning

Velocity–time (v–t)	Acceleration	Displacement (area below the t-axis counts as negative)
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- On an s–t graph a horizontal section means the particle is at rest. A straight sloping line means constant velocity.
- On a v–t graph a horizontal section means constant velocity. A straight sloping line means constant acceleration.
- **Total distance** travelled = the sum of all areas between the v–t graph and the t-axis, each taken as positive. **Displacement** = area above the axis minus area below it.

Worked example 4: a journey in three stages (v–t graph)

A car starts from rest and accelerates uniformly at 2 m s^{-2} for 6 s. It then travels at constant speed for 20 s, then decelerates uniformly to rest in 8 s. Sketch the velocity–time graph and find the total distance travelled and the deceleration.

Step 1: top speed. $v = u + at = 0 + 2 \times 6 = 12 \text{ m s}^{-1}$.

Graph in words: a trapezium. A straight line from (0, 0) up to (6, 12), a horizontal line from (6, 12) to (26, 12), then a straight line down to (34, 0).

Step 2: area of each part.

- First triangle: $\frac{1}{2} \times 6 \times 12 = 36 \text{ m}$
- Rectangle: $20 \times 12 = 240 \text{ m}$
- Last triangle: $\frac{1}{2} \times 8 \times 12 = 48 \text{ m}$

Total distance = $36 + 240 + 48 = 324 \text{ m}$.

Step 3: deceleration. Gradient of the last section = $(0 - 12) \div 8 = -1.5$, so the deceleration is 1.5 m s^{-2} .

Worked example 5: a displacement–time graph

A cyclist rides in a straight line from her home to a shop 300 m away, at constant speed, taking 60 s. She waits at the shop for 40 s, then rides straight home at constant speed in 50 s. (a) Describe the displacement–time graph. (b) Find her velocity on each stage. (c) Find her average speed and her average velocity for the whole trip.

(a) Graph in words (s in metres from home, t in seconds): a straight line from (0, 0) up to (60, 300), a horizontal line from (60, 300) to (100, 300), then a straight line down to (150, 0).

(b) Velocity = gradient of the s–t graph.

- Stage 1: $300 \div 60 = 5 \text{ m s}^{-1}$ away from home.
- Stage 2: horizontal line, so velocity = 0 (at rest).
- Stage 3: $(0 - 300) \div 50 = -6 \text{ m s}^{-1}$, which means 6 m s^{-1} towards home.

(c) Total distance = $300 + 300 = 600 \text{ m}$ and total time = 150 s. Average speed = $600 \div 150 = 4 \text{ m s}^{-1}$.

She ends where she started, so her displacement is 0 and her **average velocity is 0**. This shows why speed and velocity must not be confused.

Worked example 6: vertical motion

A ball is thrown vertically upwards at 15 m s^{-1} from a point 20 m above the ground. Find (a) the greatest height of the ball above the ground, (b) the time before it hits the ground, (c) its speed as it hits the ground.

Take upwards as positive, so $a = -10$.

(a) At the top, $v = 0$. Using $v^2 = u^2 + 2as$: $0 = 225 - 20s$, so $s = 11.25 \text{ m}$. Height above ground = $20 + 11.25 = \mathbf{31.25 \text{ m}}$.

(b) The ground is 20 m *below* the start, so $s = -20$.

$$s = ut + \frac{1}{2}at^2 \text{ gives } -20 = 15t - 5t^2$$

$$5t^2 - 15t - 20 = 0, \text{ so } t^2 - 3t - 4 = 0$$

$$(t - 4)(t + 1) = 0, \text{ so } \mathbf{t = 4 \text{ s}}$$
 (reject $t = -1$).

(c) $v = u + at = 15 - 40 = -25$. The negative sign means it is moving downwards. **Speed = 25 m s^{-1}** .

Check with $v^2 = u^2 + 2as$: $225 + 2(-10)(-20) = 625$, and $\sqrt{625} = 25$, which agrees.

Variable acceleration (calculus)

When the acceleration is not constant, the suvat equations do **not** apply. Use calculus instead:

To go from	To	Do this
s	v	$v = ds/dt$ (differentiate)
v	a	$a = dv/dt = d^2s/dt^2$ (differentiate)
a	v	$v = \int a \, dt$ (integrate, then find the constant)
v	s	$s = \int v \, dt$ (integrate, then find the constant)

- "Instantaneously at rest" means $v = 0$.
- A maximum or minimum velocity occurs where $a = 0$ (a stationary point of v). If the question gives a time interval, also check the velocity at the ends of the interval.
- Always use the initial conditions to find the constant of integration, or use a definite integral.
- For total distance, find where $v = 0$ inside the time interval and integrate each part separately.

Worked example 7: calculus kinematics

A particle P moves in a straight line, starting from a fixed point O. Its velocity after t seconds is $v = t^2 - 8t + 12 \text{ m s}^{-1}$. Find (a) the times when P is instantaneously at rest, (b) the acceleration of P when $t = 3$, (c) the minimum velocity of P, (d) the total distance travelled in the first 6 seconds.

(a) $v = 0$: $t^2 - 8t + 12 = 0$, so $(t - 2)(t - 6) = 0$. **$t = 2$ and $t = 6$** .

(b) $a = dv/dt = 2t - 8$. At $t = 3$, $a = 6 - 8 = \mathbf{-2 \text{ m s}^{-2}}$.

(c) Minimum v when $a = 0$: $2t - 8 = 0$, so $t = 4$. $v = 16 - 32 + 12 = \mathbf{-4 \text{ m s}^{-1}}$. (P is moving back towards O at 4 m s^{-1} .)

(d) $s = \int(t^2 - 8t + 12) dt = (1/3)t^3 - 4t^2 + 12t + c$. At $t = 0$, $s = 0$, so $c = 0$.

- $s(2) = 8/3 - 16 + 24 = 32/3$ m
- $s(6) = 72 - 144 + 72 = 0$ m

P moves $32/3$ m away from O, then turns and travels $32/3$ m back to O.

Total distance = $32/3 + 32/3 = 64/3 = 21.3$ m (3 s.f.). The displacement after 6 s is zero.

4.3 Momentum

Learn this: momentum

The **momentum** of a particle of mass m moving with velocity v is mv . It is a vector quantity, in the same direction as the velocity. Units: kg m s^{-1} (equivalently N s).

Principle of conservation of momentum: when two particles collide, and no external force acts in the direction of motion, the total momentum before the collision equals the total momentum after it.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

- **Signs matter.** Choose a positive direction. A particle moving the other way has negative velocity, so it has negative momentum.
- **Coalescing** particles stick together and move off as one particle: $m_1u_1 + m_2u_2 = (m_1 + m_2)v$.
- If your answer for an unknown velocity is negative, the particle moves in the negative direction. Say so in words.
- Momentum is conserved in these collisions but kinetic energy is usually **lost**. Loss of KE = total KE before – total KE after, and this should never come out negative.

Worked example 8: a direct collision

Two particles A and B move towards each other along the same straight line on a smooth horizontal surface. A has mass 0.3 kg and speed 5 m s^{-1} . B has mass 0.2 kg and speed 2 m s^{-1} . After they collide, A moves at 1 m s^{-1} in its original direction. Find the velocity of B after the collision and the loss of kinetic energy.

Diagram in words: A on the left moving right; B on the right moving left. Take right as positive.

Step 1: momentum before. $0.3 \times 5 + 0.2 \times (-2) = 1.5 - 0.4 = 1.1 \text{ kg m s}^{-1}$.

Step 2: momentum after. $0.3 \times 1 + 0.2v = 0.3 + 0.2v$.

Step 3: conserve. $0.3 + 0.2v = 1.1$, so $0.2v = 0.8$ and $v = 4$.

B moves at 4 m s^{-1} in the direction A was originally moving. Its direction has reversed.

Step 4: kinetic energy.

- Before: $\frac{1}{2}(0.3)(5^2) + \frac{1}{2}(0.2)(2^2) = 3.75 + 0.4 = 4.15$ J
- After: $\frac{1}{2}(0.3)(1^2) + \frac{1}{2}(0.2)(4^2) = 0.15 + 1.6 = 1.75$ J

Loss of KE = $4.15 - 1.75 = 2.4$ J.

Worked example 9: successive coalescing collisions

Three particles P, Q and R lie in a straight line on a smooth horizontal surface. P (2 kg) moves at 6 m s^{-1} towards Q (3 kg), which is at rest. P and Q coalesce. The combined particle then collides with R (1 kg), which is moving towards it at 4 m s^{-1} , and they coalesce. Find the final velocity.

Take P's original direction as positive.

First collision: $2 \times 6 + 3 \times 0 = 5v$, so $v = 12 \div 5 = 2.4 \text{ m s}^{-1}$.

Second collision: $5 \times 2.4 + 1 \times (-4) = 6w$

$12 - 4 = 6w$, so $w = 8 \div 6 = 4/3$.

Final velocity = **1.33 m s^{-1}** (3 s.f.) in P's original direction.

4.4 Newton's laws of motion

Learn this: Newton's three laws

- First law:** a particle stays at rest, or keeps moving with constant velocity, unless a resultant force acts on it.
- Second law:** the resultant force on a particle equals its mass times its acceleration, **$F = ma$** . The acceleration is in the direction of the resultant force.
- Third law:** if body A exerts a force on body B, then B exerts a force on A that is equal in magnitude and opposite in direction.

1 newton is the resultant force that gives a mass of 1 kg an acceleration of 1 m s^{-2} .

Method for any $F = ma$ problem

- Draw a force diagram for each particle, with the direction of acceleration marked.
- Resolve perpendicular to the motion. There is no acceleration in that direction, so the forces balance. This usually gives R.
- Resolve in the direction of the acceleration: (forces in that direction) – (forces against it) = ma .
- For connected particles, write one equation for each particle. You can also write one for the whole system, as a check, where internal tensions cancel.
- Solve, then use suvat if a distance, speed or time is needed.

Standard situations

Situation	Key equation or fact
Person of mass m standing in a lift accelerating upwards at a	$R - mg = ma$ (R is greater than mg)
Person in a lift accelerating downwards at a	$mg - R = ma$ (R is less than mg)
Lift moving at constant speed	$R = mg$
Particle sliding down a smooth plane at angle α	$a = g \sin \alpha$

Particle sliding down a rough plane	$mg \sin \alpha - \mu mg \cos \alpha = ma$
Car towing a trailer with a light rigid tow bar	Same acceleration for both; the tow bar is in tension if pulling, thrust if pushing (e.g. when braking)
Two particles joined by a light inextensible string over a smooth pulley	Same tension T on each side; same magnitude of acceleration while the string is taut

A lift accelerating downwards could be moving down and speeding up, or moving up and slowing down. The direction of the acceleration is what matters, not the direction of motion.

Worked example 10: a lift

A lift of mass 400 kg carries a passenger of mass 60 kg. The lift accelerates upwards at 1.5 m s^{-2} . Find the tension in the lift cable and the normal contact force between the passenger and the floor.

Whole system (mass 460 kg, weight 4600 N):

$$T - 4600 = 460 \times 1.5 = 690, \text{ so } T = 5290 \text{ N.}$$

Passenger alone (weight 600 N):

$$R - 600 = 60 \times 1.5 = 90, \text{ so } R = 690 \text{ N.}$$

Check with the lift alone: $T - 4000 - R = 400 \times 1.5 = 600$, and $5290 - 4000 - 690 = 600$, which agrees. By Newton's third law, the passenger pushes down on the floor with 690 N.

Worked example 11: car and trailer

A car of mass 1200 kg tows a trailer of mass 300 kg along a straight horizontal road using a light rigid tow bar. Resistances to motion are 400 N on the car and 150 N on the trailer. The engine's driving force is 2800 N. Find the acceleration and the tension in the tow bar.

Whole system: $2800 - 400 - 150 = 1500a$, so $2250 = 1500a$ and **$a = 1.5 \text{ m s}^{-2}$** .

Trailer alone: $T - 150 = 300 \times 1.5 = 450$, so **$T = 600 \text{ N}$** .

Check with the car alone: $2800 - 400 - 600 = 1800 = 1200 \times 1.5$, which agrees.

Worked example 12: rough inclined plane and pulley

Particle A (mass 5 kg) lies on a rough plane inclined at angle α to the horizontal, where $\sin \alpha = 0.28$ and $\cos \alpha = 0.96$. The coefficient of friction between A and the plane is 0.25. A light inextensible string runs from A, parallel to a line of greatest slope, over a small smooth pulley at the top of the plane. Particle B (mass 4 kg) hangs on the other end, 1.2 m above the floor. The system is released from rest.

(a) Show that the acceleration of the system is $14/9 \text{ m s}^{-2}$, and find the tension in the string. [6]

(b) Find the speed of B just before it hits the floor. [2]

(c) B does not rebound. Find how much further A travels up the plane. [3]

Diagram in words: A sits on the slope with the string running up the slope to the pulley at the top corner. B hangs vertically below the pulley. On A: weight 50 N down, R perpendicular to the plane, T up the plane, friction down the plane (A moves up). On B: weight 40 N down and T up.

(a) Perpendicular to the plane: $R = 50 \cos \alpha = 50 \times 0.96 = 48 \text{ N}$. **B1**

A is moving, so $F = \mu R = 0.25 \times 48 = 12 \text{ N}$. **M1**

Weight component of A down the plane: $50 \sin \alpha = 50 \times 0.28 = 14 \text{ N}$.

Check the direction first: B's weight is 40 N, and $14 + 12 = 26 \text{ N}$ is the most that can resist it. $40 > 26$, so B does fall and A moves up the plane.

B (down positive): $40 - T = 4a \dots(1)$ **M1**

A (up the plane positive): $T - 14 - 12 = 5a \dots(2)$ **M1**

Add (1) and (2): $14 = 9a$, so $a = 14/9 \text{ m s}^{-2}$ ($= 1.56$ to 3 s.f.). **A1 AG**

From (1): $T = 40 - 4 \times 14/9 = 304/9 = 33.8 \text{ N}$. **A1**

Because this is "Show that", the exact value $14/9$ must appear at the end of the working. Writing only 1.56 would lose the mark.

(b) $v^2 = u^2 + 2as = 0 + 2 \times (14/9) \times 1.2 = 56/15 = 3.733$ **M1**, so $v = 1.93 \text{ m s}^{-1}$. **A1**

(c) After B lands the string goes slack, so $T = 0$. A is still moving up the plane, with both the weight component and friction acting down the plane:

$-(14 + 12) = 5a$, so $a = -5.2 \text{ m s}^{-2}$. **M1**

At A's highest point $v = 0$: $0 = 56/15 + 2(-5.2)s$ **M1**, so $s = 3.733 \div 10.4 = 14/39 = 0.359 \text{ m}$. **A1**

Extension: at rest, the weight component (14 N) is greater than the maximum friction (12 N), so A then slides back down.

4.5 Energy, work and power

Learn this: work, energy and power

- **Work done** by a constant force = force $\times$ distance moved in the direction of the force. If the force acts at angle θ to the direction of motion, work done = **$Fd \cos \theta$** . Unit: joule (J).
- **Kinetic energy** = $\frac{1}{2}mv^2$.
- **Gravitational potential energy** gained when a mass m is raised through a vertical height h = **mgh** .
- **Work–energy principle**: the change in kinetic energy of a particle equals the total work done on it by all the forces.
- **Power** is the rate at which work is done. Unit: watt (W), where $1 \text{ W} = 1 \text{ J s}^{-1}$. For a force F moving its point of application at speed v in the direction of the force, **$P = Fv$** .

Using energy

A reliable form of the energy equation is:

Initial KE + initial PE + work done by driving forces = final KE + final PE + work done against resistances

- Only the **vertical** height change counts for PE. On a slope of length d at angle α , $h = d \sin \alpha$.
- Forces perpendicular to the motion (such as R) do no work.

- If the only force that does work is weight (smooth surfaces, no resistance), mechanical energy is conserved: $KE + PE$ stays constant.
- Work done against friction = $F \times \text{distance}$, so it is a quick way to find F or μ .
- Energy is a scalar, so you do not need to worry about direction. This is why energy is often quicker than $F = ma$ when the path is not straight (for example a curved slide), or when a vehicle works at constant power, where work done by the engine = power $\times$ time.

Power and vehicles

- A vehicle working at power P and moving at speed v has driving force $D = P \div v$.
- Then apply Newton's second law: $D - \text{resistance} (-mg \sin \alpha \text{ on an uphill slope}) = ma$.
- At **maximum (steady) speed**, $a = 0$, so the driving force equals the total resistance.
- Remember to convert kW to W ($\times 1000$) before substituting.

Worked example 13: work–energy with friction

A crate of mass 15 kg is pulled from rest across rough horizontal ground by a rope inclined at 20° above the horizontal. The tension in the rope is 60 N and the coefficient of friction is 0.3. Use an energy method to find the speed of the crate after it has moved 8 m.

Diagram in words: the crate sits on a horizontal line. Weight 150 N acts straight down and R acts straight up. The 60 N tension points up and to the right at 20° to the ground. Friction F acts to the left along the ground. The crate moves right.

Step 1: normal contact force. $R + 60 \sin 20^\circ = 150$, so $R = 150 - 20.52 = 129.48$ N.

Step 2: friction (sliding). $F = 0.3 \times 129.48 = 38.84$ N.

Step 3: work done by the tension. $60 \cos 20^\circ \times 8 = 451.05$ J.

Step 4: work done against friction. $38.84 \times 8 = 310.75$ J.

Step 5: work–energy principle. Gain in KE = $451.05 - 310.75 = 140.30$ J.

$\frac{1}{2} \times 15 \times v^2 = 140.30$, so $v^2 = 18.71$ and $v = 4.33 \text{ m s}^{-1}$.

Worked example 14: finding a coefficient of friction from energy

A particle of mass 0.5 kg slides down a line of greatest slope of a rough plane inclined at 30° to the horizontal. It passes point X at 2 m s^{-1} and point Y, 6 m further down the slope, at 7 m s^{-1} . Find the frictional force (assumed constant) and the coefficient of friction.

Diagram in words: a slope rising to the left at 30° to the horizontal. X is higher up the slope and Y is 6 m further down it. The particle moves down the slope from X to Y. Its weight (5 N) acts vertically down, R acts perpendicular to the slope, and friction acts up the slope, against the motion.

Step 1: height lost. $h = 6 \sin 30^\circ = 3$ m, so PE lost = $0.5 \times 10 \times 3 = 15$ J.

Step 2: KE gained. $\frac{1}{2} \times 0.5 \times (7^2 - 2^2) = 0.25 \times 45 = 11.25$ J.

Step 3: work done against friction. $15 - 11.25 = 3.75$ J.

Step 4: friction. $F \times 6 = 3.75$, so $F = 0.625$ N.

Step 5: $\mu \cdot R = 0.5 \times 10 \times \cos 30^\circ = 4.330 \text{ N}$. $\mu = 0.625 \div 4.330 = \mathbf{0.144}$ (3 s.f.).

Worked example 15: power of a car

A car of mass 1000 kg has a maximum power of 30 kW. The resistance to motion is a constant 800 N.

- Find the maximum speed of the car on a horizontal road. [2]
- Find the acceleration of the car on a horizontal road when it is working at maximum power and moving at 20 m s^{-1} . [3]
- The car now climbs a hill inclined at θ to the horizontal, where $\sin \theta = 1/20$. Find its maximum speed up the hill. [2]

Diagram in words: the car moves to the right. The driving force D acts forwards and the 800 N resistance acts backwards. In part (c) the road slopes up to the right, and the weight component $1000 \times 10 \times \sin \theta$ also acts down the slope.

(a) At maximum speed, driving force = resistance = 800 N. $v = P \div D = 30\,000 \div 800$ **M1** = $\mathbf{37.5 \text{ m s}^{-1}}$. **A1**

(b) $D = 30\,000 \div 20 = 1500 \text{ N}$. **M1** Then $1500 - 800 = 1000a$ **M1**, so $\mathbf{a = 0.7 \text{ m s}^{-2}}$. **A1**

(c) Weight component down the hill = $1000 \times 10 \times 1/20 = 500 \text{ N}$. At maximum speed, $D = 800 + 500 = 1300 \text{ N}$. **M1**

$v = 30\,000 \div 1300 = \mathbf{23.1 \text{ m s}^{-1}}$ (3 s.f.). **A1**

The common error in (c) is to forget the weight component. That loses the M1, and the A1 goes with it.

Exam technique summary

- "Show that"**: every step must be written down and the given answer must appear at the end. Give more figures than the target in your working.
- "Find" / "Calculate"**: give the final answer to 3 s.f. unless it is exact. Keep full accuracy in your working, and only round at the end.
- "State"**: a short answer with no working is needed.
- "Explain"**: give a reason linked to a principle, for example "the string is inextensible, so...".
- "Determine whether"**: do the calculation, compare, then write a clear conclusion.
- Method marks (M1) are for a correct method, so always write down the equation you are using before you substitute. Accuracy marks (A1) usually need the M1 first.
- Include units. Speeds are in m s^{-1} , not m s^{-2} .
- When a question asks for a velocity, give its direction as well as its size.

Key terms

Term	Definition
Particle	A model of a body whose dimensions are ignored, so it has mass but no size.

Resultant force	The single force that has the same effect as all the forces acting together; their vector sum.
Equilibrium	The state in which the resultant force on a particle is zero.
Normal contact force	The force exerted by a surface on a body in contact with it, acting perpendicular to the surface.
Coefficient of friction (μ)	The constant μ in $F \leq \mu R$; the ratio of limiting friction to the normal contact force.
Limiting equilibrium	Equilibrium in which the body is on the point of moving, so friction has its maximum value μR .
Tension / thrust	The pulling force in a string or rod / the pushing force in a rod.
Displacement	Position relative to a fixed origin, including direction.
Velocity	Rate of change of displacement, $v = ds/dt$.
Acceleration	Rate of change of velocity, $a = dv/dt$.
Instantaneously at rest	Having zero velocity at a particular instant.
Momentum	Mass $\times$ velocity, a vector measured in kg m s^{-1} or N s .
Coalesce	Of two colliding particles, to join together and move on as a single particle.
Weight	The gravitational force on a body, mg , acting vertically downwards.
Work done	Force $\times$ distance moved in the direction of the force, measured in joules.
Kinetic energy	Energy a body has because it is moving, $\frac{1}{2}mv^2$.
Gravitational potential energy	Energy a body has because of its height, mgh relative to a chosen level.
Power	Rate of doing work, measured in watts; $P = Fv$ for a force F moving at speed v .

Quick check

Use $g = 10 \text{ m s}^{-2}$. Give non-exact answers to 3 significant figures.

- Write down the weight of a particle of mass 0.8 kg . [1]
- A cyclist slows uniformly from 24 m s^{-1} to 9 m s^{-1} in 5 s . Find the deceleration and the distance travelled in this time. [3]
- A block of mass 5 kg rests on rough horizontal ground, where $\mu = 0.4$. A horizontal force of 15 N is applied to it. Determine whether the block moves, and state the frictional force acting on it. [3]
- A particle is in equilibrium under three forces. Two of them are 10 N and 24 N and act at right angles to each other. Find the magnitude of the third force and the angle it makes with the 24 N force. [3]
- A stone is dropped from rest from the top of a cliff and hits the sea 2.4 s later. Find the height of the cliff and the speed of the stone as it hits the sea. [3]
- A trolley of mass 1.5 kg moving at 4 m s^{-1} collides with a stationary trolley of mass 2.5 kg , and the two coalesce. Find their common speed and the loss of kinetic energy. [4]

7. A crane lifts a load of mass 250 kg vertically at a constant speed of 0.4 m s^{-1} . Find the power of the crane and the work it does in 30 s. [3]
8. A person of mass 70 kg stands in a lift. The normal contact force from the floor is 630 N. Find the magnitude and direction of the lift's acceleration, and describe two possible motions of the lift. [4]
9. A particle starts from rest at O and moves in a straight line with acceleration $a = 6 - 2t \text{ m s}^{-2}$. Find the maximum velocity of the particle and its displacement from O at that instant. [5]
10. Particles of masses 5 kg and 3 kg are joined by a light inextensible string passing over a smooth fixed pulley, with both hanging vertically. They are released from rest. Find the acceleration, the tension in the string, and the speed of the 5 kg particle after it has fallen 0.9 m. [6]
11. A particle of mass 4 kg is on a rough plane inclined at 35° to the horizontal, with $\mu = 0.3$. A force $P \text{ N}$ acts on it up the plane, parallel to a line of greatest slope. Find (a) the least value of P that stops the particle sliding down, (b) the value of P for which the particle is about to move up the plane. [6]
12. A car of mass 1200 kg climbs a straight hill inclined at θ to the horizontal, where $\sin \theta = 0.05$. Resistance to motion is a constant 600 N. At one instant the car's speed is 15 m s^{-1} and its acceleration is 0.4 m s^{-2} . (a) Find the power of the engine at this instant. (b) The engine keeps working at this power. Find the car's maximum speed up the hill. [6]

Answers

1. $W = 0.8 \times 10 = 8 \text{ N}$.
2. $a = (9 - 24) \div 5 = -3$, so the deceleration is 3 m s^{-2} . $s = \frac{1}{2}(24 + 9) \times 5 = 82.5 \text{ m}$.
3. $R = 50 \text{ N}$, so maximum friction $= 0.4 \times 50 = 20 \text{ N}$. $15 < 20$, so **the block does not move**. It is in equilibrium, so **friction = 15 N**, opposite to the applied force.
4. The resultant of the two forces is $\sqrt{(10^2 + 24^2)} = \sqrt{676} = 26 \text{ N}$, so the third force is **26 N**, acting directly opposite that resultant. The resultant makes angle $\tan^{-1}(10/24) = 22.6^\circ$ with the 24 N force, so the third force makes an angle of $180^\circ - 22.6^\circ = 157.4^\circ$ with the 24 N force.
5. $s = \frac{1}{2} \times 10 \times 2.4^2 = 28.8 \text{ m}$. $v = 10 \times 2.4 = 24 \text{ m s}^{-1}$.
6. $1.5 \times 4 = 4v$, so $v = 1.5 \text{ m s}^{-1}$. KE before $= \frac{1}{2} \times 1.5 \times 16 = 12 \text{ J}$. KE after $= \frac{1}{2} \times 4 \times 1.5^2 = 4.5 \text{ J}$. Loss $= 7.5 \text{ J}$.
7. At constant speed the lifting force equals the weight, 2500 N. $P = 2500 \times 0.4 = 1000 \text{ W (1 kW)}$. Work in 30 s $= 1000 \times 30 = 30\,000 \text{ J (30 kJ)}$.
8. Taking up as positive: $630 - 700 = 70a$, so $a = -1$. The acceleration is **1 m s^{-2} downwards**. The lift is either **moving down and speeding up** or **moving up and slowing down**.
9. $v = \int (6 - 2t) dt = 6t - t^2 + c$. At $t = 0$, $v = 0$, so $c = 0$. Maximum v when $a = 0$: $t = 3$, so **$v_{\max} = 18 - 9 = 9 \text{ m s}^{-1}$** . $s = 3t^2 - \frac{1}{3}t^3$ ($s = 0$ at $t = 0$). At $t = 3$, **$s = 27 - 9 = 18 \text{ m}$** .
10. 5 kg: $50 - T = 5a$. 3 kg: $T - 30 = 3a$. Adding: $20 = 8a$, so **$a = 2.5 \text{ m s}^{-2}$** . $T = 30 + 7.5 = 37.5 \text{ N}$. $v^2 = 2 \times 2.5 \times 0.9 = 4.5$, so **$v = 2.12 \text{ m s}^{-1}$** .
11. $R = 40 \cos 35^\circ = 32.77 \text{ N}$, so $\mu R = 9.830 \text{ N}$. Weight component down the plane $= 40 \sin 35^\circ = 22.94 \text{ N}$.

(a) The particle is about to slide down, so friction acts up the plane: $P + 9.830 = 22.94$, so **$P = 13.1 \text{ N}$** .

(b) The particle is about to move up, so friction acts down the plane: $P = 22.94 + 9.830 =$
 32.8 N .

12. (a) Weight component down the hill $= 1200 \times 10 \times 0.05 = 600 \text{ N}$. $D - 600 - 600 = 1200 \times$
 $0.4 = 480$, so $D = 1680 \text{ N}$. $P = 1680 \times 15 =$ **$25\,200 \text{ W (25.2 kW)}$** .

(b) At maximum speed $a = 0$, so $D = 600 + 600 = 1200 \text{ N}$. $v = 25\,200 \div 1200 =$ **21 m s^{-1}** .