IB DP MATHEMATICS

Analysis and Approaches (AA)

Topic 1 Number and Algebra

Revision Notes · Standard and Higher Level

Fahad H. Ahmad

+92 323 509 4443 | Megalecture.com

Original notes prepared for the IB Diploma Programme Mathematics: Analysis and Approaches course (first assessment 2021)

What Topic 1 requires

Number and Algebra is the foundation topic of AA. It is examined on both papers, and its ideas — sequences, exponents, logarithms and the binomial theorem — reappear throughout calculus, functions and statistics. Use the map below as a final checklist. Rows marked **(HL)** are examined at Higher Level only.

Sub-topic	You should be able to...
Standard form & accuracy	Write numbers as $a \times 10^k$; round to decimal places and significant figures; find percentage error.
Arithmetic sequences & series	Use $u_n = u_1 + (n-1)d$ and S_n; model real situations.
Geometric sequences & series	Use $u_n = u_1 r^{n-1}$, S_n, and S_∞ when $\|r\| < 1$; compound interest, growth and decay.
Sigma notation	Read, write and evaluate series written with $\sum$.
Exponents & logarithms	Apply the laws of indices and logarithms; change of base; solve exponential equations.
Binomial theorem	Expand $(a+b)^n$ for $n \in \mathbb{Z}^+$; use Pascal's triangle and nCr; find a particular term.
(HL) Counting	Factorials, permutations and combinations; extended binomial for $n \in \mathbb{Q}$.
(HL) Proof	Mathematical induction, proof by contradiction, and disproof by counterexample.
(HL) Complex numbers	Cartesian, polar and Euler form; Argand diagram; De Moivre's theorem; roots.
(HL) Linear systems	Solve up to three equations in three unknowns; classify the solution set.
(HL) Partial fractions	Split a rational function into a sum of simpler fractions.

> **Exam note:** SL and HL share the whole SL core. The six HL rows above are the extra Higher Level content; they are also examined in longer, multi-step questions and combined freely with other topics.

1. Standard form, rounding and error

Standard form (scientific notation) writes a number as $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$. It compresses very large and very small numbers and makes their order of magnitude obvious.

$$6\,240\,000 = 6.24 \times 10^6 \qquad 0.000\,087 = 8.7 \times 10^{-5}$$

When multiplying or dividing, handle the decimal parts and the powers of ten separately, then re-normalise so that $1 \le a < 10$.

$$(3.0 \times 10^8) \times (4.0 \times 10^{-3}) = 12 \times 10^5 = 1.2 \times 10^6$$

Rounding and significant figures

- **Decimal places (d.p.)** count digits after the decimal point.
- **Significant figures (s.f.)** count from the first non-zero digit. Leading zeros are never significant; trailing zeros after a decimal point are.

- Round the final answer only; keep full precision in intermediate steps to avoid rounding error building up. IB answers are normally given to **three significant figures** unless told otherwise.

Percentage error

If v_A is an approximate (measured or rounded) value and v_E is the exact value, the percentage error is

$$\varepsilon = \left| (v_A - v_E) / v_E \right| \times 100\%$$

The vertical bars make the error non-negative regardless of whether the estimate is high or low.

Worked example 1 — percentage error

A student measures a rod as 24.8 cm; its true length is 25.0 cm. Find the percentage error.

$\varepsilon = | (24.8 - 25.0) / 25.0 | \times 100\% = (0.2 / 25.0) \times 100\% =$ **0.8%**.

Common pitfall: Always divide by the *exact* value v_E, not by the approximation. Dividing by the wrong quantity is a frequent lost mark.

2. Arithmetic sequences and series

An **arithmetic sequence** has a constant **common difference** d between consecutive terms: $d = u_{n+1} - u_n$. Each term is the previous term plus d.

$$u_n = u_1 + (n - 1)d$$

A **series** is the sum of the terms of a sequence. For the first n terms of an arithmetic sequence:

$$S_n = n/2\,(2u_1 + (n - 1)d) = n/2\,(u_1 + u_n)$$

The second form is quickest when the first and last terms are both known: the sum equals the number of terms times the average of the first and last term.

Worked example 2 — arithmetic sequence and series

The 7th term of an arithmetic sequence is 32 and the 12th term is 52. Find u_1, d, and the sum of the first 20 terms.

$u_7 = u_1 + 6d = 32$ and $u_{12} = u_1 + 11d = 52$.

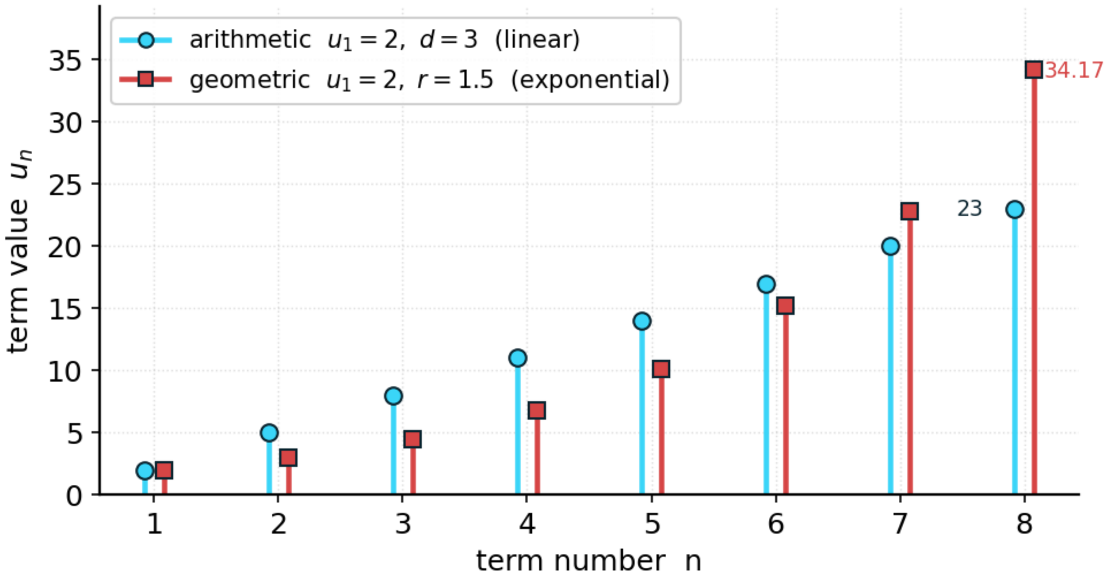
Subtracting: $5d = 20$, so $d = 4$, and $u_1 = 32 - 6(4) =$ **8**.

$S_{20} = 20/2\,(2\times8 + 19\times4) = 10(16 + 76) = 10 \times 92 =$ **920**.

Exam technique: Two facts about an arithmetic sequence give two equations in u_1 and d. Subtract them to eliminate u_1 and find d first.

3. Geometric sequences and series

A **geometric sequence** has a constant **common ratio** r between consecutive terms: $r = u_{n+1} / u_n$. Each term is the previous term multiplied by r.



First eight terms: the arithmetic sequence (u_1=2, d=3) rises in a straight line, while the geometric sequence (u_1=2, r=1.5) curves upward — linear versus exponential growth.

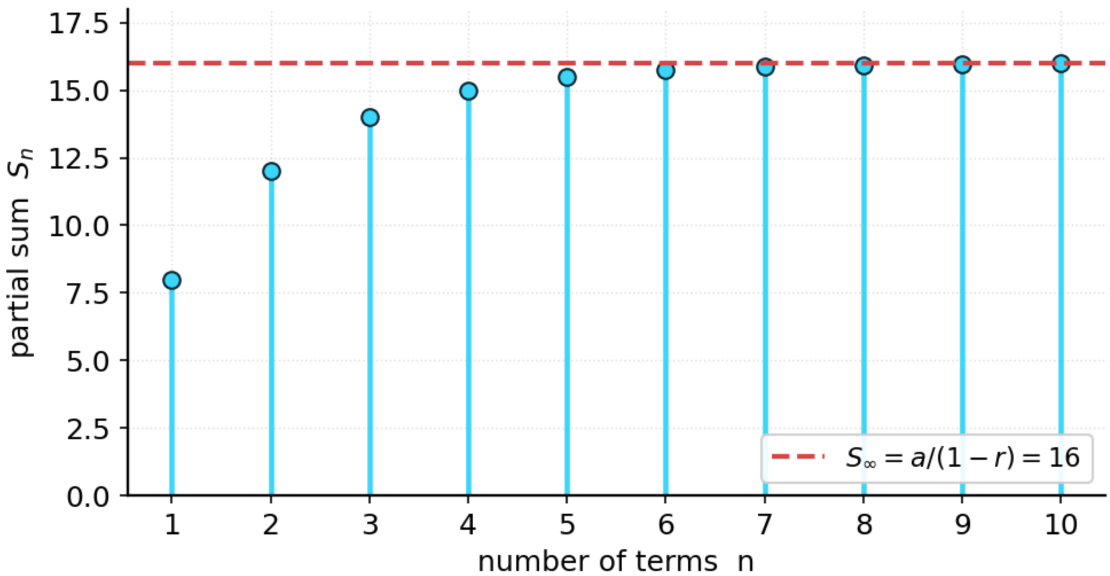
$$u_n = u_1 r^{n-1} \qquad S_n = u_1(r^n - 1)/(r - 1) = u_1(1 - r^n)/(1 - r),\ r \neq 1$$

Use whichever form of S_n keeps the arithmetic positive: the left form when r > 1, the right form when r < 1.

Sum to infinity

If the common ratio satisfies |r| < 1, the terms shrink towards zero and the partial sums approach a finite limit, the **sum to infinity**:

$$S_\infty = u_1 / (1 - r), \text{ valid only for } |r| < 1$$



Partial sums S_n of 8 + 4 + 2 + … (a=8, r=1/2) climb toward the limit S_∞ = a/(1−r) = 16 (dashed line).

If |r| ≥ 1 the series **diverges** and no sum to infinity exists — a condition examiners love to test.

Applications: growth and decay

Geometric sequences model anything that changes by a fixed *percentage* each step. For **compound interest** at rate i per period on principal P, the value after n periods is $P(1 + i)^n$; a decaying quantity (depreciation, radioactive material, cooling) uses a ratio $r < 1$.

Worked example 3 — sum to infinity and compound interest

(a) A geometric series has first term 24 and common ratio 1/2. Find its sum to infinity.

Since $|r| = 1/2 < 1$, $S_\infty = 24 / (1 - 1/2) = 24 / (1/2) =$ **48**.

(b) \$2000 is invested at 4.5% per year, compounded annually. Find its value after 6 years.

Value $= 2000(1.045)^6 = 2000 \times 1.30226 =$ **\$2604.52** (2 d.p.).

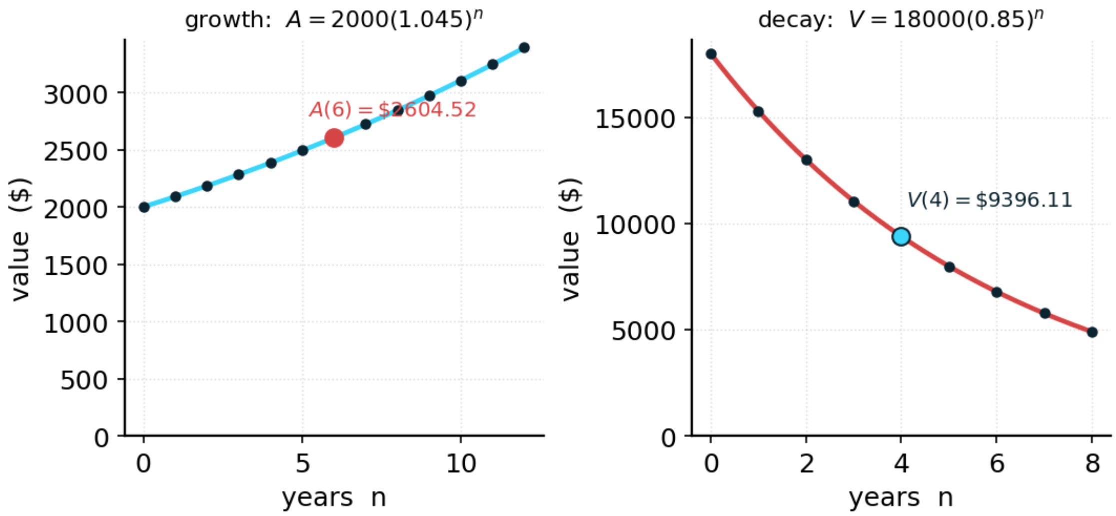
Worked example 4 — depreciation (geometric decay)

A car bought for \$18 000 loses 15% of its value each year. Find its value after 4 years and the total value lost.

Each year the value is multiplied by $r = 1 - 0.15 = 0.85$ (a geometric ratio < 1).

Value after 4 years $= 18000(0.85)^4 = 18000 \times 0.52200625 \approx$ **\$9396.11**.

Value lost $= 18000 - 9396.11 =$ **\$8603.89**.



Exponential models: compound interest $2000(1.045)^n$ grows (left) while a car's value $18000(0.85)^n$ decays (right). The marked points reproduce Worked examples 3(b) and 4.

Common pitfall: Never apply $S_\infty = u_1/(1-r)$ without first checking $|r| < 1$. If $|r| \geq 1$ the correct answer is that the sum does not exist.

4. Sigma notation

Sigma notation is a compact way of writing a sum. The Greek capital sigma Σ means 'add up'; the letter below is the index and its starting value, the number on top is where it stops.

$$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \dots + u_n$$

For example $\sum_{r=1}^{4} (2r + 1) = 3 + 5 + 7 + 9 = 24$. The index letter is a dummy: $\sum_{r=1}^{n}$ and $\sum_{k=1}^{n}$ mean the same thing. Sigma notation is simply a way to *write* a series — the arithmetic and geometric sum formulae still do the actual work.

Worked example 5 — evaluating a series in sigma notation

Evaluate $\sum_{r=1}^{8} 3(2)^{r-1}$.

The general term $3(2)^{r-1}$ is geometric with $u_1 = 3$, $r = 2$, and there are $n = 8$ terms.

$S_8 = 3(2^8 - 1)/(2 - 1) = 3(256 - 1) = 3 \times 255 =$ **765**.

Exam technique: If the general term u_r is linear in r the series is arithmetic; if it contains a constant raised to the power r (such as 2^r) it is geometric. Identify which, read off u_1, d or r, and the number of terms, then apply S_n.

5. Exponents and logarithms

The **laws of exponents** (indices) hold for any non-zero base a and rational powers:

Law	Statement
Product	$a^m \times a^n = a^{m+n}$
Quotient	$a^m \div a^n = a^{m-n}$
Power of a power	$(a^m)^n = a^{mn}$
Zero and negative	$a^0 = 1$; $a^{-n} = 1 / a^n$
Fractional (roots)	$a^{1/n} = \sqrt[n]{a}$; $a^{m/n} = (\sqrt[n]{a})^m$

A **logarithm** answers 'to what power must the base be raised?' It is the inverse of exponentiation:

$$a^x = b \;\Rightarrow\; x = \log_a b \quad (a > 0, a \neq 1, b > 0)$$

The **laws of logarithms** mirror the index laws:

Law	Statement
Product	$\log_a(xy) = \log_a x + \log_a y$
Quotient	$\log_a(x/y) = \log_a x - \log_a y$
Power	$\log_a(x^m) = m \log_a x$
Special values	$\log_a 1 = 0$; $\log_a a = 1$
Change of base	$\log_a x = (\log_b x) / (\log_b a)$

Natural logarithms use base $e \approx 2.71828$ and are written $\ln x = \log_e x$. The change-of-base rule lets any logarithm be evaluated on a calculator as $(\ln x)/(\ln a)$ or $(\log x)/(\log a)$.

To solve an equation with the unknown in the exponent, take logs of both sides and use the power law to bring the exponent down.

Worked example 6 — solving with logarithms

(a) Solve $5^x = 20$, giving x to 3 s.f.

Take logs: $x \log 5 = \log 20$, so $x = \log 20 / \log 5 = 1.30103 / 0.69897 = \mathbf{1.86}$.

(b) Solve $\log_2 x + \log_2(x - 2) = 3$.

Product law: $\log_2[x(x - 2)] = 3$, so $x(x - 2) = 2^3 = 8$.

$x^2 - 2x - 8 = 0 \Rightarrow (x - 4)(x + 2) = 0$, so $x = 4$ or $x = -2$.

Reject $x = -2$ (a logarithm needs a positive argument). $\therefore$ **$x = 4$**.

Common pitfall: $\log(x + y)$ is **not** $\log x + \log y$. The product law applies to $\log(xy)$, not to the log of a sum. Always check the domain of a log equation and reject solutions that make any argument zero or negative.

6. The binomial theorem

The **binomial theorem** expands a power of a two-term bracket without multiplying it out term by term. For a positive integer n:

$$(a + b)^n = \sum_{r=0}^{n} nCr\, a^{n-r}\, b^r$$

where the **binomial coefficient** is

$$nCr = \binom{n}{r} = n! / (r!(n - r)!)$$

The coefficients are the rows of **Pascal's triangle**: each entry is the sum of the two above it. Row n (starting at $n = 0$) gives the coefficients of $(a + b)^n$.

n	Row of Pascal's triangle	gives coefficients of
0	1	$(a+b)^0$
1	1 1	$(a+b)^1$
2	1 2 1	$(a+b)^2$
3	1 3 3 1	$(a+b)^3$
4	1 4 6 4 1	$(a+b)^4$

Worked example 7 — a full expansion

Expand $(2x - 3)^4$ completely.

Use Pascal's row for $n = 4$: 1, 4, 6, 4, 1, with $a = 2x$ and $b = -3$.

$(2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$

$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$.

Finding a particular term

The **general term** (the term containing b^r) is $nCr\ a^{n-r}\ b^r$. To pick out one specific term, find the value of r that gives the required power, then evaluate just that term — there is no need to expand the whole bracket.

Worked example 8 — a particular term

Find the coefficient of x^3 in the expansion of $(2 + 3x)^7$.

General term: $7Cr\ (2)^{7-r}\ (3x)^r$. The power of x is r, so for x^3 take $r = 3$.

Term $= 7C3\ (2)^4\ (3)^3\ x^3 = 35 \times 16 \times 27\ x^3 = 15120\ x^3$.

The coefficient is **15120**.

Exam technique: For a *constant* term, set the total power of the variable to zero and solve for r. Watch signs carefully when the bracket contains a minus, e.g. $(x - k)^n$ gives $(-k)^r$ in the general term.

7. (HL) Counting: permutations and combinations

(HL) The **factorial** $n! = n(n-1)(n-2)\ldots3 \cdot 2 \cdot 1$ counts the number of ways of arranging n distinct objects in a row, with $0! = 1$ by definition.

A **permutation** counts *ordered* selections; a **combination** counts *unordered* selections of r objects from n distinct objects:

$$nPr = n! / (n - r)! \qquad nCr = n! / (r!(n - r)!)$$

Because order does not matter for a combination, we divide the permutation count by r! (the number of orderings of the chosen r). Combinations are exactly the binomial coefficients of Topic 1.6.

Worked example 9 — combinations with a restriction

A committee of 4 is chosen from 7 women and 5 men. (a) How many committees are possible? (b) How many contain exactly 2 women and 2 men?

(a) Order does not matter, so choose 4 from 12: $12C4 = 12!/(4!\ 8!) =$ **495**.

(b) Choose 2 of 7 women and 2 of 5 men, then multiply: $7C2 \times 5C2 = 21 \times 10 =$ **210**.

(HL) Extended binomial theorem

For a **fractional or negative** exponent $n \in \mathbb{Q}$, the binomial series is an infinite series, valid only for $|x| < 1$:

$$(1 + x)^n = 1 + nx + n(n-1)/2!\, x^2 + n(n-1)(n-2)/3!\, x^3 + \ldots$$

Unlike the positive-integer case it does not terminate, and the $|x| < 1$ condition is essential for the series to converge. To expand $(a + x)^n$, first factor out a^n to reach the standard (1 + something) form.

> **Common pitfall:** The extended expansion of $(1 + x)^n$ is only valid for $|x| < 1$. State the range of validity — it is often worth a mark of its own.

8. (HL) Proof

(HL) A **proof** establishes a result for *all* cases by watertight reasoning; a single numerical check is never a proof. Three methods are examined.

Proof by mathematical induction

Used to prove a statement P(n) for all integers $n \geq$ some starting value. There are three steps:

- **Base case:** show P(1) (or the first relevant value) is true.
- **Inductive step:** *assume* P(k) is true (the inductive hypothesis) and use it to prove P(k+1).
- **Conclusion:** since P(1) holds and $P(k) \Rightarrow P(k+1)$, by induction P(n) is true for all $n \geq 1$.

Worked example 10 — proof by induction

Prove that $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$ for all $n \in \mathbb{Z}^+$.

Base case (n = 1): LHS = $1^2 = 1$; RHS = $1\times2\times3/6 = 1$. True.

Inductive step: assume $\sum_{r=1}^{k} r^2 = k(k+1)(2k+1)/6$. Then adding the next term $(k+1)^2$:

$\sum_{r=1}^{k+1} r^2 = k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)[k(2k+1) + 6(k+1)] / 6$.

The bracket is $2k^2 + 7k + 6 = (k+2)(2k+3)$, so the sum = $(k+1)(k+2)(2k+3)/6$, which is the formula with n = k+1.

Conclusion: true for n = 1 and $P(k) \Rightarrow P(k+1)$, so by induction the result holds for all $n \in \mathbb{Z}^+$.

Proof by contradiction

Assume the *negation* of what you want to prove, then derive a logical impossibility. The classic example is that $\sqrt{2}$ is irrational: assume $\sqrt{2} = p/q$ in lowest terms, square to get $p^2 = 2q^2$, deduce that p and q are both even, contradicting 'lowest terms'.

Disproof by counterexample

To show a universal statement is **false**, one explicit counterexample is enough. For instance, '$n^2 + n + 41$ is prime for all $n \in \mathbb{N}$' fails at n = 41, where the value is 41^2, which is not prime.

> **Exam technique:** In an induction answer you must (i) write the inductive hypothesis explicitly, (ii) show clearly where you use it, and (iii) finish with a full concluding sentence. Marks are awarded for this structure, not only for the algebra.

9. (HL) Complex numbers

(HL) The imaginary unit is defined by $i^2 = -1$. A **complex number** in **Cartesian form** is $z = a + bi$, with real part $\mathrm{Re}(z) = a$ and imaginary part $\mathrm{Im}(z) = b$. The set of all complex numbers is $\mathbb{C}$.

Arithmetic works like ordinary algebra with the single rule $i^2 = -1$. The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$; multiplying a number by its conjugate gives the real number $zz^* = a^2 + b^2$, which is used to divide (realise the denominator).

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \qquad z\,z^* = a^2 + b^2$$

Worked example 11 — dividing complex numbers

Express $(3 + 2i)/(1 - i)$ in the form $a + bi$.

Multiply top and bottom by the conjugate of the denominator, $1 + i$:

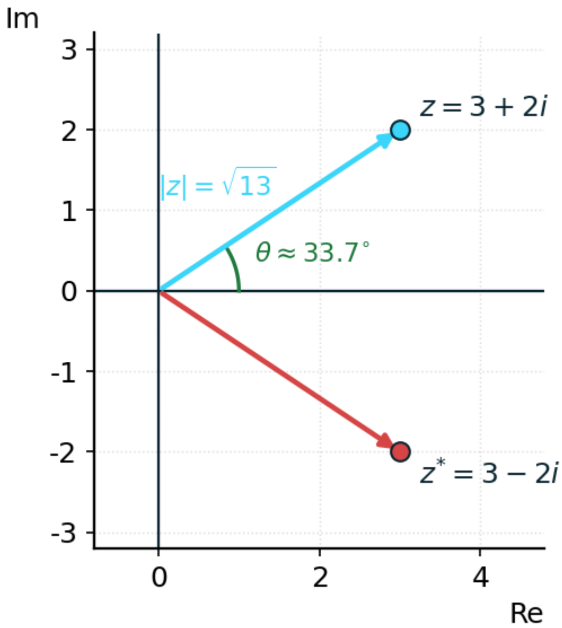
$(3 + 2i)(1 + i) / ((1 - i)(1 + i)) = (3 + 3i + 2i + 2i^2) / (1 + 1)$.

Using $i^2 = -1$: numerator $= 3 + 5i - 2 = 1 + 5i$, so the result is $(1 + 5i)/2 =$ **$0.5 + 2.5i$**.

Modulus, argument and the Argand diagram

On an **Argand diagram** $z = a + bi$ is the point (a, b). Its **modulus** is the distance from the origin and its **argument** is the angle from the positive real axis:

$$|z| = \sqrt{a^2 + b^2} \qquad \arg(z) = \theta, \text{ where } \tan\theta = b/a$$



Argand diagram of z = 3 + 2i and its conjugate z = 3 − 2i. The modulus |z| = √13 ≈ 3.61 and the argument θ ≈ 33.7° are shown; conjugation reflects z in the real axis.*

The argument is usually given in the range $-\pi < \theta \le \pi$ (principal argument); always use a sketch to place z in the correct quadrant.

Polar and Euler form; De Moivre's theorem

Writing $r = |z|$ and $\theta = \arg(z)$, the same number is

$$\mathbf{z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta = r\,e^{i\theta}}$$

Multiplication multiplies the moduli and adds the arguments; this leads to **De Moivre's theorem**, the engine for powers and roots:

$$\mathbf{[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)}$$

Worked example 12 — De Moivre's theorem

Let $z = 1 + i\sqrt{3}$. Find $|z|$ and $\arg(z)$, and hence evaluate z^6.

$|z| = \sqrt{(1^2 + (\sqrt{3})^2)} = \sqrt{(1 + 3)} = 2$.

$\tan\theta = \sqrt{3} / 1$ and z is in the first quadrant, so $\theta = \pi/3$.

$z = 2\,\mathrm{cis}(\pi/3)$, so by De Moivre $z^6 = 2^6\,\mathrm{cis}(6 \times \pi/3) = 64\,\mathrm{cis}(2\pi)$.

$\mathrm{cis}(2\pi) = \cos 2\pi + i\sin 2\pi = 1$, so $z^6 =$ **64**.

Roots of a complex number

A non-zero complex number has exactly n distinct **n-th roots**, found by writing arguments as $\theta + 2\pi k$ and applying De Moivre:

$$\mathbf{z^{1/n} = r^{1/n}\,\mathrm{cis}((\theta + 2\pi k)/n),\ k = 0, 1, \ldots, n-1}$$

The n roots are equally spaced by $2\pi/n$ around a circle of radius $r^{1/n}$ centred at the origin — they form a regular n-gon on the Argand diagram.

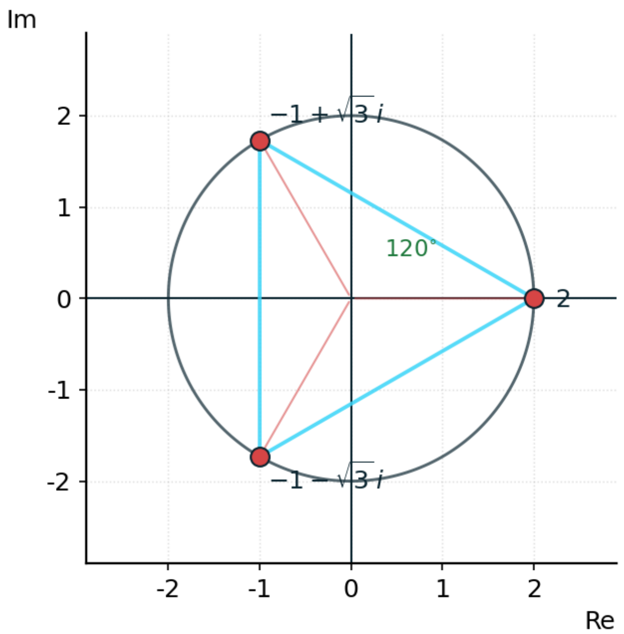
Worked example 13 — the cube roots of a number

Find the three cube roots of 8.

Write 8 in polar form: $8 = 8 \text{ cis } 0$, so $r^{1/3} = 2$ and the arguments are $(0 + 2\pi k)/3$ for $k = 0, 1, 2$.

$k = 0$: $2 \text{ cis } 0 = 2$. $k = 1$: $2 \text{ cis}(2\pi/3) = -1 + i\sqrt{3}$. $k = 2$: $2 \text{ cis}(4\pi/3) = -1 - i\sqrt{3}$.

The three roots **2**, $\mathbf{-1 + i\sqrt{3}}$, $\mathbf{-1 - i\sqrt{3}}$ sit at the vertices of an equilateral triangle on a circle of radius 2.



The three cube roots of 8 lie on a circle of radius 2 at arguments 0°, 120°, 240° (spaced 2π/3 apart), forming the vertices of an equilateral triangle.

Common pitfall: $i^2 = -1$, not $+1$. When finding roots, add $2\pi k$ to the argument *before* dividing by n, or you will find only one of the n roots. Check your quadrant with a sketch before quoting an argument.

10. (HL) Systems of linear equations

(HL) A linear system in up to three unknowns is solved by **elimination** (Gaussian reduction): use one equation to eliminate a variable from the others, repeat, then back-substitute. The solution set is one of three types:

Type	Geometric meaning (3 planes)	Algebra you will see
Unique solution	Planes meet at a single point	Each variable is pinned to one value
No solution (inconsistent)	No common point of intersection	A false row such as 0 = 5

Type	Geometric meaning (3 planes)	Algebra you will see
Infinitely many	Planes meet in a common line	A row reduces to 0 = 0; answer has a parameter

Worked example 14 — three equations, three unknowns

Solve $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - 3z = -4$.

Eliminate x. (Eq2 − 2×Eq1): $-3y - z = -9$. (Eq3 − Eq1): $y - 4z = -10$.

From the first of these, $z = 9 - 3y$. Substitute: $y - 4(9 - 3y) = -10 \Rightarrow 13y - 36 = -10 \Rightarrow y = 2$.

Then $z = 9 - 6 = 3$, and $x = 6 - y - z = 6 - 2 - 3 = 1$.

Unique solution: **$x = 1, y = 2, z = 3$** (check: all three equations are satisfied).

> **Exam technique:** Always substitute your answer back into the original equations. If a variable cancels to give 0 = 0 the system has infinitely many solutions; if it gives a false statement, there is no solution.

11. (HL) Partial fractions

(HL) A single rational expression with a factorable denominator can be split into a sum of simpler fractions — the reverse of adding fractions. This is essential for integration later in the course. For distinct linear factors:

$$(px + q)/((x - a)(x - b)) = A/(x - a) + B/(x - b)$$

Multiply through by the denominator, then find A and B either by substituting the values $x = a$ and $x = b$ (which make one bracket zero) or by comparing coefficients.

Worked example 15 — partial fractions

Express $(3x + 5)/((x - 1)(x + 2))$ in partial fractions.

Write $(3x + 5)/((x - 1)(x + 2)) = A/(x - 1) + B/(x + 2)$, so $3x + 5 = A(x + 2) + B(x - 1)$.

Let $x = 1$: $8 = 3A \Rightarrow A = 8/3$. Let $x = -2$: $-1 = -3B \Rightarrow B = 1/3$.

Therefore $(3x + 5)/((x - 1)(x + 2)) = 8/(3(x - 1)) + 1/(3(x + 2))$.

> **Exam technique:** The 'cover-up' substitution (choosing x to kill one factor) is the fastest route for distinct linear factors. Check the degree of the numerator is less than that of the denominator first; if not, divide out the polynomial part.

12. Common exam pitfalls

- Using $S_\infty = u_1/(1-r)$ when $|r| \geq 1$ — the sum to infinity only exists for $|r| < 1$.

- Writing $\log(x + y) = \log x + \log y$. The laws apply to products and quotients, never to a sum inside the logarithm.
- Confusing nPr (order matters) with nCr (order does not). A team is a combination; a ranked podium is a permutation.
- Forgetting $i^2 = -1$ and treating i as an ordinary variable when multiplying complex numbers.
- Losing a sign in a binomial expansion of $(a - b)^n$: the general term contains $(-b)^r$.
- Treating one numerical check as a proof, or omitting the concluding statement in an induction argument.
- Rounding intermediate values too early and carrying the error into the final answer.

13. Quick-reference formulae

Result	Formula
Arithmetic n-th term	$u_n = u_1 + (n - 1)d$
Arithmetic sum	$S_n = n/2\ (2u_1 + (n-1)d) = n/2\ (u_1 + u_n)$
Geometric n-th term	$u_n = u_1\, r^{n-1}$
Geometric sum	$S_n = u_1(r^n - 1)/(r - 1),\ r \neq 1$
Sum to infinity	$S_\infty = u_1/(1 - r),\ \lvert r\rvert < 1$
Binomial ($n \in \mathbb{Z}^+$)	$(a+b)^n = \sum_{r=0}^{n} nCr\ a^{n-r}\, b^r$
Binomial coefficient	$nCr = n! / (r!(n - r)!)$
Change of base	$\log_a x = (\log_b x)/(\log_b a)$
(HL) Permutations	$nPr = n!/(n - r)!$
(HL) Modulus / argument	$\lvert z\rvert = \sqrt{(a^2+b^2)}$; $z = r \operatorname{cis} \theta = r\, e^{i\theta}$
(HL) De Moivre	$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis} n\theta$

14. Test yourself

Attempt all ten without notes, then check against the full solutions that follow. Questions 7–10 are Higher Level.

1. Evaluate $(6.0 \times 10^4) \times (5.0 \times 10^{-7})$, giving the answer in standard form.

2. An arithmetic sequence has $u_1 = 5$ and $d = 3$. Find u_{10} and S_{10}.

3. A geometric sequence has $u_1 = 40$ and $r = 0.8$. Find its sum to infinity.

4. Evaluate $\sum_{r=1}^{5} (3r - 1)$.

5. Solve $4 \times 3^x = 100$, giving x to 3 s.f.

6. Find the coefficient of x^2 in the expansion of $(1 + 2x)^6$.

7. (HL) In how many ways can a president, a secretary and a treasurer be chosen from a club of 10 members (all posts different)?

8. (HL) Prove by induction that $\sum_{r=1}^{n} r = n(n+1)/2$.

9. (HL) Write $z = -1 + i$ in modulus–argument form.

10. (HL) Express $(5x - 1)/((x + 1)(x - 2))$ in partial fractions.

Full solutions

1. Multiply the numbers and add the powers: $6.0 \times 5.0 = 30$ and $10^4 \times 10^{-7} = 10^{-3}$, giving $30 \times 10^{-3} =$ $\mathbf{3.0 \times 10^{-2}}$.

2. $u_{10} = 5 + 9(3) =$ **32**. $S_{10} = 10/2\ (5 + 32) = 5 \times 37 =$ **185**.

3. $|r| = 0.8 < 1$, so $S_\infty = 40/(1 - 0.8) = 40/0.2 =$ **200**.

4. Terms are 2, 5, 8, 11, 14 (arithmetic, $u_1 = 2$, $d = 3$, $n = 5$). Sum $= 5/2\ (2 + 14) =$ **40**.

5. $3^x = 25$, so $x = \log 25 / \log 3 = 1.39794/0.47712 =$ **2.93** (3 s.f.).

6. General term $6Cr\ (2x)^r$; for x^2 take $r = 2$: $6C2\ (2)^2 = 15 \times 4 =$ **60**.

7. (HL) Order matters, so $10P3 = 10!/7! = 10 \times 9 \times 8 =$ **720** ways.

8. (HL) Base $n = 1$: LHS = 1, RHS = $1(2)/2 = 1$. Assume $\sum_{r=1}^{k} r = k(k+1)/2$. Then $\sum_{r=1}^{k+1} r = k(k+1)/2 + (k+1) = (k+1)(k+2)/2$, the formula with $n = k+1$. By induction it holds for all $n \in \mathbb{Z}^+$.

9. (HL) $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. z is in the second quadrant, so $\arg(z) = 3\pi/4$. Thus $z = \sqrt{2}\ \mathrm{cis}(3\pi/4)$.

10. (HL) Set $= A/(x + 1) + B/(x - 2)$, so $5x - 1 = A(x - 2) + B(x + 1)$. $x = -1$: $-6 = -3A \Rightarrow A = 2$. $x = 2$: $9 = 3B \Rightarrow B = 3$. Answer: $2/(x + 1) + 3/(x - 2)$.