IB DP MATHEMATICS

Analysis and Approaches (AA)

Topic 3 Geometry and Trigonometry

Revision Notes · Standard and Higher Level

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Original notes prepared for the IB Diploma Programme Mathematics: Analysis and Approaches course (first assessment 2021)

What Topic 3 requires

Topic 3 is the geometry and trigonometry strand of Analysis and Approaches. Items marked **(HL)** are examined only at Higher Level; everything else is common to SL and HL. Use this checklist for a final revision sweep.

You should be able to...	Where it is covered
Find distances, midpoints and angles inside three-dimensional solids, and compute volume and surface area.	Section 1
Apply right-angled trigonometry, angles of elevation and depression, and bearings.	Section 2
Use the sine rule (including the ambiguous case), the cosine rule and the area formula ½ab sin C.	Section 3
Work in radians: convert to and from degrees, and use arc length and sector area.	Section 4
Read the unit circle, quote exact values and apply the CAST sign rule.	Section 5
Use the Pythagorean, quotient and double-angle identities.	Section 6
Solve trigonometric equations over a stated interval.	Section 7
Sketch and transform sin, cos and tan graphs, and build models.	Section 8
(HL) Use sec, csc, cot and the related Pythagorean identities.	Section 9
(HL) Work with arcsin, arccos, arctan and their domains and ranges.	Section 10
(HL) Apply the compound-angle identities.	Section 11
(HL) Handle vectors: dot and cross products, lines and planes.	Section 12

> **Exam note:** The formula booklet lists the sine rule, cosine rule, area formula, arc length, sector area and (HL) the compound-angle and vector results. The unit-circle exact values and the double-angle formulae are expected to be reproduced from memory.

1. Geometry in three dimensions

A point in space is fixed by three coordinates. The distance and midpoint results extend the two-dimensional formulae by simply adding a *z*-term.

For points A(x_1, y_1, z_1) and B(x_2, y_2, z_2):

$$\mathbf{AB} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$\textbf{Midpoint} = \left(\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2),\ \tfrac{1}{2}(z_1+z_2) \right)$$

Angles inside a solid

To find an angle inside a prism, pyramid or cone, identify a right-angled triangle that contains it, usually one lying in a vertical plane through the required line. Two lengths that repeatedly appear are the diagonal of a rectangular face and the space diagonal of a cuboid.

Volume and surface area

Solid	Volume	Total surface area
Cuboid (a, b, c)	abc	$2(ab+bc+ca)$
Prism (cross-section A, length L)	AL	$2A$ + (perimeter)$\times L$
Cylinder (r, h)	$\pi r^2 h$	$2\pi r^2 + 2\pi rh$
Cone (r, height h, slant l)	$\frac{1}{3}\pi r^2 h$	$\pi r^2 + \pi rl$
Pyramid (base area A, height h)	$\frac{1}{3}Ah$	base + triangular faces
Sphere (r)	$(4/3)\pi r^3$	$4\pi r^2$

For a cone the slant height is $l = \sqrt{(r^2 + h^2)}$. The curved (lateral) surface alone is πrl.

Worked example 1 — angle in a cuboid

A cuboid measures 6 cm × 4 cm × 3 cm (length, width, height). Find (a) the length of the space diagonal, (b) the angle it makes with the base.

(a) Base diagonal $d = \sqrt{(6^2+4^2)} = \sqrt{52} = 7.21$ cm. Space diagonal = $\sqrt{(52 + 3^2)} = \sqrt{61}$ = **7.81 cm**.

(b) The diagonal, the base diagonal and the vertical edge form a right-angled triangle, so tan θ = height / base diagonal = 3 / 7.21 = 0.4160.

Hence θ = arctan(0.4160) = **22.6°**.

Distance and midpoint in practice

For A(1, 2, −3) and B(4, −2, 3): AB = $\sqrt{[(4-1)^2 + (-2-2)^2 + (3+3)^2]} = \sqrt{(9+16+36)} = \sqrt{61} = 7.81$, and the midpoint is (2.5, 0, 0).

Worked example — composite solid

A toy is a cone of base radius 3 cm and height 4 cm sitting on a hemisphere of radius 3 cm. Find the total volume and the slant height of the cone.

Cone: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(9)(4) = 12\pi$ cm^3.

Hemisphere: $V = (2/3)\pi r^3 = (2/3)\pi(27) = 18\pi$ cm^3.

Total = 12π + 18π = 30π = **94.2 cm^3**. Slant height $l = \sqrt{(3^2 + 4^2)} = \sqrt{25}$ = **5 cm**.

2. Right-angled trigonometry

In a right-angled triangle the three primary ratios are remembered by **SOH CAH TOA**, where the sides are named relative to the angle θ:

sin θ = opposite / hypotenuse, cos θ = adjacent / hypotenuse, tan θ = opposite / adjacent

Angles of elevation and depression

- The **angle of elevation** of an object is measured upward from the horizontal to the line of sight.
- The **angle of depression** is measured downward from the horizontal. By alternate angles it equals the angle of elevation seen from the other end.

Bearings

A bearing is a direction measured **clockwise from north**, always written with three figures (for example due east is 090°, south-west is 225°). Bearing problems usually reduce to a triangle solved with the sine or cosine rule; sketch the north lines carefully to find the interior angle.

Worked example 2 — bearings

A ship sails 40 km on a bearing of 060°, then 30 km on a bearing of 150°. Find its distance and bearing from the start.

Resolve each leg into east (E) and north (N) components.

Leg 1: $E = 40 \sin 60° = 34.64$, $N = 40 \cos 60° = 20.00$.

Leg 2: $E = 30 \sin 150° = 15.00$, $N = 30 \cos 150° = -25.98$.

Totals: $E = 49.64$, $N = -5.98$. Distance $= \sqrt{(49.64^2 + 5.98^2)} =$ **50.0 km**.

Because N is negative and E positive the direction is in the south-east quadrant: bearing $= 180° - \arctan(49.64 / 5.98) = 180° - 83.1° =$ **097**°.

Worked example — angles of elevation

From a point on level ground the angle of elevation of the top of a building is 28°. From a point 40 m closer it is 45°. Find the height of the building.

Let the height be h and the nearer horizontal distance be d. Then $\tan 45° = h/d$, so $h = d$; and $\tan 28° = h/(d+40)$.

Substituting $h = d$: $d = (d+40)(0.5317)$, so $d(1 - 0.5317) = 21.27$, giving $d = 45.4$ m.

Therefore the building is **45.4 m** tall.

3. The sine and cosine rules

For any triangle with sides a, b, c opposite angles A, B, C:

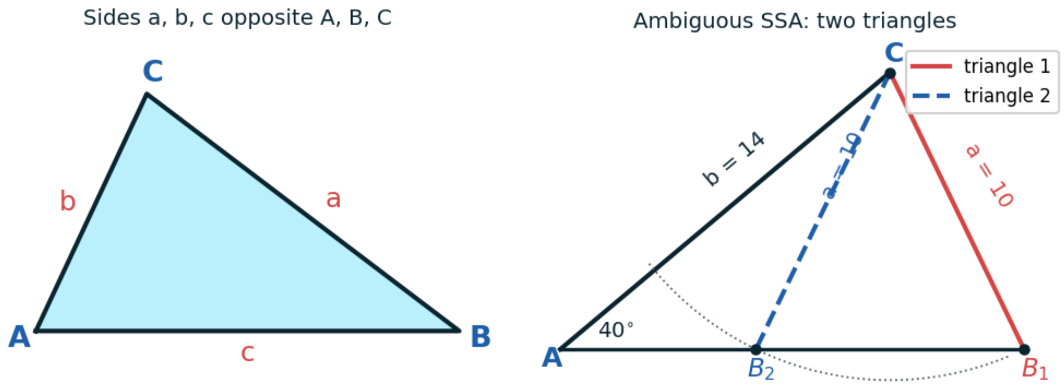

Figure 1. Left: a general (non-right) triangle with sides a, b, c opposite angles A, B, C. Right: the ambiguous SSA case with A = 40°, a = 10, b = 14. Since b sin A = 9.00 < a < b, the side a reaches the base in two places (B_1 and B_2), giving two valid triangles (B = 64.2° or 115.8°).

Sine rule: a / sin A = b / sin B = c / sin C

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Area: Area = ½ab sin C

Choose the sine rule when you have a matching side-angle pair; choose the cosine rule for three sides (SSS) or two sides and the included angle (SAS). The cosine rule rearranges to $\cos A = (b^2 + c^2 - a^2) / (2bc)$ when you need an angle from three sides.

The ambiguous case (SSA)

When you use the sine rule to find an angle, remember that sin is positive in the first and second quadrants, so a value such as sin B = 0.90 gives two possibilities: B = 64.2° or B = 180° − 64.2° = 115.8°. Keep the obtuse option only if the three angles still sum to less than 180°.

Worked example 3 — cosine rule then sine rule

In triangle ABC, b = 8, c = 5 and A = 60°. Find a and angle B.

$a^2 = 8^2 + 5^2 - 2(8)(5) \cos 60° = 64 + 25 - 80(0.5) = 49$, so a = **7**.

Sine rule: sin B = b sin A / a = 8 sin 60° / 7 = 6.928 / 7 = 0.9897.

This gives B = 81.8° or 98.2°. To decide, use the cosine rule: cos B = (49 + 25 − 64)/(2×7×5) = 10/70 > 0, so B is acute: B = **81.8**°.

Worked example 4 — sine rule, ambiguous case

In triangle ABC, A = 40°, a = 10 and b = 14. Find angle B.

sin B = b sin A / a = 14 sin 40° / 10 = 8.999 / 10 = 0.8999.

B = 64.2° or B = 180° − 64.2° = 115.8°.

Both survive the angle-sum test (40°+64.2° = 104.2° and 40°+115.8° = 155.8°, each less than 180°), so **two triangles exist**. Since $b > a$, this SSA data genuinely has two solutions.

Choosing a rule

You are given...	You want...	Use
Two angles and a side (AAS / ASA)	another side	sine rule
Two sides and a non-included angle (SSA)	an angle	sine rule (check the ambiguous case)
Two sides and the included angle (SAS)	the third side	cosine rule
Three sides (SSS)	an angle	cosine rule, rearranged for cos
Two sides and the included angle (SAS)	the area	Area = ½*ab* sin C

4. Radian measure, arcs and sectors

One radian is the angle subtended at the centre of a circle by an arc equal in length to the radius. A full turn is 2π radians, so:

180° = π rad → degrees → radians: × π/180; radians → degrees: × 180/π

For a sector of radius *r* with central angle θ **in radians**:

Arc length $s = r\theta$ Sector area $A = \frac{1}{2}r^2\theta$

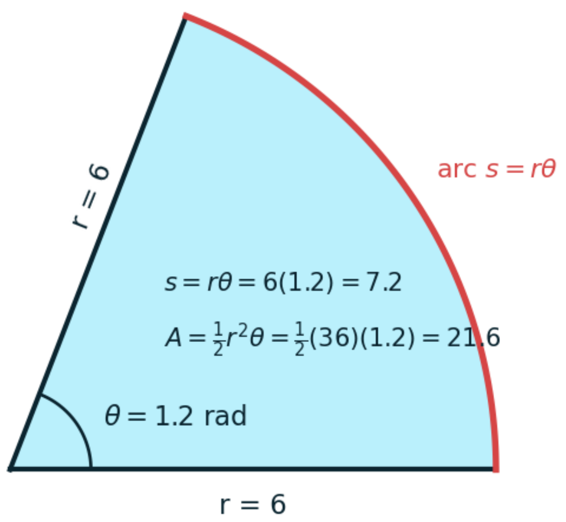

Figure 2. A sector of radius r = 6 with central angle θ = 1.2 rad. Arc length s = rθ = 6(1.2) = 7.2, and sector area A = ½r²θ = ½(36)(1.2) = 21.6. Both formulae require θ in radians.

> **Radians or degrees?** The formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ are valid only with θ in radians. Set your calculator to the matching mode; a mismatch here is one of the most common lost-mark errors in the whole topic.

Worked example 5 — arc and sector

A sector has radius 12 cm and central angle 1.2 radians. Find (a) the arc length, (b) the sector area, (c) the perimeter of the sector.

(a) $s = r\theta = 12 \times 1.2 =$ **14.4 cm**.

(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 144 \times 1.2 =$ **86.4 cm²**.

(c) Perimeter = arc + two radii = 14.4 + 2(12) = **38.4 cm**.

Area of a segment

A chord divides a circle into a minor and a major segment. The area of the segment between a chord and its arc is the sector area minus the triangle area:

$$\textbf{Segment area} = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta)$$

Worked example — segment area

A chord subtends an angle of 1.5 rad at the centre of a circle of radius 10 cm. Find the area of the minor segment.

Segment $= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}(100)(1.5 - \sin 1.5)$.

$\sin 1.5 = 0.9975$, so segment $= 50(1.5 - 0.9975) = 50(0.5025) =$ **25.1 cm²**.

5. The unit circle and exact values

On the unit circle (radius 1), a point at angle θ measured anticlockwise from the positive x-axis has coordinates $(\cos\theta, \sin\theta)$. Then $\tan\theta = \sin\theta / \cos\theta$ is the gradient of the radius. This definition extends sin, cos and tan to all angles, not just those in a right triangle.

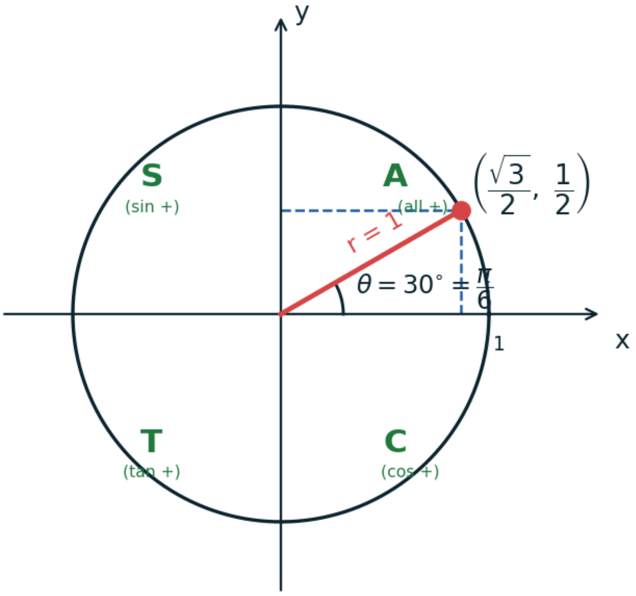

Figure 3. The unit circle. The point at θ = 30° = π/6 has coordinates (cos 30°, sin 30°) = (√3/2, ½), shown with radius 1 and its x- and y-projections. The letters A, S, T, C mark the CAST rule: All ratios are positive in quadrant I, then Sin, Tan and Cos only in II, III, IV.

θ (rad)	θ (deg)	sin θ	cos θ	tan θ
0	0°	0	1	0
π/6	30°	1/2	√3 / 2	1 / √3
π/4	45°	√2 / 2	√2 / 2	1
π/3	60°	√3 / 2	1/2	√3
π/2	90°	1	0	undefined

Signs by quadrant (CAST)

Reading anticlockwise, the letters C, A, S, T mark which ratio (and its reciprocal) is positive in each quadrant. Starting from quadrant I: **A**ll, **S**in, **T**an, **C**os.

Quadrant	Angle range	Positive ratios
I	0 to π/2	all (sin, cos, tan)
II	π/2 to π	sin only
III	π to 3π/2	tan only
IV	3π/2 to 2π	cos only

To evaluate, say, cos 150°: it lies in quadrant **II** where cos is negative, and its reference angle is 30°, so cos 150° = − cos 30° = −√3 / 2.

Related angles and symmetry

The symmetry of the unit circle gives relationships that save re-deriving values:

- sin(−θ) = − sin θ and cos(−θ) = cos θ (sin is odd, cos is even).

- $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$.
- $\sin(\theta + 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \cos\theta$; both have period 2π, while tan has period π.

6. Trigonometric identities

Identities let you rewrite an expression so that an equation becomes solvable, usually by reducing it to a single ratio.

$$\textbf{Pythagorean: } \sin^2\theta + \cos^2\theta = 1$$

$$\textbf{Quotient: } \tan\theta = \sin\theta / \cos\theta$$

Double-angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

The three forms of $\cos 2\theta$ are equivalent through $\sin^2\theta + \cos^2\theta = 1$. Choose whichever form leaves the equation in terms of a single ratio.

Worked example 6 — double angle

Given $\sin\theta = 3/5$ with θ acute, find $\sin 2\theta$ and $\cos 2\theta$.

Since θ is acute, $\cos\theta = \sqrt{(1 - 9/25)} = 4/5$.

$\sin 2\theta = 2\sin\theta\cos\theta = 2(3/5)(4/5) = \mathbf{24/25}$.

$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2(9/25) = \mathbf{7/25}$. (Check: $\cos^2\theta - \sin^2\theta = 16/25 - 9/25 = 7/25$.)

7. Solving trigonometric equations

A trigonometric equation usually has infinitely many solutions; the question restricts you to a stated interval. A dependable procedure:

- Use identities to reduce to a single ratio, then factorise or take a square root as needed.
- Find the principal value from the inverse function.
- Use symmetry (the unit circle or CAST) to generate every solution inside the interval – remember there are usually two per revolution.
- If the argument is a multiple such as $2x$, first widen the interval to match $2x$, solve, then divide.

Worked example 7 — quadratic in cos

Solve $2\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$.

Factorise as a quadratic in $\cos x$: $(2\cos x + 1)(\cos x - 1) = 0$.

So $\cos x = -1/2$ or $\cos x = 1$.

$\cos x = 1$ gives $x = 0$ and $x = 2\pi$. $\cos x = -1/2$ (quadrants II and III) gives $x = 2\pi/3$ and $x = 4\pi/3$.

Solutions: $\boldsymbol{x = 0, 2\pi/3, 4\pi/3, 2\pi}$.

Watch the interval: For an equation in $2x$ over $0 \le x \le 2\pi$, the angle $2x$ runs over 0 to 4π, so expect roughly twice as many solutions. Solve for $2x$ first, then halve.

Worked example — using a double angle

Solve $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

Write $\sin 2x = 2\sin x\cos x$, so $2\sin x\cos x - \sin x = 0$, i.e. $\sin x(2\cos x - 1) = 0$.

$\sin x = 0$ gives $x = 0, \pi, 2\pi$. And $\cos x = ½$ gives $x = \pi/3$ and $x = 5\pi/3$.

Solutions: $\boldsymbol{x = 0, \pi/3, \pi, 5\pi/3, 2\pi}$. Never divide both sides by $\sin x$ — that would discard the $\sin x = 0$ roots.

8. Graphs and transformations

The graphs of sin and cos are smooth waves of amplitude 1 and period 2π, differing only by a horizontal shift (cos leads sin by $\pi/2$). The tan graph has period π and vertical asymptotes where $\cos\theta = 0$, that is at $\theta = \pi/2 + n\pi$.

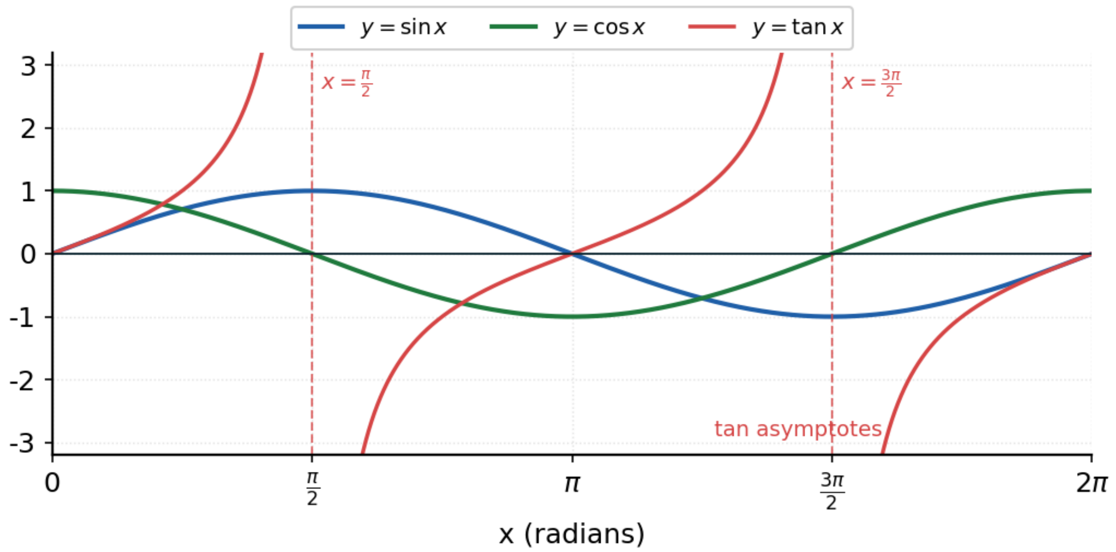

Figure 4. The three basic graphs on $0 \le x \le 2\pi$ (radians). sin and cos have amplitude 1 and period 2π; tan has period π with vertical asymptotes (dashed) at $x = \pi/2$ and $x = 3\pi/2$, where $\cos x = 0$.

For the general sinusoid

$$y = a\sin(b(x - c)) + d$$

Parameter	Effect	Value / formula
a	Amplitude — vertical stretch	amplitude = $\lvert a \rvert$
b	Horizontal stretch — sets the period	period = $2\pi / \lvert b \rvert$
c	Horizontal (phase) shift	shift c to the right if $c > 0$
d	Vertical shift — the midline	midline $y = d$

The maximum is $d + |a|$ and the minimum is $d - |a|$, so the range is $[d - |a|, d + |a|]$. For a real-world model, $|a|$ is half the peak-to-trough gap, d is the average level, and the period is the time for one full cycle.

Worked example 8 — reading a sinusoid

Describe $y = 3\sin(2(x - \pi/4)) + 1$ and state its range.

Amplitude = 3; period = $2\pi / 2 = \pi$; phase shift = $\pi/4$ to the right; vertical shift = +1 (midline $y = 1$).

Maximum = 1 + 3 = 4, minimum = 1 − 3 = −2, so the range is $\mathbf{-2 \le y \le 4}$.

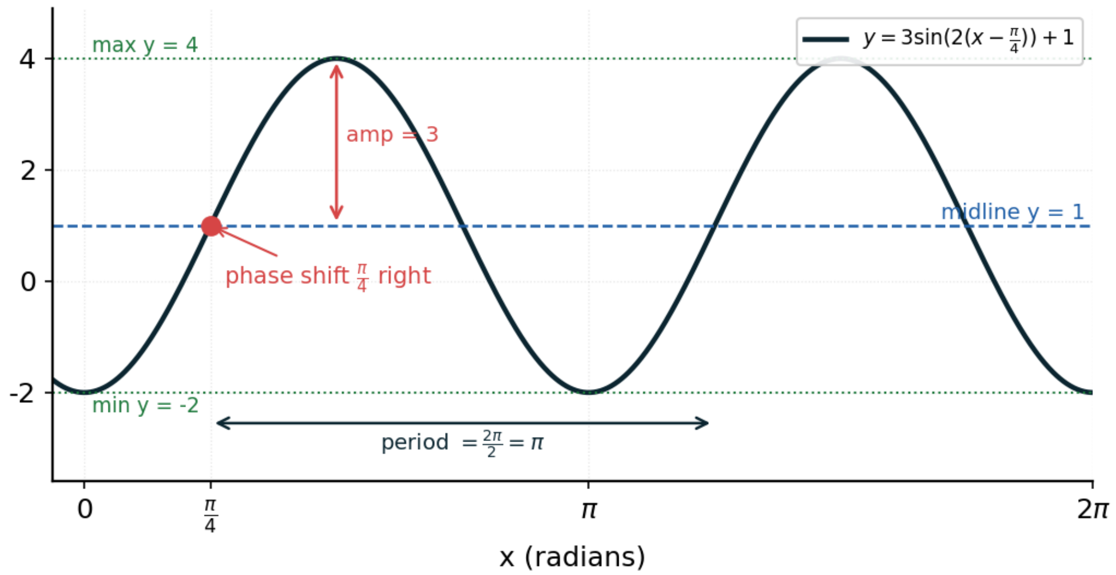

Figure 5. The transformed sinusoid $y = 3\sin(2(x - \pi/4)) + 1$. Amplitude 3 (from midline $y = 1$ up to $y = 4$); period $2\pi/2 = \pi$; phase shift $\pi/4$ to the right; midline $y = 1$. Maximum 4, minimum −2.

Sketching in five steps

1. Draw the midline $y = d$. 2. Mark the maximum $d + |a|$ and the minimum $d - |a|$. 3. Compute the period $2\pi/|b|$ and split it into quarters. 4. Apply the horizontal shift c. 5. Plot one full cycle, then repeat it.

Worked example — modelling with a sinusoid

A Ferris wheel of diameter 40 m has its lowest point 2 m above the ground and turns once every 60 s. A rider boards at the lowest point. Model the height h (m) after t seconds.

Radius 20 m gives amplitude $a = 20$; the centre is $2 + 20 = 22$ m up, so $d = 22$. A period of 60 s gives $b = 2\pi/60 = \pi/30$.

Boarding at the minimum suggests a negative cosine: $h = 22 - 20\cos((\pi/30)t)$.

Check: at $t = 0$, $h = 22 - 20 = 2$ m ✓; at $t = 30$ s, $h = 22 + 20 = 42$ m, the top. ✓

9. Reciprocal ratios (HL)

Three reciprocal ratios complete the set:

$$\sec\theta = 1 / \cos\theta,\ \csc\theta = 1 / \sin\theta,\ \cot\theta = 1 / \tan\theta = \cos\theta / \sin\theta$$

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ and then by $\sin^2\theta$ produces two further Pythagorean identities:

$$1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$$

Naming trap: The reciprocal of cos is **sec** (not csc), and the reciprocal of sin is **csc**. A quick aid: the pair cos-sec is the exception — memorise that sec pairs with cos and csc pairs with sin.

Worked example 9 — using an identity

Given $\sec\theta = 13/5$ with θ acute, find the other five ratios.

$\cos\theta = 5/13$. Then $\sin\theta = \sqrt{(1 - 25/169)} = \sqrt{(144/169)} = 12/13$ (positive, θ acute).

$\tan\theta = (12/13)/(5/13) =$ **12/5**; $\cot\theta =$ **5/12**; $\csc\theta =$ **13/12**.

Check: $1 + \tan^2\theta = 1 + 144/25 = 169/25 = \sec^2\theta$. ✓

10. Inverse trigonometric functions (HL)

Because sin, cos and tan are many-to-one, each is restricted to a domain on which it is one-to-one before an inverse is defined. The inverse then returns the single **principal value**.

Function	Domain	Range (principal values)
arcsin x	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
arccos x	$-1 \le x \le 1$	$0 \le y \le \pi$
arctan x	$x \in \mathbb{R}$ (all reals)	$-\pi/2 < y < \pi/2$

When solving an equation, the calculator's inverse gives only the principal value; you must then use symmetry to recover every solution in the required interval. For arccos the range is the upper half-plane, so arccos of a negative number is an obtuse angle (for example $\arccos(-1/2) = 2\pi/3$).

Worked example — using principal values (HL)

Find every x with $0 \le x \le 2\pi$ for which $\sin x = -0.5$.

$\arcsin(-0.5) = -\pi/6$ is the principal value, but it lies outside $[0, 2\pi]$. sin is negative in quadrants III and IV.

Using the reference angle $\pi/6$: $x = \pi + \pi/6 = 7\pi/6$ (QIII) and $x = 2\pi - \pi/6 = 11\pi/6$ (QIV).

Solutions: $\boldsymbol{x = 7\pi/6, 11\pi/6}$.

11. Compound-angle identities (HL)

These expand a function of a sum or difference of two angles:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = (\tan A \pm \tan B) / (1 \mp \tan A \tan B)$$

Watch the sign switch: in cos(A + B) the middle sign is minus, and the tan formula's denominator sign is opposite to the numerator's. Setting A = B recovers the double-angle formulae. These identities also give exact values for angles such as 15° and 75° by splitting them into 45° ± 30°.

Worked example 10 — exact value

Find the exact value of cos 15°.

$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$.

$= (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = \sqrt{6}/4 + \sqrt{2}/4 = \mathbf{(\sqrt{6} + \sqrt{2})/4}$.

Numerically this is 0.966, which agrees with cos 15° on a calculator.

12. Vectors (HL)

A vector has magnitude and direction. In three dimensions write it in component form $\mathbf{v} = (v_1, v_2, v_3)$ or as $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, where **i**, **j**, **k** are the unit vectors along the axes.

Basic operations

- **Magnitude:** $|\mathbf{v}| = \sqrt{(v_1^2 + v_2^2 + v_3^2)}$.
- **Unit vector:** $\mathbf{v} / |\mathbf{v}|$ has magnitude 1 in the direction of **v**.
- **Addition and scalar multiple:** add componentwise; $k\mathbf{v}$ scales length by $|k|$ and reverses direction if $k < 0$.

Scalar (dot) product

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$$

The dot product is a **scalar**. It gives the angle between two vectors through cos θ = (**a** · **b**) / (|**a**| |**b**|); the vectors are perpendicular exactly when **a** · **b** = 0.

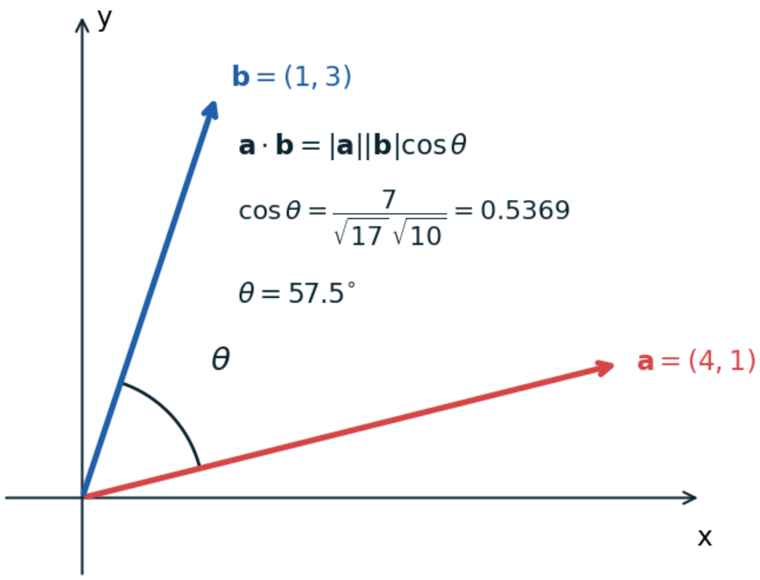

*Figure 6. The angle between two vectors from the dot product. For **a** = (4, 1) and **b** = (1, 3): **a**·**b** = 4(1) + 1(3) = 7, |**a**| = √17, |**b**| = √10, so cos θ = 7 / (√17 √10) = 0.5369 and θ = 57.5°, verifying cos θ = (**a**·**b**) / (|**a**||**b**|).*

Vector (cross) product

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

The cross product is a **vector** perpendicular to both **a** and **b**, with magnitude |**a** × **b**| = |**a**| |**b**| sin θ. Hence the parallelogram on **a** and **b** has area |**a** × **b**|, and the triangle has area ½|**a** × **b**|.

> **Dot vs cross:** The dot product outputs a **number** and measures how parallel two vectors are (uses cos θ); the cross product outputs a **vector** and measures area (uses sin θ). Never write **a** · **b** when you mean **a** × **b**.

Worked example 11 — dot product and angle

Find the angle between **a** = (2, −1, 3) and **b** = (1, 4, −2).

a · **b** = 2(1) + (−1)(4) + 3(−2) = 2 − 4 − 6 = −8.

|**a**| = √(4+1+9) = √14, |**b**| = √(1+16+4) = √21.

cos θ = −8 / (√14 √21) = −8 / √294 = −0.4666.

Hence θ = arccos(−0.4666) = **117.8°** (2.06 rad).

Lines and planes

A line through point **a** with direction **b** has vector equation **r** = **a** + λ**b**, where λ ∈ ℝ is a parameter. A plane can be written as **r** · **n** = **a** · **n**, i.e. $n_1x + n_2y + n_3z = d$, where **n** is a normal vector.

- **Angle between two lines** — use the dot product of their direction vectors.
- **Angle between two planes** — use the dot product of their normals.
- **Angle between a line and a plane** — if φ is the angle between the line's direction and the normal, the line-plane angle is $90° - \varphi$; use $\sin(\text{angle}) = |\mathbf{b} \cdot \mathbf{n}| / (|\mathbf{b}||\mathbf{n}|)$.

Two lines in space either meet, are parallel, or are **skew** (neither parallel nor intersecting). To test for intersection, set the parametric coordinates equal, solve two of the three equations, then check the third: if it holds the lines meet, otherwise they are skew.

Worked example 12 — intersection of two lines

Do the lines $\mathbf{r} = (1, 0, -1) + \lambda(1, 1, 1)$ and $\mathbf{r} = (0, -4, 4) + \mu(1, 2, -1)$ intersect? If so, where?

Equate components: $1 + \lambda = \mu$; $\lambda = -4 + 2\mu$; $-1 + \lambda = 4 - \mu$.

From the first, $\mu = 1 + \lambda$. Substitute into the second: $\lambda = -4 + 2(1 + \lambda) = -2 + 2\lambda$, so $\lambda = 2$ and $\mu = 3$.

Check the third equation: $-1 + 2 = 1$ and $4 - 3 = 1$. ✓ Consistent, so the lines meet.

Point of intersection (put $\lambda = 2$ into line 1): $(1+2, 0+2, -1+2) =$ **(3, 2, 1)**.

Worked example — angle between two planes (HL)

Find the acute angle between the planes $2x - y + 2z = 5$ and $x + 2y + 2z = 1$.

The angle between two planes equals the angle between their normals: $\mathbf{n}_1 = (2, -1, 2)$ and $\mathbf{n}_2 = (1, 2, 2)$.

$\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 + 4 = 4$; $|\mathbf{n}_1| = \sqrt{9} = 3$, $|\mathbf{n}_2| = \sqrt{9} = 3$.

$\cos\theta = 4 / (3 \times 3) = 4/9 = 0.444$, so $\theta =$ **63.6°**.

13. Common pitfalls

- **Radians vs degrees.** Arc-length and sector-area formulae need radians; always match the calculator mode to the question.
- **The ambiguous case.** When the sine rule gives an angle, test the obtuse alternative $180° - \theta$ against the angle sum before discarding it.
- **Period of a sinusoid.** The period of $\sin(bx)$ is $2\pi / |b|$, not $2\pi b$.
- **Dot vs cross (HL).** The dot product is a scalar (angle, perpendicularity); the cross product is a vector (area, normal direction).
- **Losing solutions.** An inverse function returns only the principal value; use symmetry to find every root in the interval.
- **Multiple-angle intervals.** For an equation in $2x$, extend the interval for $2x$, solve, then divide by 2.
- **Rounding early.** Keep full accuracy through multi-step problems and round only the final answer to 3 significant figures.

14. Quick reference

Result	Statement
Sine rule	$a/\sin A = b/\sin B = c/\sin C$
Cosine rule	$a^2 = b^2 + c^2 - 2bc \cos A$
Area of a triangle	$\frac{1}{2}ab \sin C$
Arc / sector (θ in rad)	$s = r\theta$; $A = \frac{1}{2}r^2\theta$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$; (HL) $1+\tan^2\theta = \sec^2\theta$; $1+\cot^2\theta = \csc^2\theta$
Double angle	$\sin 2\theta = 2 \sin \theta \cos \theta$; $\cos 2\theta = 1 - 2 \sin^2\theta$
General sinusoid	$y = a \sin(b(x-c))+d$; amp $\lvert a\rvert$, period $2\pi/\lvert b\rvert$
(HL) Compound	$\sin(A\pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos(A\pm B) = \cos A \cos B \mp \sin A \sin B$
(HL) Dot product	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos \theta$ (scalar)
(HL) Cross product	$\lvert\mathbf{a} \times \mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin \theta$ (area)
(HL) Line	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

15. Test yourself

Attempt all ten without notes, then check against the full solutions below.

1. A triangle has two sides 6 and 10 with an included angle of 120°. Find the third side and the area.

2. In triangle ABC, A = 30°, $a = 5$ and $b = 8$. Find the possible value(s) of angle B.

3. A sector of a circle has radius 9 cm and arc length 15 cm. Find the central angle (in radians) and the sector area.

4. Convert 135° to radians and $5\pi/6$ rad to degrees.

5. Find the exact value of $\sin(2\pi/3) + \cos(\pi/6)$.

6. Solve $2 \cos^2 x + \sin x - 1 = 0$ for $0 \le x \le 2\pi$.

7. For $y = 3 \sin(2(x - \pi/4)) + 1$, state the amplitude, period, phase shift, vertical shift and range.

8. Given $\cos \theta = -1/3$ with $\pi/2 < \theta < \pi$, find $\sin 2\theta$ and $\cos 2\theta$.

9. **(HL)** Given $\sec \theta = 13/5$ with θ acute, find the remaining five ratios.

10. **(HL)** For $\mathbf{a} = (3, -1, 2)$ and $\mathbf{b} = (-1, 4, 1)$, find $\mathbf{a} \cdot \mathbf{b}$, the angle between them, and the area of the triangle with these as two sides.

Full solutions

1. $a^2 = 6^2 + 10^2 - 2(6)(10) \cos 120° = 136 - 120(-0.5) = 196$, so $a = 14$. Area $= \frac{1}{2}(6)(10) \sin 120° = 30(0.8660) = 26.0$.

2. $\sin B = 8 \sin 30° / 5 = 8(0.5)/5 = 0.8$, so B = 53.1° or 126.9°. Both satisfy the angle sum (30°+126.9° = 156.9° < 180°), so **two triangles** are possible.

3. $\theta = s/r = 15/9 = 1.67$ rad. Area = $\frac{1}{2}rs = \frac{1}{2}(9)(15) = 67.5$ cm^2 (equivalently $\frac{1}{2}r^2\theta$).

4. $135° = 135 \times \pi/180 = 3\pi/4$ rad. $5\pi/6 \times 180/\pi = 150°$.

5. $\sin(2\pi/3) = \sqrt{3}/2$ and $\cos(\pi/6) = \sqrt{3}/2$, so the sum is $\sqrt{3} \approx 1.73$.

6. Replace $\cos^2 x = 1 - \sin^2 x$: $2 - 2\sin^2 x + \sin x - 1 = 0 \rightarrow 2\sin^2 x - \sin x - 1 = 0 \rightarrow (2\sin x + 1)(\sin x - 1) = 0$. So $\sin x = 1$ ($x = \pi/2$) or $\sin x = -1/2$ ($x = 7\pi/6, 11\pi/6$). Solutions: $x = \pi/2, 7\pi/6, 11\pi/6$.

7. Amplitude 3; period $2\pi/2 = \pi$; phase shift $\pi/4$ right; vertical shift +1; range $-2 \le y \le 4$.

8. In quadrant II $\sin\theta > 0$: $\sin\theta = \sqrt{(1 - 1/9)} = 2\sqrt{2}/3$. $\sin 2\theta = 2(2\sqrt{2}/3)(-1/3) = -4\sqrt{2}/9$. $\cos 2\theta = 2\cos^2\theta - 1 = 2/9 - 1 = -7/9$.

9. (HL) $\cos\theta = 5/13$; $\sin\theta = 12/13$; $\tan\theta = 12/5$; $\cot\theta = 5/12$; $\csc\theta = 13/12$. (Check $1+\tan^2\theta = 169/25 = \sec^2\theta$.)

10. (HL) $\mathbf{a} \cdot \mathbf{b} = -3 - 4 + 2 = -5$. $|\mathbf{a}| = \sqrt{14}$, $|\mathbf{b}| = \sqrt{18} = 3\sqrt{2}$. $\cos\theta = -5/(\sqrt{14} \cdot 3\sqrt{2}) = -0.315$, so $\theta = 108.4°$. Cross product = $(-9, -5, 11)$, $|\mathbf{a} \times \mathbf{b}| = \sqrt{227} = 15.07$, so the triangle area = $\frac{1}{2}(15.07) = 7.53$.