IB DP MATHEMATICS

Analysis and Approaches (AA)

Topic 2 Functions

Revision Notes · Standard and Higher Level

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What Topic 2 requires

Functions is the backbone of the AA course: nearly every later topic (calculus, trigonometry, sequences) is expressed through functions. Use this checklist to confirm you can do each item before the exam. Rows marked **(HL)** are additional Higher Level content.

Skill	You should be able to...
Function & notation	State domain and range, evaluate f(x), and decide whether a relation is a function (vertical line test).
Composite & inverse	Form $f \circ g$, find f^{-1}, and relate the graph of f^{-1} to f by reflection in y = x.
Key graph features	Read intercepts, symmetry, asymptotes and turning points from a graph or a GDC.
Transformations	Apply translations, stretches and reflections, and combine them in the correct order.
Quadratics	Move between the three forms, use the discriminant $\Delta = b^2 - 4ac$, and solve with the formula.
Rational (ax+b)/(cx+d)	Find vertical and horizontal asymptotes and sketch the reciprocal-type curve.
Exponential & log	Graph a^x, e^x and log functions, identify asymptotes and solve equations.
Polynomials (HL)	Use the factor and remainder theorems, root sum/product, and multiplicity of roots.
Advanced rational (HL)	Handle quadratic over linear (and similar) and find oblique asymptotes.
Symmetry & modulus (HL)	Test odd/even and self-inverse, and work with |x| in equations and inequalities.
Transform families (HL)	Sketch y = 1/f(x), y = f(|x|), y = |f(x)| and y = f(ax+b).
Inequalities (HL)	Solve analytically and graphically, respecting sign changes at zeros and undefined points.

Exam note: SL and HL share the core of Topic 2. At HL the same ideas are examined in longer multi-step questions and combined with calculus, so master the SL foundations first.

1. The function concept, domain and range

A **function** is a rule that assigns to **each** input exactly **one** output. If f sends x to y we write y = f(x): x is the input (independent variable) and f(x) is the output (dependent variable). The letter f names the rule; f(x) is the value produced when the rule acts on x.

Domain, range and codomain

- The **domain** is the set of all permitted inputs x.
- The **range** is the set of all outputs f(x) actually produced.
- Unless told otherwise, take the **largest possible** real domain (the “natural domain”).

Two rules govern the natural domain: you may never divide by zero, and (in AA) you may never take the square root of a negative number. So $1/(x - 2)$ excludes $x = 2$, and $\sqrt{(x - 3)}$ requires $x \geq 3$.

Function	Natural domain	Range
$f(x) = x^2$	$x \in \mathbb{R}$	$f(x) \geq 0$
$f(x) = \sqrt{x}$	$x \geq 0$	$f(x) \geq 0$
$f(x) = 1/x$	$x \neq 0$	$f(x) \neq 0$
$f(x) = 2^x$	$x \in \mathbb{R}$	$f(x) > 0$
$f(x) = \ln x$	$x > 0$	$f(x) \in \mathbb{R}$

Mappings and the vertical line test

A relation is a function only if every input has a single output. Graphically this is the **vertical line test**: any vertical line meets the graph at most once. A circle $x^2 + y^2 = 9$ fails the test, so it is not a function of x. Mappings are also classified by how inputs pair with outputs:

- **one-to-one**: different inputs give different outputs (e.g. $f(x) = 2x + 1$).
- **many-to-one**: different inputs can share an output (e.g. $f(x) = x^2$, since 3 and -3 both map to 9).

> **Why it matters:** Only **one-to-one** functions have an inverse over their whole domain. A many-to-one function must first have its domain restricted before it can be inverted (see Section 2).

2. Composite and inverse functions

Composite functions

A **composite** applies one function to the output of another. $(f \circ g)(x) = f(g(x))$ means "do g first, then f". Work from the inside out. Order matters: in general $f \circ g \neq g \circ f$.

The composite g(x) must land inside the domain of f. So the domain of $f \circ g$ is the set of x in the domain of g for which g(x) lies in the domain of f.

Worked example 1 - composite and inverse

Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$, (c) $f^{-1}(x)$.

(a) $(f \circ g)(x) = f(x^2) = 2x^2 + 3$.

(b) $(g \circ f)(x) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$.

Parts (a) and (b) differ, confirming the order of composition matters.

(c) Write $y = 2x + 3$, swap the roles and solve for x: $x = (y - 3)/2$, so $\mathbf{f^{-1}(x) = (x - 3)/2}$. Check: $f(f^{-1}(x)) = 2 \times (x - 3)/2 + 3 = x$.

Inverse functions

The **inverse** f^{-1} reverses f: if $f(a) = b$ then $f^{-1}(b) = a$. It exists only when f is one-to-one. Key relationships:

- $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ on the appropriate domains.

- domain of f^{-1} = range of f, and range of f^{-1} = domain of f.
- the graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$; every point (a, b) becomes (b, a).

Method to find an inverse: (1) write $y = f(x)$; (2) interchange x and y; (3) solve for y; (4) rename the result $f^{-1}(x)$; (5) state its domain (= the range of f). The $^{-1}$ notation means the inverse, **not** the reciprocal 1/f.

> **Common pitfall:** For $f(x) = x^2$ there is no inverse over all of $\mathbb{R}$ because it is many-to-one. Restrict the domain, e.g. to $x \geq 0$, giving $f^{-1}(x) = \sqrt{x}$ with domain $x \geq 0$.

3. Graphing functions and their key features

When you sketch or analyse a graph, examiners expect you to label the same short list of features every time.

Feature	How to find it	What it looks like
y-intercept	Evaluate f(0)	where the curve crosses the y-axis
x-intercept(s) / zeros	Solve f(x) = 0	where the curve crosses the x-axis
Vertical asymptote	Input value where f is undefined (denominator = 0)	curve shoots to $\pm\infty$ near a vertical line
Horizontal asymptote	Behaviour as $x \to \pm\infty$	curve levels off toward a horizontal line
Maximum / minimum	Turning points (GDC or completing the square)	local peak or valley of the curve
Symmetry	Even: $f(-x) = f(x)$; odd: $f(-x) = -f(x)$	mirror in y-axis / rotational about O

An **asymptote** is a line the curve approaches arbitrarily closely but never reaches. A graph may cross a horizontal asymptote in its middle region; what matters is the behaviour far out, as $x \to \pm\infty$.

> **GDC habit:** Set a sensible window, then use the built-in tools for zeros, intersections and maxima/minima rather than reading pixels by eye. Always quote GDC answers to 3 significant figures unless told otherwise.

4. Transformations of graphs

Starting from $y = f(x)$, each algebraic change produces a predictable geometric change. Split them into changes **outside** f (they affect y, and behave the “obvious” way) and changes **inside** f (they affect x, and behave the **opposite** way to what you might expect).

Transformation	Effect on the graph	Effect on a point (x, y)
$y = f(x) + b$	translation b units up (down if $b < 0$)	$(x, y + b)$
$y = f(x - a)$	translation a units right (left if $a < 0$)	$(x + a, y)$
$y = a\,f(x)$	vertical stretch, scale factor a	$(x, a\,y)$
$y = f(x/a)$ or $f(bx)$	horizontal stretch, factor $a = 1/b$	$(a\,x, y)$
$y = -f(x)$	reflection in the x-axis	$(x, -y)$

Transformation	Effect on the graph	Effect on a point (x, y)
y = f(−x)	reflection in the y-axis	(−x, y)

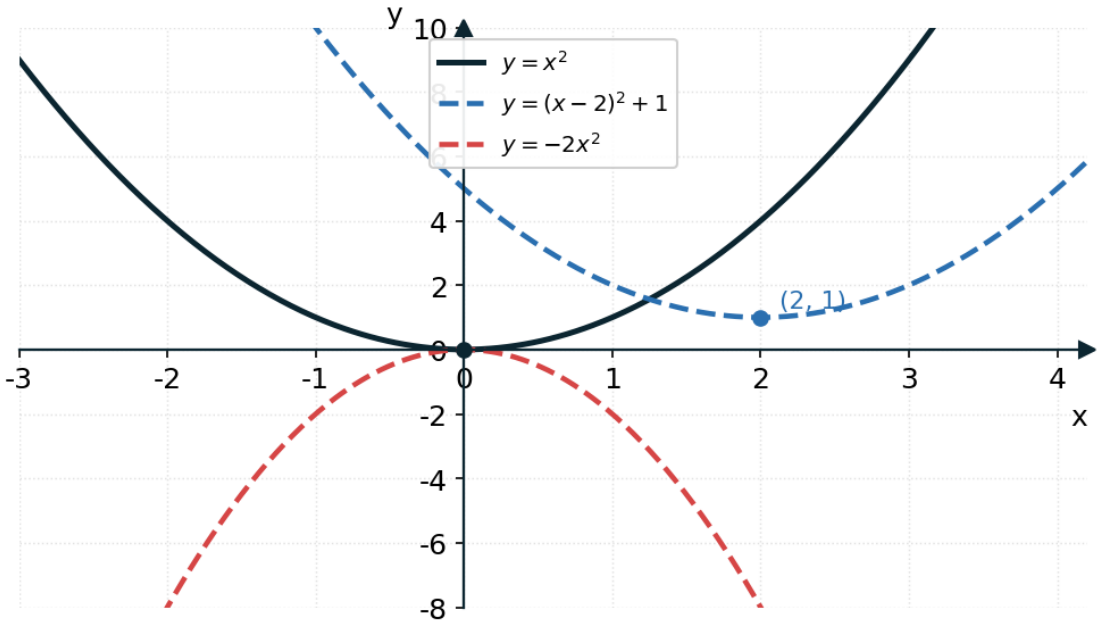

Figure 1. Transformations of $y = x^2$. The dashed curves are $y = (x - 2)^2 + 1$ (translation 2 right and 1 up, vertex (2, 1)) and $y = -2x^2$ (reflection in the x-axis with vertical stretch factor 2).

Combining transformations - order matters

For a chain such as y = a f(b(x − h)) + k, apply **horizontal** operations in the reverse of the usual algebra order, then **vertical** operations. A safe recipe: horizontal translation and stretch first (inside the bracket, working on x), then vertical stretch/reflection, then vertical translation last. Vertical and horizontal families are independent, so a vertical stretch followed by a horizontal shift can be done in either order relative to each other.

Worked example 2 - tracking a point through transformations

The graph of y = f(x) passes through the point (2, 5). Find the image of this point on (a) y = 3f(x − 1) − 4 and (b) y = f(2x).

(a) The bracket x − 1 must equal 2, so x = 3. Then y = 3(5) − 4 = 11. Image: **(3, 11)**.

(b) Here 2x must equal 2, so x = 1, and the output is unchanged: y = 5. Image: **(1, 5)** - a horizontal stretch of factor 1/2.

Notice the inside change (x − 1, 2x) acts on the x-coordinate in the opposite sense to intuition, while 3(...) − 4 acts directly on y.

Common pitfall: y = f(2x) is a horizontal stretch of factor 1/2 (a squeeze), not a stretch by 2. Inside changes always invert.

5. Quadratic functions

A quadratic $f(x) = ax^2 + bx + c$ ($a \neq 0$) has a parabola graph, opening up if $a > 0$ and down if $a < 0$. The same quadratic can be written three ways, each revealing something different.

Form	Written as	Reveals
Standard	$ax^2 + bx + c$	y-intercept c
Completed square / vertex	$a(x - h)^2 + k$	vertex (h, k); axis x = h
Factored	$a(x - p)(x - q)$	x-intercepts p and q

The **axis of symmetry** is $x = -b/(2a)$; the vertex lies on it. The vertex is the minimum point when $a > 0$ and the maximum point when $a < 0$. Completing the square converts standard form to vertex form and is the exam's preferred way to find the turning point by hand.

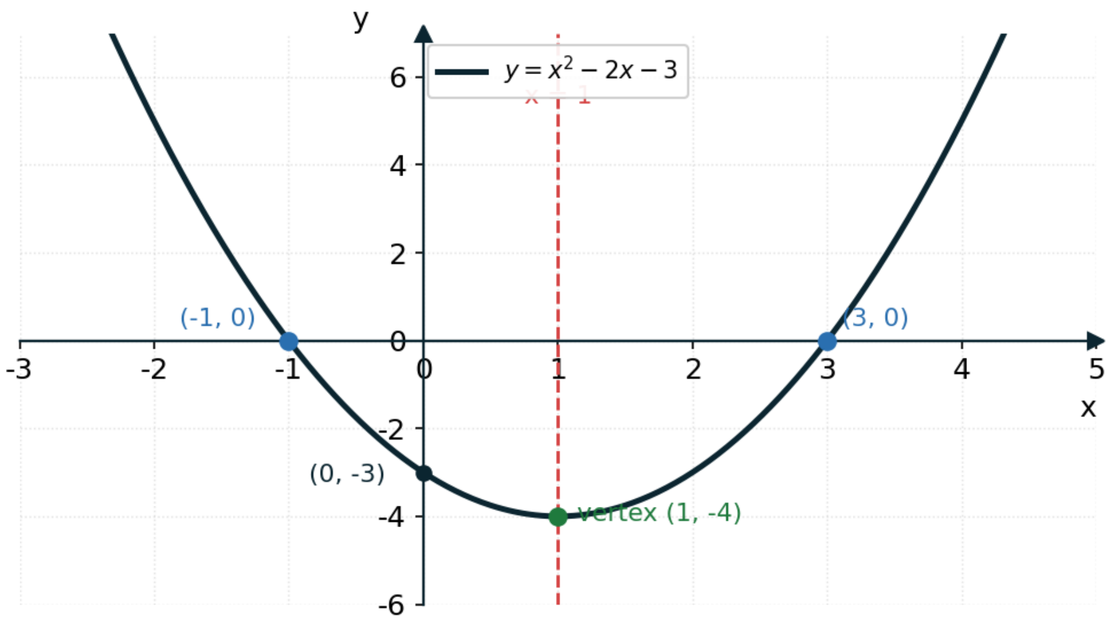

Figure 2. Key features of $y = x^2 - 2x - 3$. Roots $(-1, 0)$ and $(3, 0)$, vertex $(1, -4)$, axis of symmetry $x = 1$ (dashed) and y-intercept $(0, -3)$.

Worked example 3 - completing the square

Express $f(x) = 2x^2 - 12x + 7$ in vertex form and state the coordinates and nature of the turning point.

Take the factor 2 from the x-terms: $f(x) = 2(x^2 - 6x) + 7$.

Complete the square inside: $x^2 - 6x = (x - 3)^2 - 9$.

So $f(x) = 2[(x - 3)^2 - 9] + 7 = 2(x - 3)^2 - 18 + 7 = \mathbf{2(x - 3)^2 - 11}$.

Vertex $(3, -11)$; since $a = 2 > 0$ it is a **minimum**; axis $x = 3$; minimum value $f(3) = -11$.

The discriminant and nature of the roots

For $ax^2 + bx + c = 0$ the roots come from the quadratic formula

$$x = \left(-b \pm \sqrt{b^2 - 4ac} \right) / (2a)$$

The quantity under the root, $\Delta = b^2 - 4ac$, is the **discriminant**. It decides how many real roots there are without solving.

Discriminant	Real roots	Graph meets x-axis
$\Delta > 0$	two distinct real roots	at two points
$\Delta = 0$	one repeated (double) root	touches at the vertex
$\Delta < 0$	no real roots	does not cross the axis

Worked example 4 - using the discriminant

Find the values of k for which $2x^2 + kx + 8 = 0$ has (a) equal roots, (b) two distinct real roots.

Here $a = 2$, $b = k$, $c = 8$, so $\Delta = k^2 - 4(2)(8) = k^2 - 64$.

(a) Equal roots need $\Delta = 0$: $k^2 = 64$, so $\mathbf{k = \pm 8}$.

(b) Two distinct real roots need $\Delta > 0$: $k^2 > 64$, so **$k > 8$ or $k < -8$**.

Common pitfall: $\Delta = 0$ gives **one** (repeated) root, not zero roots. “No real roots” is $\Delta < 0$. Read the wording carefully: “equal roots” and “tangent to the x-axis” both mean $\Delta = 0$.

6. Reciprocal and simple rational functions

The reciprocal function $f(x) = 1/x$ has a rectangular hyperbola graph with two branches. Its domain is $x \neq 0$ and its range is $f(x) \neq 0$; the axes are its asymptotes: the vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$. It is an odd function, symmetric under a half-turn about the origin.

The general linear-over-linear function

$f(x) = (ax + b)/(cx + d)$ has the same hyperbola shape, shifted. Its asymptotes come from two simple rules:

- **Vertical asymptote**: where the denominator is zero, $x = -d/c$.
- **Horizontal asymptote**: the ratio of the leading coefficients, $y = a/c$ (the limiting value as $x \to \pm\infty$).
- **x-intercept**: where the numerator is zero, $x = -b/a$; **y-intercept**: $f(0) = b/d$.

Worked example 5 - asymptotes of a rational function

Sketch the key features of $f(x) = (3x - 2)/(x + 1)$.

Vertical asymptote: $x + 1 = 0$, so $\mathbf{x = -1}$.

Horizontal asymptote: leading coefficients 3 and 1, so $\mathbf{y = 3}$.

x-intercept: $3x - 2 = 0$, so $x = 2/3$. y-intercept: $f(0) = -2/1 = -2$.

The curve has two branches sitting either side of $x = -1$ and approaching $y = 3$ far from the origin.

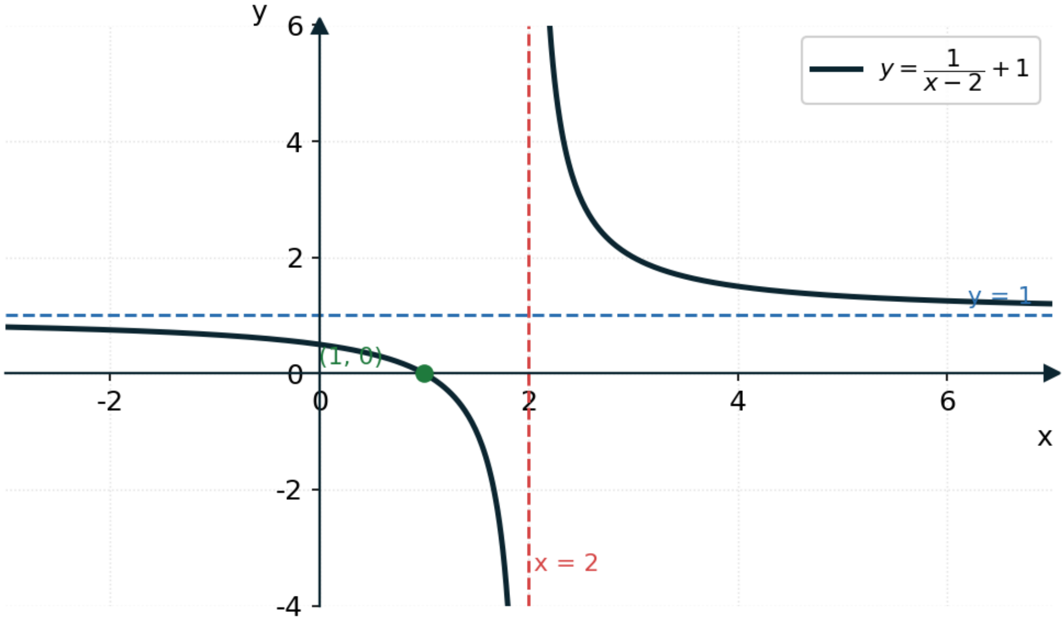

Figure 3. y = 1/(x − 2) + 1 has a vertical asymptote x = 2 and a horizontal asymptote y = 1 (both dashed); the x-intercept is (1, 0) and the y-intercept is (0, 0.5).

7. Exponential and logarithmic functions

An **exponential** function $f(x) = a^x$ ($a > 0$, $a \neq 1$) has domain $\mathbb{R}$ and range $f(x) > 0$. Every such graph passes through (0, 1), rises if $a > 1$ and falls if $0 < a < 1$, and has the x-axis ($y = 0$) as a horizontal asymptote. The special base $e \approx 2.718$ gives the natural exponential e^x.

A **logarithmic** function $f(x) = \log_a x$ is the **inverse** of a^x. Its domain is $x > 0$ and its range is $\mathbb{R}$; it passes through (1, 0) and has the y-axis ($x = 0$) as a vertical asymptote. Because they are inverses, the graphs of a^x and $\log_a x$ are reflections of each other in $y = x$. The natural logarithm $\ln x = \log_e x$.

Function	Domain	Range	Asymptote	Passes through
a^x ($a > 1$)	$x \in \mathbb{R}$	$y > 0$	$y = 0$	(0, 1)
e^x	$x \in \mathbb{R}$	$y > 0$	$y = 0$	(0, 1)
$\log_a x$	$x > 0$	$y \in \mathbb{R}$	$x = 0$	(1, 0)
$\ln x$	$x > 0$	$y \in \mathbb{R}$	$x = 0$	(1, 0)

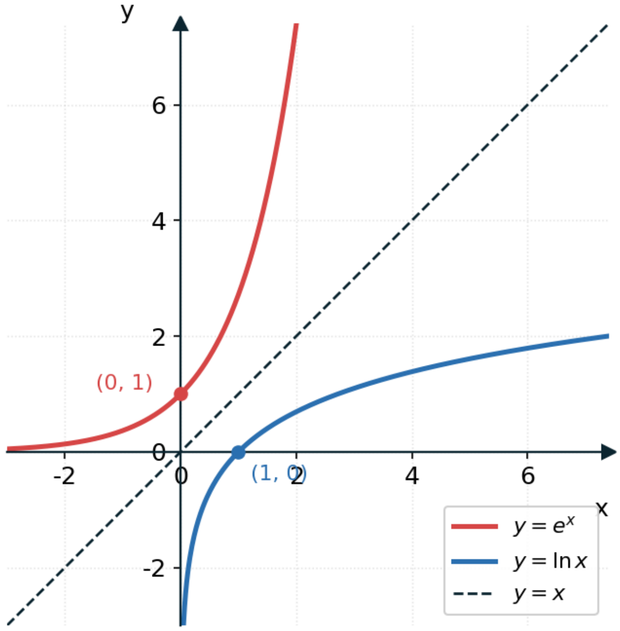

Figure 4. $y = e^x$ and its inverse $y = \ln x$ are reflections of each other in the line $y = x$ (dashed). They pass through (0, 1) and (1, 0) respectively.

Solving equations: use that $a^x = b$ means $x = \log_a b$, and that ln and e^x undo each other. For example, $2 \times 3^x = 54$ gives $3^x = 27$, so $x = 3$; and $e^x = 10$ gives $x = \ln 10 \approx 2.30$.

> **Domain alert:** log and ln accept only **positive** inputs. After solving a log equation, reject any solution that would make the argument zero or negative - a routine source of lost marks.

8. Solving equations graphically and with the GDC

Any equation $g(x) = h(x)$ can be solved by graphing $y = g(x)$ and $y = h(x)$ and reading the x-coordinates of their intersection points; equivalently, graph $y = g(x) - h(x)$ and find its zeros. This is essential when no exact algebraic method exists, for instance $e^x = 3 - x$.

- Choose a viewing window that shows all intersections you expect.
- Use the GDC's **intersect** or **zero** tools, not tracing by eye.
- Report each solution to 3 significant figures unless an exact value is asked for, and check none has been missed by scanning the whole relevant range.

> **Exam technique:** On a calculator paper, a quick GDC sketch confirms how many solutions an equation has before you commit to an algebraic route - and can rescue full marks when the algebra stalls.

9. Polynomial functions (HL)

A polynomial of degree n has the form $a_n x^n + ... + a_1 x + a_0$. Two theorems make factorising and root-finding efficient.

- **Remainder theorem:** the remainder when $p(x)$ is divided by $(x - a)$ equals $p(a)$.

- **Factor theorem:** $(x - a)$ is a factor of $p(x)$ exactly when $p(a) = 0$. (More generally $(bx - a)$ is a factor when $p(a/b) = 0$.)

Sum and product of roots

For a polynomial $a_n x^n + ... + a_0$ with roots $r_1, ..., r_n$, the sum of the roots is $-a_{n-1}/a_n$ and the product of the roots is $(-1)^n a_0/a_n$. For a quadratic $ax^2 + bx + c$: sum $= -b/a$ and product $= c/a$.

Multiplicity and the shape of the graph

If $(x - a)^m$ divides $p(x)$ exactly, then a is a root of **multiplicity** m. At a root of odd multiplicity the curve **crosses** the x-axis; at a root of even multiplicity it **touches** and turns back. A double root ($m = 2$) sits at a turning point on the axis.

Worked example 6 - factor and remainder theorems (HL)

For $p(x) = 2x^3 - 3x^2 - 11x + 6$, (a) show $(x - 3)$ is a factor, (b) factor $p(x)$ fully and state the roots, (c) find the remainder when $p(x)$ is divided by $(x - 2)$.

(a) $p(3) = 2(27) - 3(9) - 11(3) + 6 = 54 - 27 - 33 + 6 = 0$, so by the factor theorem $(x - 3)$ is a factor.

(b) Dividing gives $p(x) = (x - 3)(2x^2 + 3x - 2) = (x - 3)(2x - 1)(x + 2)$. Roots: $\mathbf{x = 3, x = 1/2, x = -2}$.

Check by root sum: $-b/a = 3/2$, and $3 + 1/2 - 2 = 3/2$.

(c) By the remainder theorem the remainder is $p(2) = 16 - 12 - 22 + 6 = \mathbf{-12}$.

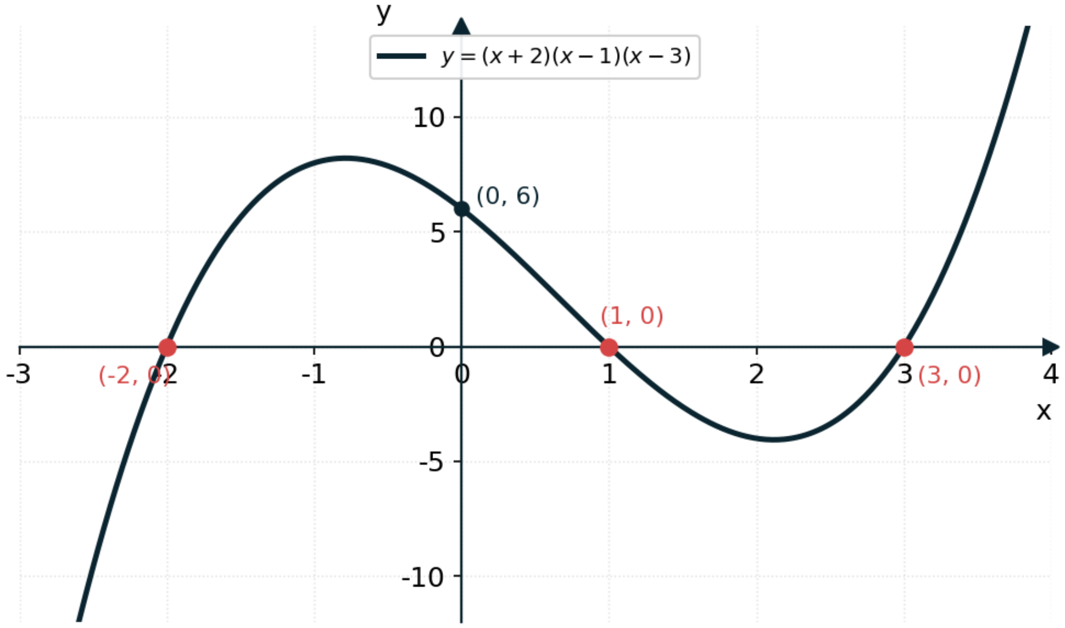

Figure 5. $y = (x + 2)(x - 1)(x - 3)$ crosses the x-axis at its three real roots $x = -2$, 1 and 3; the y-intercept is (0, 6).

10. Rational functions and oblique asymptotes (HL)

At HL the numerator or denominator may be quadratic. Locate features as follows:

- **Vertical asymptotes**: values that make the denominator zero (and do not cancel with the numerator).
- **Horizontal asymptote**: if numerator and denominator have the **same** degree, y = ratio of leading coefficients; if the numerator degree is **smaller**, $y = 0$.
- **Oblique (slant) asymptote**: if the numerator degree is exactly **one more** than the denominator, polynomial division gives a straight line $y = mx + c$ that the curve approaches as $x \to \pm\infty$.

For example $f(x) = (x^2 + 1)/(x - 1)$ divides to $x + 1 + 2/(x - 1)$. As $x \to \pm\infty$ the fraction $2/(x - 1) \to 0$, so the oblique asymptote is $\mathbf{y = x + 1}$, and the vertical asymptote is $x = 1$.

> **Method reminder:** Find an oblique asymptote by long (or synthetic) division: the quotient is the asymptote line, and the leftover fraction tends to 0 far from the origin.

11. Odd, even and self-inverse functions (HL)

- **Even:** $f(-x) = f(x)$ for all x. The graph is symmetric in the y-axis. Examples: x^2, $\cos x$, $|x|$.
- **Odd:** $f(-x) = -f(x)$ for all x. The graph has rotational symmetry of order 2 about the origin. Examples: x^3, $\sin x$, $1/x$.

Most functions are neither. To test, compute $f(-x)$ and compare it with $f(x)$ and $-f(x)$.

Self-inverse functions

A function is **self-inverse** if $f^{-1} = f$, that is $f(f(x)) = x$ for all x in the domain. Its graph is symmetric in the line $y = x$. Examples include $f(x) = 1/x$, $f(x) = a - x$, and $f(x) = (x + b)/(x - 1)$ type hyperbolas for suitable constants. To verify, show $f(f(x))$ simplifies to x.

> **Quick check:** For a self-inverse function, reflecting its graph in $y = x$ leaves it unchanged - a fast visual confirmation on the GDC.

12. The modulus (absolute value) function (HL)

The modulus $|x|$ gives the non-negative size of x: $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. The graph of $y = |x|$ is a V with vertex at the origin; $y = |x - a|$ has its vertex at $(a, 0)$. Geometrically $|x - a|$ is the distance between x and a on the number line.

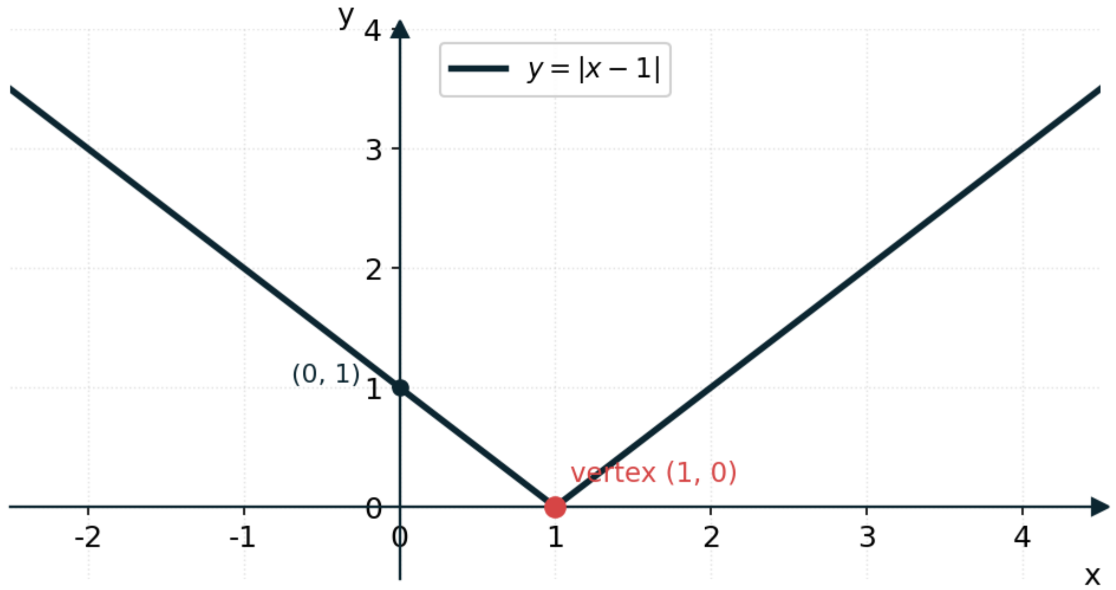

Figure 6. y = |x − 1| is a V-shape with its vertex at (1, 0) and y-intercept (0, 1).

Solving modulus equations and inequalities

- $|x| = k$ ($k \geq 0$) means $x = k$ or $x = -k$.
- $|x| < k$ means $-k < x < k$ (a single interval).
- $|x| > k$ means $x < -k$ or $x > k$ (two pieces).
- $|A| = |B|$ can be solved by squaring: $A^2 = B^2$.

Worked example 7 - modulus equation and inequality (HL)

Solve (a) $|2x - 1| = |x + 3|$ and (b) $|x - 2| < 3$.

(a) Square both sides: $(2x - 1)^2 = (x + 3)^2$.

$4x^2 - 4x + 1 = x^2 + 6x + 9$, so $3x^2 - 10x - 8 = 0$.

Factor: $(3x + 2)(x - 4) = 0$, giving **$x = -2/3$ or $x = 4$** (both check).

(b) $|x - 2| < 3$ means $-3 < x - 2 < 3$, so **$-1 < x < 5$**.

> **Common pitfall:** Squaring is valid for $|A| = |B|$ and $|A| < |B|$ type statements because both sides are non-negative. Do not square $|A| = B$ unless you have first checked $B \geq 0$.

13. Graphs of related functions (HL)

Given the graph of $y = f(x)$, you should be able to sketch four related graphs.

Graph	How to build it from y = f(x)
$y = \|f(x)\|$	reflect every part below the x-axis up above it; parts already above stay put
$y = f(\|x\|)$	keep the graph for $x \geq 0$, then reflect that part in the y-axis (the $x < 0$ side is discarded)

Graph	How to build it from $y = f(x)$
$y = 1/f(x)$	zeros of f become vertical asymptotes; where $f \to \pm\infty$, $1/f \to 0$; maxima of f become minima of 1/f and vice versa; the sign is preserved
$y = f(ax + b)$	a horizontal stretch of factor 1/a then a translation; treat it as $f(a(x + b/a))$

For $y = 1/f(x)$, a useful anchor is that points where $f(x) = 1$ or $f(x) = -1$ are unchanged, since their reciprocal is the same value. Watch signs: 1/f keeps the sign of f, so it never jumps across an asymptote without f passing through zero.

> **Order matters:** $y = |f(x)|$ reflects **outputs** (in the x-axis) while $y = f(|x|)$ reflects **inputs** (in the y-axis). They usually give different graphs - sketch both carefully.

14. Solving inequalities analytically and graphically (HL)

Inequalities are solved by finding where an expression changes sign. The sign can only change at a **zero** (numerator = 0) or at an **undefined point** (denominator = 0). Mark these critical values on a number line and test the sign of the expression in each interval.

- **Never** multiply both sides of an inequality by an expression that could be negative - the direction of the inequality may flip. Instead move everything to one side and analyse the sign of the single combined expression.
- Include a zero of the numerator when the sign is $\geq$ or $\leq$, but never include a value that makes the denominator zero.
- Graphically, solve $f(x) > g(x)$ by reading where the graph of f lies above the graph of g.

Example: to solve $(x - 1)/(x + 2) \geq 0$, the critical values are $x = 1$ (numerator zero) and $x = -2$ (denominator zero, excluded). Testing the three intervals gives the solution **$x < -2$ or $x \geq 1$**.

> **Common pitfall:** $x = -2$ is excluded even though the inequality is “greater than or equal to”, because the expression is undefined there. Use an open circle at excluded points and a filled circle at included zeros.

Quick reference

Idea	Key result
Composite	$(f \circ g)(x) = f(g(x))$; order matters, $f \circ g \neq g \circ f$
Inverse	swap x and y then solve; domain f^{-1} = range f; reflect in $y = x$
Quadratic vertex	$a(x - h)^2 + k$ has vertex (h, k), axis $x = h = -b/(2a)$
Quadratic formula	$x = (-b \pm \sqrt{(b^2 - 4ac)}) / (2a)$
Discriminant	$\Delta = b^2 - 4ac$: >0 two roots, =0 one, <0 none
(ax+b)/(cx+d)	vertical $x = -d/c$; horizontal $y = a/c$
Exponential a^x	range $y > 0$; asymptote $y = 0$; through (0, 1)

Idea	Key result
Logarithm $\log_a x$	domain $x > 0$; asymptote $x = 0$; through $(1, 0)$
Factor theorem (HL)	$(x - a)$ factor of $p(x)$ iff $p(a) = 0$
Oblique asymptote (HL)	numerator degree one more than denominator; divide to find $y = mx + c$
Even / odd (HL)	even $f(-x) = f(x)$; odd $f(-x) = -f(x)$
Modulus (HL)	$\|x\| < k$ gives $-k < x < k$; $\|x\| > k$ gives two pieces

Test yourself

Attempt all ten without notes, then check against the full solutions. Questions marked **(HL)** are Higher Level.

1. For $f(x) = 3x - 4$ and $g(x) = x^2 + 1$, find $(f \circ g)(2)$ and $(g \circ f)(x)$.

2. Find the inverse of $f(x) = (2x + 1)/(x - 3)$ and state its domain.

3. Express $f(x) = 3x^2 + 12x + 5$ in vertex form and state the minimum point.

4. Find the values of m for which $x^2 - mx + 9 = 0$ has two distinct real roots.

5. Describe the transformations mapping $y = x^2$ to $y = -(x - 1)^2 + 4$ and state the vertex.

6. State the vertical and horizontal asymptotes and both intercepts of $y = (3x - 2)/(x + 1)$.

7. Solve $e^{2x} - 5e^x + 6 = 0$.

8. (HL) Given $p(x) = x^3 - 4x^2 + x + 6$, show $(x + 1)$ is a factor and hence solve $p(x) = 0$.

9. (HL) Solve $|3x - 2| = 4$ and $|x + 1| \le 2$.

10. (HL) Solve the inequality $(x - 1)/(x + 2) \ge 0$.

Worked answers

1. $g(2) = 4 + 1 = 5$, so $(f \circ g)(2) = f(5) = 3(5) - 4 =$ **11**. And $(g \circ f)(x) = (3x - 4)^2 + 1 =$ $\mathbf{9x^2 - 24x + 17}$.

2. Put $y = (2x + 1)/(x - 3)$; then $y(x - 3) = 2x + 1$, so $xy - 3y = 2x + 1$ and $x(y - 2) = 3y + 1$. Hence $\mathbf{f^{-1}(x) = (3x + 1)/(x - 2)}$, domain $\mathbf{x \ne 2}$.

3. $3x^2 + 12x + 5 = 3(x^2 + 4x) + 5 = 3[(x + 2)^2 - 4] + 5 =$ $\mathbf{3(x + 2)^2 - 7}$; minimum point **(−2, −7)**.

4. $\Delta = (-m)^2 - 4(1)(9) = m^2 - 36 > 0$, so $m^2 > 36$, giving $\mathbf{m > 6}$ **or** $\mathbf{m < -6}$.

5. Translate 1 right, reflect in the x-axis, then translate 4 up; the parabola opens downward with vertex **(1, 4)** (a maximum).

6. Vertical asymptote $\mathbf{x = -1}$; horizontal asymptote $\mathbf{y = 3}$; x-intercept **(2/3, 0)**; y-intercept **(0, −2)**.

7. Let $u = e^x$: $u^2 - 5u + 6 = 0$, so $(u - 2)(u - 3) = 0$. Then $e^x = 2$ or $e^x = 3$, giving $\mathbf{x = \ln 2}$ **or** $\mathbf{x = \ln 3}$ (both valid since $u > 0$).

8. (HL) $p(-1) = -1 - 4 - 1 + 6 = 0$, so $(x + 1)$ is a factor. Division gives $p(x) = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$, so $\mathbf{x = -1, 2, 3}$.

9. (HL) $|3x - 2| = 4$ gives $3x - 2 = \pm 4$, so $\mathbf{x = 2}$ **or** $\mathbf{x = -2/3}$. $|x + 1| \le 2$ gives $-2 \le x + 1 \le 2$, so $\mathbf{-3 \le x \le 1}$.

10. (HL) Critical values $x = 1$ (zero) and $x = -2$ (excluded). Sign testing gives the expression ≥ 0 on **$x < -2$ or $x \geq 1$** (with $x = -2$ excluded and $x = 1$ included).