Chapter 1 : Algebra

Notes by : Ashar Ejaz

Subtopic 1 : Partial Fractions

- If the power of any variable in the denominator is equal or less than the power of any variable in numerator, the fraction is called improper fraction.

Examples :

(i) $\frac{x+1}{x-1}$ (same powers) (ii) $\frac{x^2+4}{(x+1)(x+4)}$ (same powers) (iii) $\frac{x^3+5}{x^2+4x+2}$ (larger power)

- If the power of any variable in denominator is larger than any numerator variable's power then it is called proper fraction

Examples :

(i) $\frac{x+1}{x^2+5}$ (ii) $\frac{x^2+6}{(x+1)(x+2)(x+3)}$

- We use long division method to convert an improper fraction to a proper fraction

Examples

(i) $\frac{6x}{2x+1}$ (ii) $\frac{3x^3-2}{2x-1}$ (iii) $\frac{x^3+2x^2-3x+1}{x^2+x+3}$

(i)

$$\begin{array}{r|l} & 3 \\ \hline 2x+1 & 6x \\ & -(6x+3) \\ \hline & -3 \end{array}$$

$\frac{6x}{2x} = 3$

$$\begin{array}{r|l} & c \\ \hline a & b \\ & -d \\ \hline & e \end{array}$$

we will write the fraction in clockwise

$$\boxed{c + \frac{e}{a}}$$

$$3 - \frac{3}{2x+1} \quad \text{Ans}$$

(ii)

$$\begin{array}{r|l} & \frac{3}{2}x^2 + \frac{3}{4}x + \frac{3}{8} \\ \hline 2x-1 & 3x^3 + 0x^2 + 0x - 2 \\ & -(3x^3 - \frac{3}{2}x^2) \\ \hline & \frac{3}{2}x^2 + 0x - 2 \\ & -(\frac{3}{2}x^2 - \frac{3}{4}x) \\ \hline & \frac{3}{4}x - 2 \\ & -(\frac{3}{4}x - \frac{3}{8}) \\ \hline & -\frac{13}{8} \end{array}$$

- $\frac{3x^3}{2x} = \frac{3}{2}x^2$
- $\frac{3x^2/2}{2x} = \frac{3x^2}{4x} = \frac{3}{4}x$
- $\frac{3x/4}{2x} = \frac{3x}{8x} = \frac{3}{8}$

$-2 + \frac{3}{8} = -\frac{13}{8}$

$$\frac{3x^2}{2} + \frac{3}{4}x + \frac{3}{8} - \frac{13/8}{2x-1} \quad \text{Ans}$$

(iii)

$$\begin{array}{r|l} & x+1 \\ \hline x^2+x+3 & x^3+2x^2-3x+1 \\ & -(x^3+x^2+3x) \\ \hline & x^2-6x+1 \\ & -(x^2+x+3) \\ \hline & -7x-2 \end{array}$$

$\frac{x^3}{x^2} = x$

$\frac{x^2}{x^2} = 1$

$$x+1 - \frac{7x-2}{x^2+x+3} \quad \text{Ans}$$

CASES OF PARTIAL FRACTIONS

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• CASE 1 : LINEAR AND DISTINCT FACTORS

$$\frac{x+1}{(x-3)(x+4)}$$

- We will always first check if the fraction is proper or improper. If it is improper then first convert the fraction to proper fraction.
  → In the case above its proper so we will directly solve

- The roots $(x-3)(x+4)$ are in the linear form "$mx+c$" and the roots are distinct ($x-3 \neq x+4$) in this case

- we will apply same number of constants as number of factors

- $$\frac{x+1}{(x-3)(x+4)} \longrightarrow \frac{A}{(x-3)} + \frac{B}{(x+4)}$$

- $$\frac{x^2+6}{(x+1)(x+2)(x-3)} \longrightarrow \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}$$

- Now we will solve using the cover method

$$\frac{x+1}{(x-3)(x+4)} = \frac{A(x+4)+B(x-3)}{(x-3)(x+4)}$$

$$x+1 = A(x+4)+B(x-3)$$

To find A, make B zero by putting $x=3$

To find B, make A zero by putting $x=-4$

$(3)+1 = A[(3)+4] + B[(3)-3]$

$4 = A(7) + B(0)$

$4 = 7A$

$A = 4/7$

$(-4)+1 = A[(-4)+4] + B[(-4)-3]$

$-3 = A(0) + B(-7)$

$-3 = -7B$

$B = 3/7$

$$\frac{A}{(x-3)} + \frac{B}{(x+4)}$$

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$$\frac{4/7}{(x-3)} + \frac{3/7}{(x+4)}$$ Ans

- CASE 2 : LINEAR AND REPEATED FRACTIONS

$$\frac{4x}{(x-1)(x+1)^2}$$

- In this case, the factors will be repeated

$$\frac{4x}{(x-1)(x+1)(x+1)}$$

- We will apply same number of constants as number of factors and then solve

- $$\frac{6x^2}{(x-1)(x+4)^3}$$

$$\frac{6x^3}{(x-1)(x+4)(x+4)(x+4)} = \frac{A}{(x-1)} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2} + \frac{D}{(x+4)^3}$$

- $$\frac{4x}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$$

Example :

$$\frac{2x+1}{(1+3x)(x-2)^2} = \frac{A}{(1+3x)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$\frac{2x+1}{(1+3x)(x-2)^2} = \frac{A(x-2)(x-2)^2 + B(1+3x)(x-2)^2 + C(1+3x)(x-2)}{(1+3x)(x-2)(x-2)^2}$$

$$= \frac{\cancel{(x-2)}\left[A(x-2)^2 + B(1+3x)(x-2) + C(1+3x)\right]}{(1+3x)\cancel{(x-2)}(x-2)^2}$$

$$\frac{2x+1}{(1+3x)(x-2)^2} = \frac{A(x-2)^2 + B(1+3x)(x-2) + C(1+3x)}{(1+3x)(x-2)^2}$$

$$2x+1 = A(x-2)^2 + B(1+3x)(x-2) + C(1+3x)$$

To find A, make B and C zero by putting $x = -1/3$

$$2(-1/3)+1 = A[(-1/3)-2]^2 + B(1+3(-1/3))((-1/3)-2) + C(1+3(-1/3))$$

$$\frac{1}{3} = A(-7/3)^2 + B(0) + C(0)$$

$$\frac{1}{3} = \frac{49}{9}A$$

$$A = 3/49$$

To find C, make A and B zero by putting $x = 2$

$$2(2)+1 = A(2-2)^2 + B(1+3(2))(2-2) + C(1+3(2))$$

$$5 = A(0) + B(0) + C(7)$$

$$5 = 7C$$

$$C = 5/7$$

To find C, firstly we put the values of A and C in the equation and then put any value of x of our choice which has not been used above. (for e.g we cant use $x = \frac{-1}{3}$ or $x = 2$)

$A = 3/49 \quad C = 5/7 \quad x = 0$

$$2(0)+1 = A(0-2)^2 + B(1+3(0))(0-2) + C(1+3(0))$$

$$1 = A(4) + B(-2) + C(1)$$

$$1 = \left(\frac{3}{49} \times 4\right) + \left(\frac{5}{7} \times 1\right) + B(-2)$$

$$1 = \frac{47}{49} - 2B$$

$$\frac{2}{49} = -2B$$

$$B = -1/49$$

$$\frac{A}{(1+3x)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$\frac{3/49}{1+3x} - \frac{1/49}{x-2} + \frac{5/7}{(x-2)^2}$$ Ans

- CASE 3 : IRREDUCIBLE QUADRATIC FACTORS

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$$a^2 - b^2 = (a+b)(a-b)$$

$$a^2 + b^2 = (a+ib)(a-ib)$$

- The expression a^2+b^2 can only expand in complex numbers hence it is a irreducible quadratic factor

Example :

$$\frac{4x}{(x^2+9)(1+x)} = \frac{Ax+B}{x^2+9} + \frac{C}{1+x}$$

$$\frac{4x}{(x^2+9)(1+x)} = \frac{(Ax+B)(1+x)+C(x^2+9)}{(x^2+9)(1+x)}$$

$$4x = (Ax+B)(1+x)+C(x^2+9)$$

$$4x = Ax + Ax^2 + B + Bx + Cx^2 + 9C$$

$$4x = Ax^2 + Cx^2 + Bx + Ax + B + 9C$$

$$0x^2 + 4x + 0 = x^2(A+C) + x(B+A) + B+9C$$

$$0x^2 = x^2(A+C) \qquad 4x = x(B+A) \qquad 0 = B+9C$$

$$0 = A+C \qquad 4 = B+A \qquad B = -9C$$

$$A = -C \qquad A = 4-B$$

$$-C = 4-(-9C)$$

$$-C = 4+9C$$

$$-4 = 10C$$

$$C = -4/10$$

$$C = -2/5$$

$$A = -(-2/5) \qquad B = -9(-2/5)$$

$$A = 2/5 \qquad B = 18/5$$

* This part can directly be calculated using calculator as these steps are not required to show

$$\frac{Ax+B}{x^2+9} + \frac{C}{1+x} \rightarrow \boxed{\frac{\frac{2}{5}x+\frac{18}{5}}{x^2+9} - \frac{2/5}{1+x}}$$

ANS