MEGA LECTURE

IB DP PHYSICS

Theme A: Space, Time and Motion

A.4 Rigid Body Mechanics

Revision Notes · Higher Level only

This topic is examined at HIGHER LEVEL (HL) only

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Original notes prepared for the IB Diploma Programme Physics course (first assessment 2025)

What the syllabus requires (HL only)

A.4 Rigid Body Mechanics is a **Higher Level only** topic. By the end of it you should be able to work confidently with each of the following. Use this list as a final checklist before the exam.

Understanding	You should be able to...
Torque	Calculate $\tau = Fr \sin \theta$ about an axis, including for couples, and find the net (resultant) torque on a body.
Rotational equilibrium	Apply the two conditions of equilibrium (zero net force **and** zero net torque) to beams, ladders, bridges and pivots.
Moment of inertia	Use $I = \Sigma mr^2$ for systems of point masses, and apply given formulas for standard shapes; explain how mass distribution affects I.
Rotational kinematics	Use $\omega = \omega_0 + \alpha t$, $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$ for constant angular acceleration, and the links $v = \omega r$, $a = \alpha r$.
Newton's second law for rotation	Apply $\tau = I\alpha$, including in combined translation-rotation problems (masses on strings over massive pulleys, flywheels).
Angular momentum	Use $L = I\omega$; apply conservation of angular momentum when the net external torque is zero; use angular impulse $\Delta L = \tau\Delta t$ and the area under a torque-time graph.
Rotational kinetic energy	Use $E_k = \frac{1}{2}I\omega^2$, and analyse rolling without slipping, where the total kinetic energy is translational plus rotational.

> **HL only:** A.4 appears only in Higher Level papers. It builds directly on A.1-A.3: every rotational quantity in this topic is the exact analogue of a linear quantity you already know. Learn the translation dictionary in the quick-reference table at the end and half the topic is done.

1. From point particles to rigid bodies

Up to A.3 every object was treated as a **point particle**: all of its mass at one point, so a force could only make it accelerate in a straight line. A **rigid body** is an extended object whose shape does not change — the distances between its particles are fixed. For a rigid body, *where* a force is applied matters as much as how big it is.

The general motion of a rigid body splits neatly into two parts:

- **Translational motion** of the centre of mass, governed by $F = ma$ exactly as before.
- **Rotational motion** about an axis, governed by the new rotational quantities of this topic.

A rolling wheel, a gymnast in mid-air and a spinning satellite all do both at once. The **centre of mass** is the single point that moves as if the whole weight and all external forces acted there; for a uniform symmetric body it is at the geometric centre. In torque problems the entire weight of a body is drawn as one arrow acting at the centre of mass.

> **Exam note:** Rotational quantities use radians. Angular displacement $\Delta\theta$ is in rad, angular velocity ω in rad s^{-1} and angular acceleration α in rad s^{-2}. Convert revolutions using 1 rev = 2π rad, and rpm using $\omega = 2\pi N/60$ where N is in revolutions per minute.

2. Torque

Torque (moment of a force) measures the turning effect of a force about a chosen axis. For a force F applied at distance r from the axis, at angle θ between the force and the line joining the axis to the point of application:

$$\tau = F\,r\,\sin\theta$$

The unit is the newton metre (N m). It is *not* the joule, even though 1 N m = 1 J dimensionally — torque is a turning effect, not energy. The factor $r \sin\theta$ is the **perpendicular distance** from the axis to the line of action of the force (the "lever arm"). Two equivalent readings:

- (component of F perpendicular to r) × r = $F \sin\theta \times r$
- F × (perpendicular distance from axis to line of action) = $F \times r \sin\theta$

Torque is maximum when θ = 90° (push at right angles to a door) and zero when the line of action passes through the axis (θ = 0), however large the force — pushing along a door's plane, or pulling directly towards the hinge, produces no rotation.

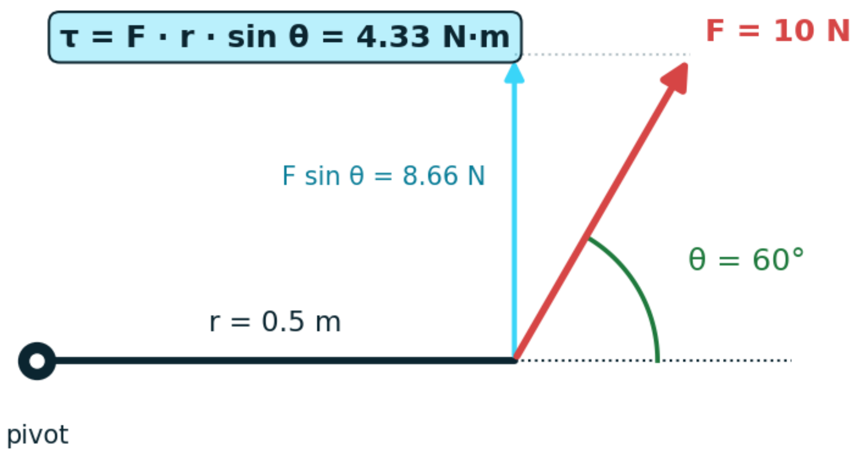

Figure 1. Torque τ = F·r·sin θ, the turning effect of a force about a pivot. With F = 10 N, r = 0.5 m and θ = 60°, τ ≈ 4.33 N·m. Only the component F sin θ perpendicular to r turns the body.

Sign convention and net torque

Choose one sense of rotation (usually anticlockwise) as positive and state it. Torques turning the body that way are positive; the other way, negative. The **net torque** is the algebraic sum of the torques of all forces about the *same* axis. Only the net torque decides whether the rotation of the body changes.

Couples

A **couple** is a pair of forces that are equal in magnitude, opposite in direction and do not share a line of action — for example two hands on a steering wheel. The net force is zero, so the centre of mass does not accelerate, but the net torque is not zero: the body purely rotates. For forces of magnitude F whose lines of action are separated by perpendicular distance d:

$$\tau_{\text{couple}} = F\,d$$

A useful property: the torque of a couple has the same value about *every* axis perpendicular to the plane of the forces.

Strictly, torque is a vector pointing along the rotation axis (by the right-hand rule), which is why torques about a single axis can be handled with plus and minus signs alone. IB problems are planar, so the sign convention is all you need — but state it every time.

Worked example 1 — torque on a spanner

A mechanic applies a 40 N force to the end of a spanner 0.25 m from the nut. The force makes an angle of 60° with the handle. Calculate the torque about the nut, and state the maximum torque available with the same force.

$\tau = Fr \sin \theta = 40 \times 0.25 \times \sin 60° = 10 \times 0.866 =$ **8.7 N m**.

Maximum torque occurs at $\theta = 90°$: $\tau_{max} = 40 \times 0.25 =$ **10 N m**. This is why you instinctively push perpendicular to the handle.

3. Rotational equilibrium: beams, ladders and pivots

A rigid body is in equilibrium only when **both** conditions hold:

- **Translational equilibrium:** the resultant force is zero ($\Sigma F = 0$ in every direction).
- **Rotational equilibrium:** the resultant torque about *any* axis is zero ($\Sigma\tau = 0$).

The two conditions are independent: a couple satisfies the first but not the second. If a body is in equilibrium, the net torque is zero about every axis, so you are free to take torques about whichever point kills the most unknowns — usually the point where an unknown force acts.

A reliable method for statics problems

1. Draw a free-body diagram of the rigid body, with the weight acting at the centre of mass. 2. Mark all distances from a clearly chosen pivot. 3. Take torques about the point through which the most unknown forces pass (their torques are then zero) and set $\Sigma\tau = 0$. 4. Use $\Sigma F = 0$ horizontally and vertically for the remaining unknowns. 5. Check: answers should be positive in the assumed directions and the totals should balance.

Worked example 2 — plank on two supports

A uniform beam of length 4.0 m and weight 200 N rests horizontally on supports at its two ends, A (left) and B (right). A person of weight 500 N stands 1.0 m from A. Find the force exerted by each support.

The beam is uniform, so its weight acts at the midpoint, 2.0 m from A.

Torques about A (anticlockwise positive): $R_B \times 4.0 - 200 \times 2.0 - 500 \times 1.0 = 0$

$R_B = (400 + 500) / 4.0 =$ **225 N** $\approx 2.3 \times 10^2$ N.

Vertical forces: $R_A + R_B = 200 + 500 = 700$ N, so $R_A = 700 - 225 =$ **475 N** $\approx 4.8 \times 10^2$ N.

Check: the person stands nearer A, so A carries more load — as found.

Worked example 3 — ladder against a smooth wall

A uniform ladder of mass 12 kg and length 5.0 m leans against a frictionless vertical wall, making 60° with the horizontal ground. Find the force from the wall and the friction force required at the ground. ($g = 9.8$ m s^{-2})

Weight $W = 12 \times 9.8 = 117.6$ N acting at the ladder's midpoint. Forces: normal N (up) and friction f (towards the wall) at the base; horizontal normal force N_w from the smooth wall.

Take torques about the base (both base forces then have zero torque). The weight's lever arm is $(L/2)\cos 60°$; the wall force's lever arm is $L \sin 60°$:

$N_w \times 5.0 \sin 60° = 117.6 \times 2.5 \cos 60°$

$N_w = (117.6 \times 2.5 \times 0.500) / (5.0 \times 0.866) = 147.0 / 4.33 =$ **34 N**.

Horizontal equilibrium: $f = N_w =$ **34 N**; vertical: $N = W =$ **118 N**. A steeper ladder needs less friction — this is why ladders slip when set at shallow angles.

Worked example 4 — rod, wire and hanging sign

A uniform rod of weight 60 N and length 1.5 m is hinged to a wall at one end. A wire from the far end runs back up to the wall, making 40° with the rod. A sign of weight 80 N hangs from the far end. Find the tension in the wire and the horizontal and vertical components of the hinge force.

Take torques about the hinge, so the unknown hinge force drops out. The wire's lever arm is 1.5 sin 40°; the rod's weight acts at its centre, 0.75 m out:

$T \times 1.5 \sin 40° = 60 \times 0.75 + 80 \times 1.5 = 45 + 120 = 165$ N m

$T = 165 / (1.5 \times 0.643) = 165 / 0.964 =$ **171 N ≈ 170 N**.

Vertical equilibrium: $V + T \sin 40° = 60 + 80$, so $V = 140 - 171 \times 0.643 = 140 - 110 =$ **30 N upward**.

Horizontal equilibrium: $H = T \cos 40° = 171 \times 0.766 =$ **131 N ≈ 130 N**, pushing away from the wall (it balances the horizontal pull of the wire on the rod).

Stability and toppling

Torque ideas also decide whether a body tips over. A body standing on a surface topples when the vertical line through its centre of mass falls outside its base of support: the weight then exerts a net torque about the pivoting edge instead of restoring the body. A low centre of mass and a wide base mean a large tilt is needed before this happens — which is why racing cars are built low and wide and why a tractor on a side slope is at risk long before a car would be.

> **Exam technique:** In “find the force” statics questions, the marks are usually: one for taking torques about a sensible point, one for correct lever arms (watch the sines and cosines), one for the answer. Writing the torque equation with a stated pivot earns credit even if arithmetic slips.

4. Moment of inertia

Mass measures resistance to linear acceleration; **moment of inertia** I measures resistance to *angular* acceleration about a particular axis. For a collection of point masses:

$$I = \Sigma mr^2$$

where r is each mass's perpendicular distance from the axis. The unit is kg m^2. Because r is squared, mass far from the axis counts far more than mass near it:

- I depends on the **axis chosen**, not just the body — a rod spun about its end has four times the moment of inertia it has about its centre.
- Two bodies of equal mass and radius can have different I: a hoop (all mass at the rim) beats a uniform disc of the same M and R.
- Moments of inertia about the same axis simply add: $I_{total} = I_1 + I_2 + ...$

Body and axis	Moment of inertia
Point mass m at distance r from axis	$I = mr^2$
Thin hoop or ring, mass M, radius R, about central axis	$I = MR^2$
Uniform solid disc or cylinder, about central axis	$I = ½MR^2$
Uniform solid sphere, about a diameter	$I = (2/5)MR^2$
Thin uniform rod, length L, about centre (perpendicular to rod)	$I = (1/12)ML^2$
Thin uniform rod, about one end (perpendicular to rod)	$I = (1/3)ML^2$

> **Exam note:** You are **not** expected to memorise or derive these formulas: IB questions supply the moment of inertia expression they want you to use. You **are** expected to use $I = \Sigma mr^2$ directly for point masses, to add moments of inertia, and to reason qualitatively about mass distribution.

Worked example 5 — point masses on a light rod

Two 0.50 kg masses sit at the ends of a light (massless) rod of length 1.2 m. Calculate the moment of inertia (a) about the centre of the rod, (b) about one end.

(a) Each mass is at $r = 0.60$ m: $I = 2 \times 0.50 \times 0.60^2 = 2 \times 0.50 \times 0.36 =$ **0.36 kg m^2**.

(b) One mass at $r = 0$ (contributes nothing), one at $r = 1.2$ m: $I = 0.50 \times 1.2^2 =$ **0.72 kg m^2**.

Same body, different axis, double the moment of inertia — axis choice matters.

5. Rotational kinematics

Angular displacement $\Delta\theta$ (rad), angular velocity $\omega = \Delta\theta/\Delta t$ (rad s^{-1}) and angular acceleration $\alpha = \Delta\omega/\Delta t$ (rad s^{-2}) describe rotation exactly as s, v, a describe straight-line motion. When α is **constant**, the suvat equations carry over symbol for symbol:

$$\omega = \omega_0 + \alpha t \quad \Delta\theta = \omega_0 t + ½\alpha t^2 \quad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta \quad \Delta\theta = ½(\omega_0 + \omega)t$$

For a body rotating at constant rate, $\omega = 2\pi/T = 2\pi f$, linking to circular motion (A.2).

Linking angular and linear quantities

A point at distance r from the axis of a rotating body travels on a circle, so its speed and tangential acceleration follow from the geometry of radians:

$$v = \omega r \qquad a = \alpha r$$

All points of a rigid body share the same ω and α, but points farther from the axis move faster. (The point also has a centripetal acceleration $v^2/r = \omega^2 r$ directed towards the axis, even when $\alpha = 0$.)

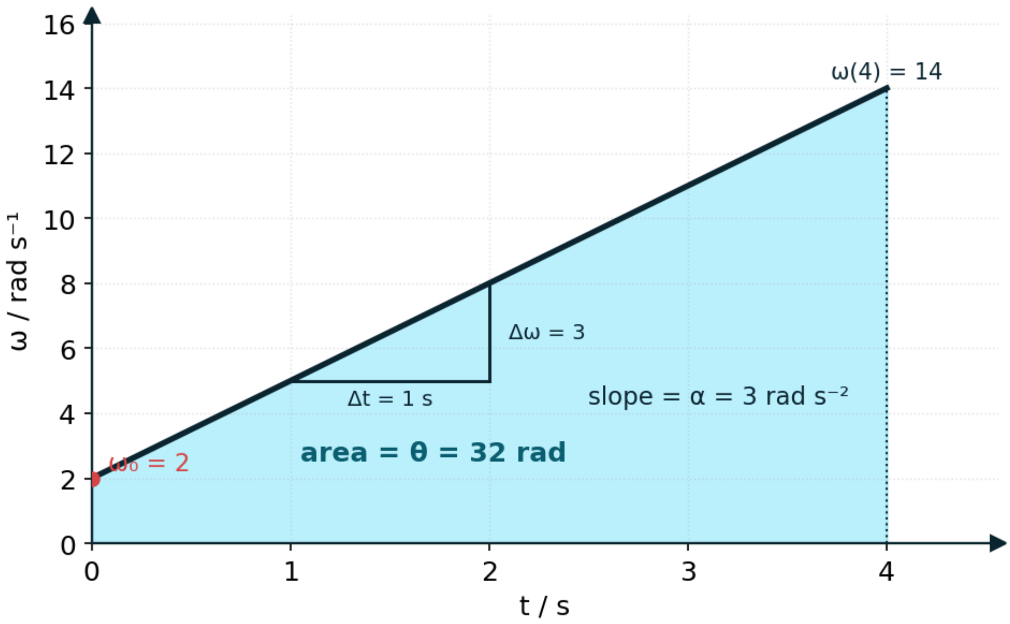

Figure 2. Angular velocity against time for constant angular acceleration α. The slope equals α and the shaded area equals the angular displacement θ. Here $\omega_0 = 2$ rad s^{-1} and $\alpha = 3$ rad s^{-2}, so over 4 s ω reaches 14 rad s^{-1} and $\theta = 32$ rad.

Worked example 6 — spinning up a flywheel

A flywheel accelerates uniformly from rest to 3000 rpm in 12 s. Find (a) the angular acceleration, (b) the number of revolutions completed in the 12 s.

(a) $\omega = 2\pi \times 3000/60 = 100\pi = 314$ rad s^{-1}. Then $\alpha = (\omega - \omega_0)/t = 314/12 =$ **26 rad s^{-2}**.

(b) $\Delta\theta = ½(\omega_0 + \omega)t = ½ \times 314 \times 12 = 1885$ rad. Revolutions = 1885/(2π) = **300 rev**.

Sanity check: average rate = 1500 rpm = 25 rev s^{-1}; over 12 s that is 25 × 12 = 300 rev. Consistent.

Worked example 7 — the wheels of an accelerating car

A car accelerates uniformly from rest to 24 m s^{-1} in 10 s. Its wheels have radius 0.30 m and roll without slipping. Find (a) the final angular velocity of a wheel, (b) its angular acceleration, (c) the number of revolutions each wheel makes.

(a) Rolling without slipping links rim speed to car speed: $\omega = v/r = 24/0.30 =$ **80 rad s^{-1}**.

(b) $\alpha = \Delta\omega/\Delta t = 80/10 =$ **8.0 rad s^{-2}** (equivalently $a/r = 2.4/0.30$).

(c) $\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t = \frac{1}{2} \times 80 \times 10 = 400$ rad; revolutions = $400/(2\pi)$ = **64 rev**.

6. Newton's second law for rotation

Just as an unbalanced force gives a mass a linear acceleration ($F = ma$), an unbalanced torque gives a rigid body an angular acceleration:

$$\tau = I\alpha$$

where τ is the **net** torque about the axis and I the moment of inertia about that same axis. Large I means sluggish response: a merry-go-round loaded at the rim is much harder to spin up than the same mass loaded at the centre.

Combined translation and rotation problems

The classic HL problem couples a falling mass to a rotating wheel by a light string. Strategy:

- Write $F = ma$ for each translating mass (weight and tension).
- Write $\tau = I\alpha$ for the wheel — the tension acting at the rim provides the torque $\tau = TR$.
- Link them with the no-slip condition $a = \alpha R$ (string does not slip on the wheel).
- Solve simultaneously. Key insight: the tension is **not** equal to the weight — if it were, nothing would accelerate.

Worked example 8 — falling mass turning a pulley

A block of mass m = 2.0 kg hangs from a light string wrapped around the rim of a uniform disc pulley of mass M = 4.0 kg and radius R = 0.20 m ($I = \frac{1}{2}MR^2$), free to rotate on a frictionless axle. Find the acceleration of the block, the string tension and the angular acceleration of the pulley. (g = 9.8 m s^{-2})

Block: $mg - T = ma$

Pulley: $TR = I\alpha = \frac{1}{2}MR^2(a/R)$ so $T = \frac{1}{2}Ma$

Substitute: $mg = ma + \frac{1}{2}Ma$ giving $a = mg/(m + \frac{1}{2}M) = (2.0 \times 9.8)/(2.0 + 2.0)$ = **4.9 m s^{-2}**.

$T = \frac{1}{2} \times 4.0 \times 4.9$ = **9.8 N** (check: $mg - T = 19.6 - 9.8 = 9.8$ N, and $ma = 2.0 \times 4.9 = 9.8$ N — consistent).

$\alpha = a/R = 4.9/0.20$ = **25 rad s^{-2}**.

Notice a < g: part of the weight's effect goes into spinning the pulley.

Common pitfall: Never write the torque on the pulley as mgR. The string exerts the **tension** on the pulley, and tension is less than the weight whenever the system accelerates.

Worked example 9 — Atwood machine with a massive pulley

Masses of 3.0 kg and 2.0 kg hang from the two ends of a light string passing over a uniform disc pulley of mass 2.0 kg and radius 0.10 m. The string does not slip. Find the acceleration and the tension on each side. (g = 9.8 m s^{-2})

Because the pulley has inertia, the two tensions are **different** — the difference is what provides the torque. Call them T_1 (under the 3.0 kg mass) and T_2.

3.0 kg mass (down positive): $3.0g - T_1 = 3.0a$

2.0 kg mass (up positive): $T_2 - 2.0g = 2.0a$

Pulley: $(T_1 - T_2)R = \frac{1}{2}MR^2(a/R)$ so $T_1 - T_2 = \frac{1}{2}Ma = 1.0a$

Adding all three: $(3.0 - 2.0)g = (3.0 + 2.0 + 1.0)a$, so $a = 9.8/6.0$ = **1.6 m s^{-2}**.

$T_1 = 3.0(9.8 - 1.63)$ = **25 N**; $T_2 = 2.0(9.8 + 1.63)$ = **23 N**. With a massless pulley these would be equal — quoting a single “the tension” here loses marks.

7. Angular momentum and its conservation

The rotational analogue of momentum $p = mv$ is **angular momentum**:

$$\boldsymbol{L = I\omega}$$

with unit kg m^2 s^{-1}. Newton's second law for rotation can be rewritten $\tau = \Delta L/\Delta t$, which leads to the central result:

> **Conservation of angular momentum:** If the net external torque on a system is zero, its total angular momentum is constant: $I_1\omega_1 = I_2\omega_2$. This holds even when the body changes shape and so changes its own moment of inertia.

- **Ice skater:** pulling the arms in reduces I, so ω increases; extending them slows the spin. No external torque acts about the vertical axis (friction is negligible).
- **Collapsing star:** when a star's core collapses to a neutron star, its radius shrinks enormously; since $I \propto r^2$, ω grows by the same huge factor — producing pulsars spinning many times per second.
- **Diver or gymnast:** tucking mid-somersault speeds the rotation; opening out before entry slows it for a clean finish.

Angular momentum of a moving particle

A point particle of mass m moving with speed v along a line whose perpendicular distance from the axis is r also carries angular momentum $L = mvr$ (this is just $I\omega$ with $I = mr^2$ and $\omega = v/r$). This is what lets you handle a lump of clay landing on a turntable, or a child jumping onto a roundabout: add the particle's mvr (or its mr^2 to I) and conserve the total.

Angular impulse

A torque acting for a time changes angular momentum by the **angular impulse**:

$$\Delta \boldsymbol{L} = \tau \Delta \boldsymbol{t}$$

exactly parallel to $\Delta p = F\Delta t$. If the torque varies, the change in angular momentum equals the **area under the torque-time graph** — a favourite graph-reading question.

Worked example 10 — the spinning skater

A skater spinning at 2.0 rev s^{-1} with arms outstretched has moment of inertia 5.0 kg m^2. She pulls her arms in, reducing it to 2.0 kg m^2. Find (a) her new rate of spin, (b) the ratio of her final to initial rotational kinetic energy.

(a) No external torque, so $I_1\omega_1 = I_2\omega_2$: f_2 = (5.0 × 2.0)/2.0 = **5.0 rev $\mathbf{s^{-1}}$** (rev s^{-1} works here because the 2π factors cancel).

(b) $E_k = ½I\omega^2 = ½(I\omega)\omega = ½L\omega$. Since L is unchanged, the energy ratio equals the ω ratio: E_2/E_1 = 5.0/2.0 = **2.5**.

Kinetic energy *increases* — the skater does work pulling her arms inward against the tendency to fly outward. Momentum conservation never guarantees energy conservation.

Worked example 11 — braking a flywheel

A flywheel of moment of inertia 2.0 kg m^2 spins at 300 rad s^{-1}. A brake applies a constant friction torque of 12 N m. Find (a) the time taken to stop, (b) the number of revolutions made while stopping.

(a) Angular impulse: $\tau\Delta t = \Delta L = I\omega_0 = 2.0 \times 300 = 600$ kg m^2 s^{-1}, so $t = 600/12 =$ **50 s**.

(b) $\alpha = \tau/I = 12/2.0 = 6.0$ rad s^{-2} (deceleration). Using $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$: $\Delta\theta = 300^2/(2 \times 6.0) =$ 7500 rad = 7500/(2π) ≈ **1.2×10^3 revolutions**.

Check via average rate: mean ω = 150 rad s^{-1} for 50 s gives 7500 rad — consistent.

8. Rotational kinetic energy and rolling

A body rotating at angular speed ω stores kinetic energy:

$$E_{k,rot} = ½I\omega^2$$

A body that both translates and rotates — a rolling ball — has both kinds at once:

$$E_{k,total} = ½mv^2 + ½I\omega^2$$

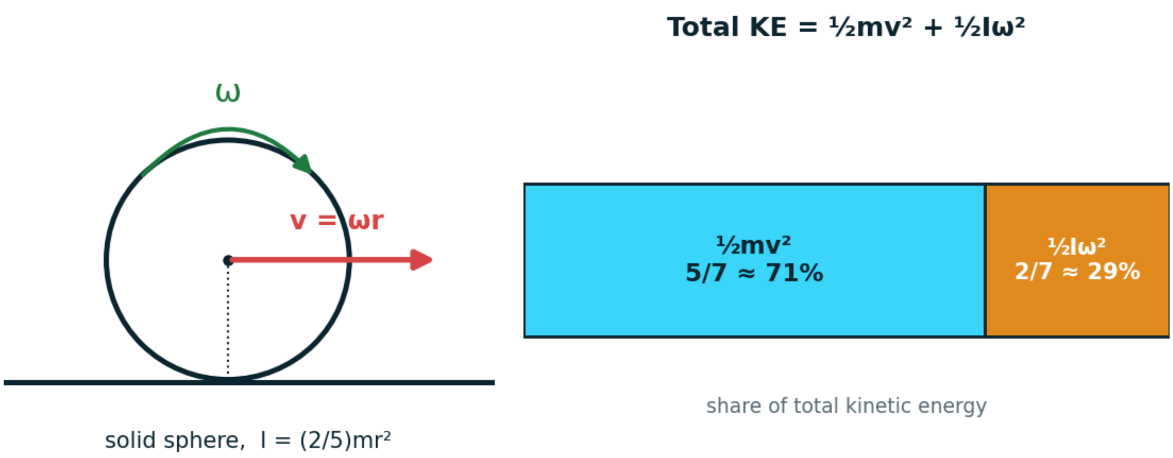

Figure 3. A solid sphere ($I = (2/5)mr^2$) rolling without slipping obeys $v = \omega r$. Its kinetic energy divides $½mv^2 : ½I\omega^2 = 5/7 : 2/7$, so 71% is translational and 29% rotational.

Rolling without slipping

If a wheel of radius *r* rolls without slipping, the contact point is momentarily at rest and the centre of mass speed is locked to the spin:

$$v = \omega r$$

Friction at the contact point is what prevents slipping, but because the contact point does not slide, this friction does **no work**: mechanical energy is conserved for ideal rolling. For a shape with $I = kmr^2$ rolling at speed *v*:

$$E_{k,total} = ½mv^2 + ½(kmr^2)(v/r)^2 = ½mv^2(1 + k)$$

Rolling downhill and the race of shapes

Releasing a shape from rest at height h on a ramp, energy conservation gives $mgh = ½mv^2(1 + k)$, so:

$$v = [2gh/(1 + k)]^{1/2}$$

Mass and radius cancel. Only the *distribution* of mass (the value of $k = I/mr^2$) decides the race:

Shape	$k = I/mr^2$	Fraction of E_k translational	Speed at bottom	Finishing order
Solid sphere	2/5	5/7 ≈ 71%	$(10gh/7)^{1/2} = 1.20(gh)^{1/2}$	1st
Solid disc / cylinder	1/2	2/3 ≈ 67%	$(4gh/3)^{1/2} = 1.15(gh)^{1/2}$	2nd
Hoop / ring	1	1/2 = 50%	$(gh)^{1/2} = 1.00(gh)^{1/2}$	3rd
Frictionless sliding block	0	100%	$(2gh)^{1/2} = 1.41(gh)^{1/2}$	(beats them all)

The more of its mass a shape carries near the rim, the larger the share of energy locked up in rotation and the slower it arrives — independent of its mass or size.

Worked example 12 — sphere rolling down a ramp

A solid sphere ($I = (2/5)mr^2$) rolls without slipping from rest down a ramp of vertical height 1.4 m. Find its speed at the bottom, and compare with a frictionless sliding block. ($g = 9.8$ m s^{-2})

$mgh = ½mv^2 + ½ \times (2/5)mr^2 \times (v/r)^2 = (7/10)mv^2$

$v = (10gh/7)^{1/2} = (10 \times 9.8 \times 1.4 / 7)^{1/2} = (19.6)^{1/2} =$ **4.4 m s^{-1}**.

Sliding block: $v = (2gh)^{1/2} = (2 \times 9.8 \times 1.4)^{1/2} = (27.4)^{1/2} =$ **5.2 m s^{-1}**. The sphere is slower because 2/7 of its energy (28.6%) is rotational.

If a wheel spins on ice or a ball is struck so that it skids, the condition $v = \omega r$ no longer holds: the contact point slides, kinetic friction acts there and **does** negative work, so mechanical energy is lost until rolling is re-established. Recognising whether contact is rolling (static friction, no energy loss) or slipping (kinetic friction, energy loss) is the key discriminator in harder questions.

Flywheels as energy stores

Because $E_k = ½I\omega^2$ grows with the square of the spin rate, a compact flywheel spun very fast stores large energy: flywheel systems recover braking energy in vehicles and smooth out the delivery of engines and power grids. Doubling ω quadruples the stored energy; moving the same mass to twice the radius also quadruples it (via I).

> **Exam technique:** In rolling problems, resist plugging numbers early. Substitute $\omega = v/r$ algebraically first — r almost always cancels, which is itself a checkable prediction: the answer should not depend on the radius.

9. Common pitfalls

- Using degrees in rotational equations. All the equations of this topic assume radians.

- Taking torques about different points in the same equation — every torque in $\Sigma\tau = 0$ must use the same axis.
- Forgetting the weight of the beam or ladder itself, or placing it anywhere other than the centre of mass.
- Using $F\,r$ instead of $F\,r \sin\theta$ when the force is not perpendicular to the lever.
- Setting string tension equal to the hanging weight in accelerating pulley systems.
- Quoting a moment of inertia without an axis. “The moment of inertia of the rod” is meaningless until the axis is stated.
- Claiming kinetic energy is conserved whenever angular momentum is (the skater's KE rises; a clay lump landing on a turntable loses KE).
- Dropping the rotational term $\frac{1}{2}I\omega^2$ from the energy equation of a rolling object.

10. Quick reference: the linear-rotational dictionary

Linear quantity / law	Rotational analogue	Link
Displacement s	Angular displacement $\Delta\theta$ (rad)	$s = \theta r$
Velocity v	Angular velocity ω (rad s^{-1})	$v = \omega r$
Acceleration a	Angular acceleration α (rad s^{-2})	$a = \alpha r$
Mass m	Moment of inertia $I = \Sigma mr^2$	depends on axis
Force F	Torque $\tau = Fr \sin\theta$	N m
$F = ma$	$\tau = I\alpha$	net torque, same axis
Momentum $p = mv$	Angular momentum $L = I\omega$	kg m^2 s^{-1}
Impulse $F\Delta t = \Delta p$	Angular impulse $\tau\Delta t = \Delta L$	area under τ-t graph
$E_k = \frac{1}{2}mv^2$	$E_k = \frac{1}{2}I\omega^2$	rolling: add both
suvat equations	$\omega = \omega_0 + \alpha t$; $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$; $\omega^2 = \omega_0{}^2 + 2\alpha\Delta\theta$	constant α only

11. Test yourself

Attempt these without notes; full answers follow. Take g = 9.8 m s^{-2}.

1. A force of 90 N is applied perpendicular to the end of a wrench 0.30 m long. Calculate the torque about the bolt.

2. Two anti-parallel 15 N forces act on opposite edges of a steering wheel of diameter 0.40 m. State the resultant force and calculate the torque of the couple.

3. A 30 kg child sits 2.0 m from the pivot of a seesaw. How far from the pivot, on the other side, must a 40 kg child sit for balance?

4. A uniform horizontal rod of weight 300 N and length 2.0 m is hinged to a wall at one end and held by a cable at the other. The cable makes 30° with the rod. Find the cable tension.

5. A wheel spinning at 20 rad s^{-1} decelerates uniformly to rest in 8.0 s. Find the angular acceleration and the angle turned through while stopping.

6. A constant torque of 4.0 N m acts for 6.0 s on a stationary disc of moment of inertia 0.50 kg m^2. Find the angular acceleration, the final angular velocity and the final angular momentum. Verify the angular momentum using angular impulse.

7. A star rotating once every 25 days collapses to a neutron star with 1/100 of its original radius, keeping its mass (treat it as a uniform sphere throughout). By what factor does its angular velocity increase, and what is its new rotation period?

8. A hoop, a uniform disc and a solid sphere of equal mass and radius are released together from rest at the top of a ramp and roll without slipping. Give the finishing order and explain it without calculation.

9. A turntable of moment of inertia 0.40 kg m^2 spins freely at 12 rad s^{-1}. A 0.50 kg lump of clay drops vertically onto it and sticks at 0.30 m from the axis. Find the new angular velocity, and state whether kinetic energy is conserved.

10. A flywheel of moment of inertia 2.0 kg m^2 rotates at 300 rad s^{-1}. How much kinetic energy does it store?

Answers

1. $\tau = Fr \sin 90° = 90 \times 0.30 =$ **27 N m**.

2. Resultant force = **zero** (equal and opposite forces). Torque of couple = $Fd = 15 \times 0.40 =$ **6.0 N m** about any axis perpendicular to the wheel.

3. Balance of torques about the pivot: $30g \times 2.0 = 40g \times d$, so $d = 60/40 =$ **1.5 m**. (g cancels.)

4. Torques about the hinge: $T \sin 30° \times 2.0 = 300 \times 1.0$ (weight at the midpoint), so $T = 300/(2 \times 0.5) \times 1.0 =$ **300 N**.

5. $\alpha = (0 - 20)/8.0 =$ **-2.5 rad s^{-2}** (magnitude 2.5 rad s^{-2}). $\Delta\theta = ½(\omega_0 + \omega)t = ½ \times 20 \times 8.0 =$ **80 rad** (about 12.7 revolutions).

6. $\alpha = \tau/I = 4.0/0.50 =$ **8.0 rad s^{-2}**. $\omega = \alpha t = 8.0 \times 6.0 =$ **48 rad s^{-1}**. $L = I\omega = 0.50 \times 48 =$ **24 kg m^2 s^{-1}**. Check: angular impulse = $\tau\Delta t = 4.0 \times 6.0 = 24$ N m s = ΔL — consistent.

7. $I \propto r^2$, so I falls by $100^2 = 10^4$. Conservation of L: ω increases by **10^4**. New period = $(25 \times 86\,400 \text{ s})/10^4 = 2.16 \times 10^6/10^4 \approx$ **220 s** (about 3.6 minutes).

8. Order: **sphere, then disc, then hoop**. All start with the same mgh and finish with $½mv^2(1 + k)$. The hoop ($k = 1$) diverts the largest share of energy into rotation, so it moves slowest; the sphere ($k = 2/5$) the smallest share, so it is fastest. Mass and radius cancel and do not matter.

9. The clay lands vertically, exerting no torque about the axis, so L is conserved: $I_2 = 0.40 + 0.50 \times 0.30^2 = 0.445$ kg m^2. $\omega_2 = (0.40 \times 12)/0.445 = 4.8/0.445 =$ **10.8 rad s^{-1}**. Kinetic energy is **not** conserved: this is a rotational "perfectly inelastic collision" (E_k drops from 28.8 J to about 25.9 J as the clay is dragged up to speed).

10. $E_k = ½I\omega^2 = ½ \times 2.0 \times 300^2 =$ **9.0×10^4 J** (90 kJ) — flywheels store energy this way in energy-recovery systems.