

Inverse trigonometric functions: practice questions with solutions

This document gives practice questions with fully worked solutions on inverse trigonometric functions, for FSc Part-I (Class 11) Mathematics as studied by students of the Pakistani intermediate boards (Punjab, Federal and other boards) under the topic of trigonometric functions and their graphs. It covers the domain, range and principal value branch of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ and the three reciprocal inverse functions; simplifying expressions such as $\sin(\cos^{-1}x)$ by a right-triangle method; proving standard identities; and solving equations involving inverse trigonometric functions. The questions are arranged in three sections of increasing difficulty — multiple choice, short answer and structured — worth a total of 63 marks.

Key ideas

- $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ are defined by restricting sine, cosine and tangent to an interval on which each is one-one (injective) and onto its range. That restricted interval is called the principal value branch.
- Domain and range (principal branch): $\sin^{-1}x$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$; $\cos^{-1}x$ has domain $[-1, 1]$ and range $[0, \pi]$; $\tan^{-1}x$ has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$.
- Reciprocal inverse functions are built from these: $\operatorname{cosec}^{-1}x = \sin^{-1}(1/x)$ and $\sec^{-1}x = \cos^{-1}(1/x)$, each with domain $|x| \geq 1$; $\sec^{-1}x$ has range $[0, \pi]$ excluding $\pi/2$, i.e. $[0, \pi/2) \cup (\pi/2, \pi]$. $\cot^{-1}x$ has domain $\mathbb{R}$ and range $(0, \pi)$; it equals $\tan^{-1}(1/x)$ for $x > 0$, and equals $\pi + \tan^{-1}(1/x)$ for $x < 0$.
- $\sin(\sin^{-1}x) = x$ for every x in $[-1, 1]$, but $\sin^{-1}(\sin\theta) = \theta$ only when θ already lies in $[-\pi/2, \pi/2]$. If it does not, replace θ by the angle in that interval with the same sine before applying $\sin^{-1}$.
- The graph of $y = \sin^{-1}x$ is the reflection of $y = \sin x$ (restricted to $[-\pi/2, \pi/2]$) in the line $y = x$: an increasing curve joining $(-1, -\pi/2)$ to $(1, \pi/2)$. $y = \cos^{-1}x$ is obtained the same way and decreases from $(-1, \pi)$ to $(1, 0)$. $y = \tan^{-1}x$ increases across all real x with horizontal asymptotes $y = \pi/2$ and $y = -\pi/2$, and passes through the origin. The three reciprocal inverse functions' graphs follow the same reflection idea from their domain/range given above; they are not detailed separately here.
- Two standard co-function identities: $\sin^{-1}x + \cos^{-1}x = \pi/2$ for $x \in [-1, 1]$, and $\tan^{-1}x + \cot^{-1}x = \pi/2$ for $x \in \mathbb{R}$.
- Odd/even behaviour: $\sin^{-1}(-x) = -\sin^{-1}x$ and $\tan^{-1}(-x) = -\tan^{-1}x$ (both odd); $\cos^{-1}(-x) = \pi - \cos^{-1}x$.
- Sum formula: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x + y)/(1 - xy)]$, valid directly provided $xy < 1$. When $xy > 1$, π must be added or subtracted according to the signs of x and y .
- To simplify an expression such as $\sin(\cos^{-1}x)$: let $\theta = \cos^{-1}x$, draw a right triangle in which θ is an acute angle with adjacent = x and hypotenuse = 1, find the missing side with the Pythagorean theorem, then read off the required ratio directly from the triangle. This picture applies directly when $x > 0$. When $x < 0$, θ is obtuse ($\theta \in (\pi/2, \pi]$); use $|x|$ as the adjacent



side of the reference triangle to find the remaining side by Pythagoras, then fix the sign of the required ratio from the quadrant of θ (as in C2(a), where $\sin\theta = \sqrt{1 - x^2}$ is taken non-negative because $\theta \in [0, \pi]$ regardless of the sign of x).

Section A: multiple choice

Each question is worth [1] mark. Choose the correct option, A, B, C or D.

1. The domain of $\sin^{-1}x$ is [1]
 - A) $\mathbb{R}$
 - B) $[-1, 1]$
 - C) $(-1, 1)$
 - D) $[0, 1]$
2. The range of the principal value branch of $\cos^{-1}x$ is [1]
 - A) $[-\pi/2, \pi/2]$
 - B) $(-\pi/2, \pi/2)$
 - C) $[0, \pi]$
 - D) $(0, \pi)$
3. The range of the principal value branch of $\tan^{-1}x$ is [1]
 - A) $[-\pi/2, \pi/2]$
 - B) $(-\pi/2, \pi/2)$
 - C) $[0, \pi]$
 - D) $\mathbb{R}$
4. The principal value of $\sin^{-1}(1/2)$ is [1]
 - A) $\pi/6$
 - B) $\pi/3$
 - C) $5\pi/6$
 - D) $-\pi/6$
5. The value of $\cos^{-1}(-1/2)$ is [1]
 - A) $\pi/3$
 - B) $2\pi/3$
 - C) $-\pi/3$
 - D) $5\pi/6$
6. The graph of $y = \sin^{-1}x$ is obtained from the graph of $y = \sin x$ (restricted to $[-\pi/2, \pi/2]$) by [1]
 - A) reflecting it in the x-axis
 - B) reflecting it in the line $y = x$
 - C) translating it up by $\pi/2$

- D) reflecting it in the y-axis
7. The domain of $\sec^{-1}x$ is [1]
- A) $[-1, 1]$
 - B) $\mathbb{R}$
 - C) $x \leq -1$ or $x \geq 1$
 - D) $x > 0$
8. The formula $\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x + y)/(1 - xy)]$ holds directly, without adding or subtracting π , provided [1]
- A) $xy > 1$
 - B) $xy < 1$
 - C) $x + y = 0$
 - D) $x = y$

Section B: short answer

1. Evaluate $\tan^{-1}(1) + \tan^{-1}(0)$, giving your answer as an exact fraction of π . [3]
2. Evaluate $\sin^{-1}(\sin(2\pi/3))$, showing why the answer is not simply $2\pi/3$. [3]
3. Using a right-angled triangle, simplify $\sin(\cos^{-1}(3/5))$ to a fraction, showing your triangle or working clearly. [3]
4. Using a right-angled triangle, simplify $\tan(\sin^{-1}(5/13))$ to a fraction, showing your working. [3]
5. If $\theta = \sin^{-1}x$, show that $\cos(2\sin^{-1}x) = 1 - 2x^2$. [3]
6. Solve $\sin^{-1}x = \pi/3$ for x , and confirm that your value of x lies in the domain of $\sin^{-1}$. [3]
7. Prove that $\sin^{-1}(-x) = -\sin^{-1}x$ for $x \in [-1, 1]$. [3]
8. Using the sum formula for inverse tangents, evaluate $\tan^{-1}(1/4) + \tan^{-1}(3/5)$ exactly, stating why the formula applies without an adjustment by π . [3]
9. Evaluate $\cot^{-1}(-1)$, and explain why the answer is not $-\pi/4$. [3]

Section C: structured questions

These structured questions build up each result in guided steps (a)–(d); this is a teaching scaffold rather than a literal reproduction of board-paper wording.

1. This question is about the reciprocal inverse function $\operatorname{cosec}^{-1}x$.
 1. (a) State the domain and the range of the principal value branch of $\operatorname{cosec}^{-1}x$. [2]
 2. (b) Evaluate $\operatorname{cosec}^{-1}(2)$ exactly, using the relation $\operatorname{cosec}^{-1}x = \sin^{-1}(1/x)$. [2]
 3. (c) Evaluate $\operatorname{cosec}^{-1}(-2)$ exactly, explaining why your answer is negative. [2]
 4. (d) State whether $\operatorname{cosec}^{-1}x$ is an odd function, an even function, or neither, giving a one-line justification. [1]
2. This question builds up a double-angle result for the inverse cosine function.

1. (a) Let $\theta = \cos^{-1}x$. State the range of θ and express $\sin\theta$ in terms of x , explaining why $\sin\theta$ is taken as non-negative. [2]
2. (b) Hence show that $\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$. [2]
3. (c) Use the result in part (b) to evaluate $\sin(2\cos^{-1}(3/5))$ as a fraction. [2]
4. (d) State the values of x for which the identity in part (b) is valid. [1]
3. This question proves the standard identity $\sin^{-1}x + \cos^{-1}x = \pi/2$.
 1. (a) State the common domain on which both $\sin^{-1}x$ and $\cos^{-1}x$ are defined. [1]
 2. (b) Let $\theta = \sin^{-1}x$. State the range of θ and write $x = \sin\theta$. [2]
 3. (c) Show that $\cos(\pi/2 - \theta) = x$, and explain carefully why this means $\cos^{-1}x = \pi/2 - \theta$. [3]
 4. (d) Using the result of part (c), write down the value of $\sin^{-1}(0.6) + \cos^{-1}(0.6)$ without further calculation. [1]
4. This question asks you to solve equations that involve inverse trigonometric functions.
 1. (a) Solve $\sin^{-1}(2x) = \pi/6$ for x , giving your answer as an exact fraction. [1]
 2. (b) Solve $\cos^{-1}(3x - 1) = \pi/3$ for x , checking that your value keeps $3x - 1$ within the domain of $\cos^{-1}$. [2]
 3. (c) Solve $\tan^{-1}(x - 1) + \tan^{-1}(x + 1) = \pi/4$ for x , using the sum formula for inverse tangents and rejecting any root for which the formula's condition fails. [3]
 4. (d) Verify your accepted solution to part (c) by estimating each inverse tangent in degrees and checking that the two angles add to 45° . [1]

Answers

Section A

A1. B) $[-1, 1]$ — sine only takes values between -1 and 1 , so its inverse can only accept inputs in this interval.

A2. C) $[0, \pi]$ — this is the principal value branch chosen for $\cos^{-1}x$ because cosine is one-one and onto $[-1, 1]$ on this interval.

A3. B) $(-\pi/2, \pi/2)$ — open at both ends, since tangent is undefined at $\pm\pi/2$.

A4. A) $\pi/6$ — since $\sin(\pi/6) = 1/2$ and $\pi/6$ lies in $[-\pi/2, \pi/2]$, the principal branch.

A5. B) $2\pi/3$ — $\cos(2\pi/3) = -1/2$ and $2\pi/3$ lies in $[0, \pi]$, the principal branch of $\cos^{-1}$, so this (not $-\pi/3$) is the required value.

A6. B) reflecting it in the line $y = x$ — this is how the graph of any inverse function is obtained from the graph of the original (restricted) function.

A7. C) $x \leq -1$ or $x \geq 1$ — because $\sec^{-1}x = \cos^{-1}(1/x)$ and $1/x$ must satisfy $|1/x| \leq 1$, which needs |

$$x| \geq 1.$$

A8. B) $xy < 1$ — when $xy > 1$ the sum of the two angles leaves the range $(-\pi/2, \pi/2)$ of a single $\tan^{-1}$, so π must be added or subtracted.

Section B

B1. $\tan^{-1}(1) = \pi/4$ (since $\tan(\pi/4) = 1$, in range) and $\tan^{-1}(0) = 0$. Sum = $\pi/4 + 0 = \pi/4$. **[3 marks: 1 for each value, 1 for correct sum]**

B2. $2\pi/3$ does not lie in $[-\pi/2, \pi/2]$, so $\sin^{-1}(\sin(2\pi/3)) \neq 2\pi/3$. Using $\sin(\pi - \theta) = \sin\theta$: $\sin(2\pi/3) = \sin(\pi - 2\pi/3) = \sin(\pi/3)$. Since $\pi/3 \in [-\pi/2, \pi/2]$, $\sin^{-1}(\sin(2\pi/3)) = \pi/3$. **[3 marks: 1 for noting $2\pi/3$ is out of range, 1 for the supplementary-angle step, 1 for the final value]**

B3. Let $\theta = \cos^{-1}(3/5)$, so $\cos\theta = 3/5$. In a right-angled triangle, adjacent = 3, hypotenuse = 5, so opposite = $\sqrt{5^2 - 3^2} = \sqrt{16} = 4$ (a 3–4–5 triangle). Hence $\sin\theta = \text{opposite/hypotenuse} = 4/5$. **[3 marks: 1 for the triangle set-up, 1 for finding the third side, 1 for the ratio]**

B4. Let $\theta = \sin^{-1}(5/13)$, so $\sin\theta = 5/13$. In a right-angled triangle, opposite = 5, hypotenuse = 13, so adjacent = $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$ (a 5–12–13 triangle). Hence $\tan\theta = \text{opposite/adjacent} = 5/12$. **[3 marks: 1 for the triangle set-up, 1 for finding the third side, 1 for the ratio]**

B5. Let $\theta = \sin^{-1}x$, so $\sin\theta = x$. Using the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$: $\cos(2\sin^{-1}x) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2x^2$. **[3 marks: 1 for substitution, 1 for quoting the correct double-angle identity, 1 for the final expression]**

B6. $\sin^{-1}x = \pi/3$ means $x = \sin(\pi/3) = \sqrt{3}/2$. Check: $\pi/3 \in [-\pi/2, \pi/2]$ ✓, and $\sqrt{3}/2 \in [-1, 1]$ ✓, so the value is valid. **[3 marks: 1 for taking sine of both sides, 1 for the value $\sqrt{3}/2$, 1 for the domain check]**

B7. Let $\theta = \sin^{-1}x$, so $\sin\theta = x$ with $\theta \in [-\pi/2, \pi/2]$. Then $\sin(-\theta) = -\sin\theta = -x$. Since $\theta \in [-\pi/2, \pi/2]$, also $-\theta \in [-\pi/2, \pi/2]$. By the definition of $\sin^{-1}$ as the unique angle in $[-\pi/2, \pi/2]$ with a given sine, $\sin^{-1}(-x) = -\theta = -\sin^{-1}x$, as required. **[3 marks: 1 for the substitution $\theta = \sin^{-1}x$, 1 for $\sin(-\theta) = -x$ with $-\theta$ in range, 1 for the concluding statement]**

B8. Here $x = 1/4$, $y = 3/5$, so $xy = 3/20 < 1$, and the sum formula applies directly. $(x + y)/(1 - xy) = (1/4 + 3/5)/(1 - 3/20) = (17/20)/(17/20) = 1$. So $\tan^{-1}(1/4) + \tan^{-1}(3/5) = \tan^{-1}(1) = \pi/4$. **[3 marks: 1 for checking $xy < 1$, 1 for the correct substitution into the formula, 1 for the final value]**

B9. Let $\theta = \cot^{-1}(-1)$, so $\theta \in (0, \pi)$ and $\cot\theta = -1$, i.e. $\tan\theta = -1$. The angle in $(0, \pi)$ with $\tan\theta = -1$ is $\theta = 3\pi/4$, since $\tan(3\pi/4) = \tan(\pi - \pi/4) = -\tan(\pi/4) = -1$. So $\cot^{-1}(-1) = 3\pi/4$. The answer is not $-\pi/4$ because $-\pi/4$ does not lie in the range $(0, \pi)$ of $\cot^{-1}$: unlike $\tan^{-1}$, whose range $(-\pi/2, \pi/2)$ includes negative angles, $\cot^{-1}$'s range contains only angles strictly between 0 and π . **[3 marks: 1 for setting $\tan\theta = -1$, 1 for finding $\theta = 3\pi/4$ in the correct range, 1 for the explanation of why $-\pi/4$ is wrong]**

Section C

C1. (a) Domain: $|x| \geq 1$ ($x \leq -1$ or $x \geq 1$). Range: $[-\pi/2, \pi/2]$ excluding 0, i.e. $[-\pi/2, 0) \cup (0, \pi/2]$. [2]

(b) $\operatorname{cosec}^{-1}(2) = \sin^{-1}(1/2) = \pi/6$, since $\sin(\pi/6) = 1/2$ and $\pi/6$ is in the allowed range. [2]

(c) $\operatorname{cosec}^{-1}(-2) = \sin^{-1}(-1/2) = -\pi/6$. The answer is negative because $\sin^{-1}$ is an odd function ($\sin^{-1}(-1/2) = -\sin^{-1}(1/2)$), and the range of $\operatorname{cosec}^{-1}$ is symmetric about 0, so a negative input gives a negative principal value. [2]

(d) $\operatorname{cosec}^{-1}x$ is an **odd function**: from part (c), $\operatorname{cosec}^{-1}(-x) = \sin^{-1}(-1/x) = -\sin^{-1}(1/x) = -\operatorname{cosec}^{-1}x$. [1]

C2. (a) $\theta = \cos^{-1}x$ has range $[0, \pi]$. On this interval sine is never negative, so $\sin\theta = \sqrt{1-x^2}$ (taken with the positive square root). [2]

(b) $\sin 2\theta = 2\sin\theta\cos\theta = 2\sqrt{1-x^2} \cdot x = 2x\sqrt{1-x^2}$. [2]

(c) With $x = 3/5$: $\sqrt{1-9/25} = \sqrt{16/25} = 4/5$. So $\sin(2\cos^{-1}(3/5)) = 2 \times (3/5) \times (4/5) = 24/25$. [2]

(d) The identity is valid for $x \in [-1, 1]$, the domain of $\cos^{-1}x$. [1]

C3. (a) Domain: $x \in [-1, 1]$, the domain common to both $\sin^{-1}x$ and $\cos^{-1}x$. [1]

(b) $\theta = \sin^{-1}x$ has range $[-\pi/2, \pi/2]$, and by definition $x = \sin\theta$. [2]

(c) By the co-function identity, $\cos(\pi/2 - \theta) = \sin\theta = x$. Since $\theta \in [-\pi/2, \pi/2]$, it follows that $\pi/2 - \theta \in [0, \pi]$, which is exactly the principal-value range of $\cos^{-1}$. Because $\cos^{-1}x$ is defined as the unique angle in $[0, \pi]$ whose cosine is x , and $\pi/2 - \theta$ is such an angle, $\cos^{-1}x = \pi/2 - \theta$. [3]

(d) Rearranging part (c): $\theta + (\pi/2 - \theta) = \pi/2$, i.e. $\sin^{-1}x + \cos^{-1}x = \pi/2$ for any valid x , so $\sin^{-1}(0.6) + \cos^{-1}(0.6) = \pi/2$. [1]

C4. (a) $2x = \sin(\pi/6) = 1/2$, so $x = 1/4$. [1]

(b) $3x - 1 = \cos(\pi/3) = 1/2$, so $3x = 3/2$ and $x = 1/2$. Check: $3(1/2) - 1 = 1/2$, which lies in $[-1, 1]$, so the value is valid. [2]

(c) Let $A = x - 1$, $B = x + 1$, so $A + B = 2x$ and $AB = (x - 1)(x + 1) = x^2 - 1$. Applying the sum formula: $2x/(1 - (x^2 - 1)) = \tan(\pi/4) = 1$, i.e. $2x/(2 - x^2) = 1$, giving $2x = 2 - x^2$, so $x^2 + 2x - 2 = 0$. By the quadratic formula, $x = (-2 \pm \sqrt{4 + 8})/2 = -1 \pm \sqrt{3}$. For $x = -1 + \sqrt{3}$ (≈ 0.732): $AB = x^2 - 1 = (4 - 2\sqrt{3}) - 1 \approx -0.46$, which is less than 1, so the sum formula applies directly and this root is valid. For $x = -1 - \sqrt{3}$ (≈ -2.732): $AB = x^2 - 1 = (4 + 2\sqrt{3}) - 1 \approx 6.46$, which is greater than 1, so the sum formula's condition fails and π would need to be subtracted; direct substitution shows the two angles actually sum to $-3\pi/4$, not $\pi/4$, so this root is rejected. The accepted solution is $x = \sqrt{3} - 1$. [3]

(d) With $x = \sqrt{3} - 1 \approx 0.732$: $x - 1 \approx -0.268$, and $\tan^{-1}(-0.268) \approx -15^\circ$. $x + 1 = \sqrt{3} \approx 1.732$, and $\tan^{-1}(\sqrt{3}) = 60^\circ$ exactly. Sum $\approx -15^\circ + 60^\circ = 45^\circ = \pi/4$, confirming the accepted solution. [1]