A LEVEL
NOTES (P1)

A LEVEL MATHEMATICS

COMPILED BY
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MS BOOKS

A-Level Mathematics Paper 1

Notes

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1.1 Quadratics

Candidates should be able to:	Notes and examples
• carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$ and use a completed square form	e.g. to locate the vertex of the graph of $y = ax^2 + bx + c$ or to sketch the graph
• find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant	e.g. to determine the number of real roots of the equation $ax^2 + bx + c = 0$. Knowledge of the term 'repeated root' is included.
• solve quadratic equations, and quadratic inequalities, in one unknown	By factorising, completing the square and using the formula.
• solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic	e.g. $x + y + 1 = 0$ and $x^2 + y^2 = 25$, $2x + 3y = 7$ and $3x^2 = 4 + 4xy$.
• recognise and solve equations in x which are quadratic in some function of x.	e.g. $x^4 - 5x^2 + 4 = 0$, $6x + \sqrt{x} - 1 = 0$, $\tan^2 x = 1 + \tan x$.

1 Pure Mathematics 1

1.2 Functions

Candidates should be able to:	Notes and examples
• understand the terms function, domain, range, one-one function, inverse function and composition of functions	
• identify the range of a given function in simple cases, and find the composition of two given functions	e.g. range of $f : x \mapsto \frac{1}{x}$ for $x \geqslant 1$ and range of $g : x \mapsto x^2 + 1$ for $x \in \mathbb{R}$. Including the condition that a composite function gf can only be formed when the range of f is within the domain of g.
• determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases	e.g. finding the inverse of $h : x \mapsto (2x + 3)^2 - 4$ for $x < -\frac{3}{2}$.
• illustrate in graphical terms the relation between a one-one function and its inverse	Sketches should include an indication of the mirror line $y = x$.
• understand and use the transformations of the graph of $y = f(x)$ given by $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, $y = f(ax)$ and simple combinations of these.	Including use of the terms 'translation', 'reflection' and 'stretch' in describing transformations. Questions may involve algebraic or trigonometric functions, or other graphs with given features.

1.3 Coordinate geometry

Candidates should be able to:	Notes and examples
• find the equation of a straight line given sufficient information	e.g. given two points, or one point and the gradient.
• interpret and use any of the forms $y = mx + c$, $y - y_1 = m(x - x_1)$, $ax + by + c = 0$ in solving problems	Including calculations of distances, gradients, midpoints, points of intersection and use of the relationship between the gradients of parallel and perpendicular lines.
• understand that the equation $(x - a)^2 + (y - b)^2 = r^2$ represents the circle with centre (a, b) and radius r	Including use of the expanded form $x^2 + y^2 + 2gx + 2fy + c = 0$.
• use algebraic methods to solve problems involving lines and circles	Including use of elementary geometrical properties of circles, e.g. tangent perpendicular to radius, angle in a semicircle, symmetry. Implicit differentiation is not included.
• understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.	e.g. to determine the set of values of k for which the line $y = x + k$ intersects, touches or does not meet a quadratic curve.

1.4 Circular measure

Candidates should be able to:	Notes and examples
• understand the definition of a radian, and use the relationship between radians and degrees	
• use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ in solving problems concerning the arc length and sector area of a circle.	Including calculation of lengths and angles in triangles and areas of triangles.

1.5 Trigonometry

Candidates should be able to:	Notes and examples
• sketch and use graphs of the sine, cosine and tangent functions (for angles of any size, and using either degrees or radians)	Including e.g. $y = 3\sin x$, $y = 1 - \cos 2x$, $y = \tan\left(x + \frac{1}{4}\pi\right)$.
• use the exact values of the sine, cosine and tangent of 30°, 45°, 60°, and related angles	e.g. $\cos 150^\circ = -\frac{1}{2}\sqrt{3}$, $\sin \frac{3}{4}\pi = \frac{1}{\sqrt{2}}$.
• use the notations $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ to denote the principal values of the inverse trigonometric relations	No specialised knowledge of these functions is required, but understanding of them as examples of inverse functions is expected.
• use the identities $\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$	e.g. in proving identities, simplifying expressions and solving equations.
• find all the solutions of simple trigonometrical equations lying in a specified interval (general forms of solution are not included).	e.g. solve $3\sin 2x + 1 = 0$ for $-\pi < x < \pi$, $3\sin^2\theta - 5\cos\theta - 1 = 0$ for $0^\circ \leqslant \theta \leqslant 360^\circ$.

1.6 Series

Candidates should be able to:	Notes and examples
• use the expansion of $(a + b)^n$, where n is a positive integer	Including the notations $\binom{n}{r}$ and $n!$ Knowledge of the greatest term and properties of the coefficients are not required.
• recognise arithmetic and geometric progressions	
• use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions	Including knowledge that numbers a, b, c are 'in arithmetic progression' if $2b = a + c$ (or equivalent) and are 'in geometric progression' if $b^2 = ac$ (or equivalent). Questions may involve more than one progression.
• use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.	

1.7 Differentiation

Candidates should be able to:	Notes and examples
• understand the gradient of a curve at a point as the limit of the gradients of a suitable sequence of chords, and use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ for first and second derivatives	Only an informal understanding of the idea of a limit is expected. e.g. includes consideration of the gradient of the chord joining the points with x coordinates 2 and $(2+h)$ on the curve $y = x^3$. Formal use of the general method of differentiation from first principles is not required.
• use the derivative of x^n (for any rational n), together with constant multiples, sums and differences of functions, and of composite functions using the chain rule	e.g. find $\frac{dy}{dx}$, given $y = \sqrt{2x^3 + 5}$.
• apply differentiation to gradients, tangents and normals, increasing and decreasing functions and rates of change	Including connected rates of change, e.g. given the rate of increase of the radius of a circle, find the rate of increase of the area for a specific value of one of the variables.
• locate stationary points and determine their nature, and use information about stationary points in sketching graphs.	Including use of the second derivative for identifying maxima and minima; alternatives may be used in questions where no method is specified. Knowledge of points of inflexion is not included.

1.8 Integration

Candidates should be able to:	Notes and examples
• understand integration as the reverse process of differentiation, and integrate $(ax+b)^n$ (for any rational n except –1), together with constant multiples, sums and differences	e.g. $\int (2x^3 - 5x + 1)\,dx$, $\int \frac{1}{(2x+3)^2}\,dx$.
• solve problems involving the evaluation of a constant of integration	e.g. to find the equation of the curve through $(1, -2)$ for which $\frac{dy}{dx} = \sqrt{2x+1}$.
• evaluate definite integrals	Including simple cases of 'improper' integrals, such as $\int_0^1 x^{-\frac{1}{2}}\,dx$ and $\int_1^\infty x^{-2}\,dx$.
• use definite integration to find – the area of a region bounded by a curve and lines parallel to the axes, or between a curve and a line or between two curves – a volume of revolution about one of the axes.	A volume of revolution may involve a region not bounded by the axis of rotation, e.g. the region between $y = 9 - x^2$ and $y = 5$ rotated about the x-axis.

Contents

COORDINATE GEOMETRY

The study of points, straight lines and curves defined by algebraic expression is called coordinate geometry.

Distance between two points or length of line segment

If $A(x_1,y_1)$ and B (x_2,y_2) are two points, then distance between A and B is AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Midpoint of a line segment

The coordinates of mid point of A (x_1, y_1) and B (x_2, y_2) are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

The diagonals of rectangle, rhombus, square and parallelogram have common mid points. To find out coordinate of fourth point of these shapes, when other three points are given, first supposed coordinate of unknown point (x,y), then put mid points of both diagram equal and find (x,y).

Gradient of straight line

Gradient is the measure of slope of line with respect to the positive x – axis.

The tangent of the angle counted anti clock wise, which a line make with positive x-axis is called gradient of line.

OR

The increase in the y – coordinate divided by the increase in the x- coordinate of two pints on the line is called gradient usually it is denoted by 'm' $m = \frac{y_2-y_1}{x_2-x_1}$

Properties of gradient

1. A line parallel to x-axis has gradient 0.
2. A line parallel to y-axis has gradient undefined. (∞)

3. If points are collinear (lie on same straight line) then their gradients are equal.
4. If two lines are parallel, then their gradients are equal. To find gradient of parallel line, first make 'y' as a subject and then take co-efficient of x as a gradient.
5. If two lines are perpendicular or normal, then product of their gradient is equal to -1, i.e. $m_1 \times m_2 = -1$
6. If gradient is +ive, then angle made by line with +ive x -axis is acute. If gradient is –ive, then angle of line with x-axis is obtuse.

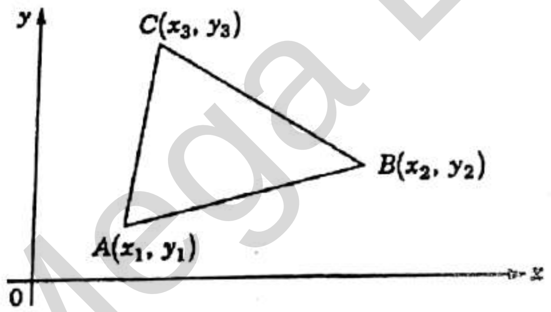
Equation of straight line

1. Equation of straight line when its gradient (m) and y – intercept (c) is given $y = mx+c$. where x and y are the variables of the equation.
2. Equation of straight line when its gradient (m) and a point (x_1,y_1) is given OR equation of line passing through two points OR equation of tangent is $y-y_1 = m(x-x_1)$.
   To show that given straight line is tangent to the curve. First substitute equation of line in equation of curve, simplify it, then show that $b^2-4ac=0$.
3. Equation of perpendicular or normal or perpendicular bisector is
   $y-y_1 = -\frac{1}{m}(x-x_1)$.
   In case of perpendicular bisector (x_1, y_1) = mid point.

Use of graph to find unknown coordinates

- To find unknown point in case of two intersecting lines, find their point of intersection of two lines by elimination or by substitution method.
- To find point of intersection of a line and curve substitute value of any variable of linear equation in equation of curve and find x, y co-ordinate.
- In quadrilateral if one diagonal is line of symmetry of given figure, then point of intersection of diagonals will be the mid point of other diagonal.

AREA OF POLYGONS



Given vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the area of ΔABC is

Area of ΔABC	$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ $= \frac{1}{2}\left\|(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\|$

} arrange the 3 points in anti-clockwise direction ends with the start point

arrange $\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ in cross pattern |\\\\\\−///|

Example

The ends of a line segment are $(p-q,p+q)$ and $(p+q,p-q)$. Find the length of the line segment, its gradient and the coordinates of its mid-point.

For the length and gradient you have to calculate

$$x_2-x_1=(p+q)-(p-q)=p+q-p+q=2q$$

and $y_2-y_1=(p-q)-(p+q)=p-q-p-q=-2q$.

The length is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(2q)^2+(-2q)^2}=\sqrt{4q^2+4q^2}=\sqrt{8q^2}$.

The gradient is $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{-2q}{2q}=-1$.

For the mid-point you have to calculate

$$x_1+x_2=(p-q)+(p+q)=p-q+p+q=2p$$

and $y_1+y_2=(p+q)+(p-q)=p+q+p-q=2p$.

The mid-point is $\left(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2)\right)=\left(\frac{1}{2}(2p),\frac{1}{2}(2p)\right)=(p,p)$.

Example

Prove that the points $A(1,1)$, $B(5,3)$, $C(3,0)$ and $D(-1,-2)$ form a parallelogram.

(using mid-points) In this method, begin by finding the mid-points of the diagonals AC and BD. If these points are the same, then the diagonals bisect each other, so the quadrilateral is a parallelogram.

The mid-point of AC is $\left(\frac{1}{2}(1+3),\frac{1}{2}(1+0)\right)$, which is $\left(2,\frac{1}{2}\right)$. The mid-point of BD is $\left(\frac{1}{2}(5+(-1)),\frac{1}{2}(3+(-2))\right)$, which is also $\left(2,\frac{1}{2}\right)$. So $ABCD$ is a parallelogram.

Example

Find the equation of the line joining the points $(3,4)$ and $(-1,2)$.

To find this equation, first find the gradient of the line joining $(3,4)$ to $(-1,2)$. Then you can use the equation $y-y_1=m(x-x_1)$.

The gradient of the line joining $(3,4)$ to $(-1,2)$ is $\dfrac{2-4}{(-1)-3}=\dfrac{-2}{-4}=\frac{1}{2}$.

The equation of the line through $(3,4)$ with gradient $\frac{1}{2}$ is $y-4=\frac{1}{2}(x-3)$. After multiplying out and simplifying you get $2y-8=x-3$, or $2y=x+5$.

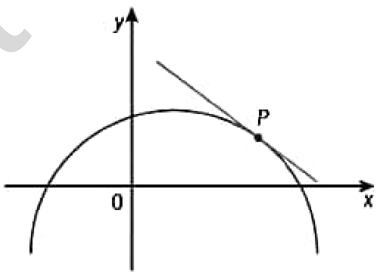
3.5 Points of intersection and circle properties

You know that the equation of a straight line is given by: $y = mx + c$, where m is the gradient and c is the y-intercept. In Chapter 1 you learnt that you can find the points of intersection of a line and a curve by solving their equations simultaneously.

When a line touches a curve in exactly one place we call that line a tangent to the curve.

This also means that there will only be one solution when solving the equations simultaneously.

If you form a quadratic equation there will be only **one (repeated)** root,
i.e. $b^2 - 4ac = 0$



Example 12

Find the centre and radius of the circle with equation $x^2 + y^2 - 10x + 12y + 12 = 0$.

Working	Explanation
$x^2 - 10x + y^2 + 12y + 12 = 0$	First rearrange the equation to this form.
$(x - 5)^2 - 25 + (y + 6)^2 - 36 + 12 = 0$	Complete the square for x and y.
$(x - 5)^2 + (y + 6)^2 = 49$	Rearrange to this form.
Centre is $(5, -6)$, radius $= 7$.	$a = 5$, $b = -6$, $r = \sqrt{49}$

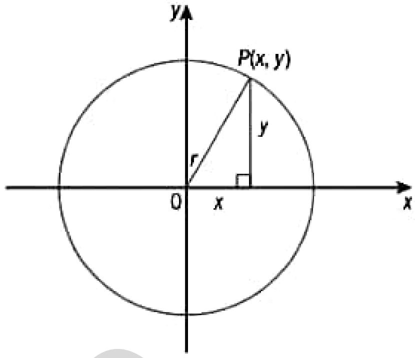
Example 13

Find the equation of the circle with centre $(4, -3)$ that passes through the point $(-2, 5)$.

Working	Explanation
$(x - 4)^2 + (y + 3)^2 = r^2$	Write the equation of circle centre $(4, -3)$.
$(-2 - 4)^2 + (5 + 3)^2 = r^2$	Substitute for $x = -2$ and $y = 5$.
$36 + 64 = r^2$	Simplify
$r^2 = 100$	Work out r^2.
Equation of the circle is $(x - 4)^2 + (y + 3)^2 = 100$	Write down the equation of the circle.

3.4 Circles

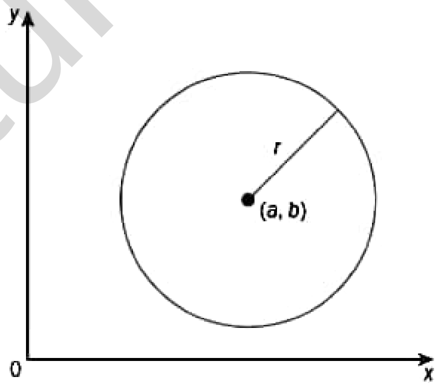
If we take any point $P(x, y)$ on the circumference of a circle, centre O, we can work out the equation of the circle using Pythagoras' theorem.



The equation of a circle, centre (0, 0), radius r, is given by $x^2 + y^2 = r^2$.

If we translate this circle by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$,

we can get the equation of any circle, centre (a, b), radius r.



The equation of a circle, centre (a, b), radius r is given by

$$(x - a)^2 + (y - b)^2 = r^2.$$

If we expand this equation, we get $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$. When this is simplified, we get the expanded form of the equation of a circle:

$x^2 + y^2 - 2ax - 2by + c = 0$ where $c = a^2 + b^2 - r^2$

Note: When the circle is given in this expanded form, we can use the method of completing the square to write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.

The expanded form of the equation of a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ where } c = f^2 + g^2 - r^2.$$

Example 11

Find the equation of the circle with centre (−5, 7) and radius 6.

$(x + 5)^2 + (y - 7)^2 = 36$ ← Substitute into $(x - a)^2 + (y - b)^2 = r^2$, where $a = -5$, $b = 7$ and $r = 6$.

Example 14

Find the set of values of k for which the line $x = y + 2$ intersects the curve $y = x^2 + 3x + k$ at two distinct points.

They meet when $x^2 + 3x + k = x - 2$.

$x^2 + 2x + (k + 2) = 0$ ← Rearrange as a quadratic equation.

There are two distinct roots when $b^2 - 4ac > 0$.

$(2)^2 - 4(1)(k + 2) > 0$ ← $a = 1$, $b = 2$ and $c = k + 2$

$-4k - 4 > 0$

$-4k > 4$ ← Use inequality signs throughout.

$k < -1$

Example 15

A line has equation $y = 4k - x$ and a curve has equation $y = 3x - kx^2$, where k is a constant.

a) Find the two values of k for which the line is a tangent to the curve.

b) Hence find the coordinates of the points where the line touches the curve.

a) The line and the curve meet when

$4k - x = 3x - kx^2$ ← Both equal y.

$kx^2 - 4x + 4k = 0$ ← Rearrange in the form $ax^2 + bx + c = 0$.

The line is a tangent so there is one repeated root.

$(-4)^2 - 4(k)(4k) = 0$ ← $b^2 - 4ac = 0$

$16 = 16k^2$, $k^2 = 1$

$k = 1$ or $k = -1$

b) When $k = 1$:

$x^2 - 4x + 4 = 0$ ← Substitute in $kx^2 - 4x + 4k = 0$.

$(x - 2)(x - 2) = 0$

$x = 2$, $y = 4k - x = 4 - 2 = 2$

When $k = -1$:

$-x^2 - 4x - 4 = 0$

$x^2 + 4x + 4 = 0$

$(x + 2)(x + 2) = 0$

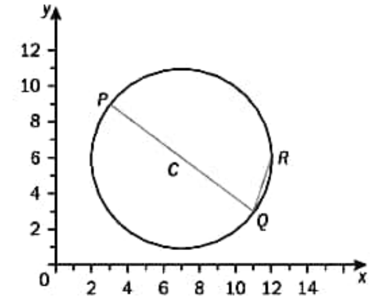
$x = -2$, $y = 4k - x = -4 + 2 = -2$

Line meets curve at (2, 2) and (−2, −2). ← The question asked for coordinates.

Example 18

The diagram shows a circle, centre C, with equation $(x-7)^2 + (y-6)^2 = 25$. PQ is a diameter, where P is (3, 9) and Q is (11, 3). $R(12, 6)$ lies on the circle.

Show that the line perpendicular to QR and passing through R goes through P.



QR has gradient $\frac{6-3}{12-11} = 3$, so the gradient of the perpendicular is $-\frac{1}{3}$. ← Gradients of perpendicular lines multiply to −1.

The equation of the perpendicular line through (12, 6) is given by

$y - 6 = -\frac{1}{3}(x - 12)$ ← Substitute the gradient and known point to get the equation of the line.

$y = -\frac{1}{3}x + 10$ ← Rearrange to standard form.

When $x = 3$, $y = -\frac{1}{3} \times 3 + 10 = 9$ so $P(3, 9)$ lies on the line. ← Always finish the answer to a 'show that' question with a statement.

Example 19

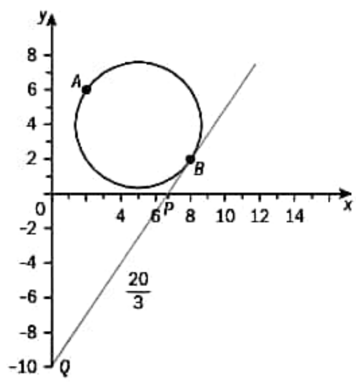
AB is the diameter of a circle, where A is (2, 6) and B is (8, 2).

The tangent to the circle at B meets the x-axis at P and the y-axis at Q. Find the coordinates of P and Q.

A tangent is perpendicular to the diameter and we have two points defining the diameter.

AB has gradient $\frac{2-6}{8-2} = -\frac{2}{3}$, so the gradient of the tangent at B is $\frac{3}{2}$.

The equation of the perpendicular line through (8, 2) is given by



$y - 2 = \frac{3}{2}(x - 8)$. ← Substitute the gradient and known point to get the equation of the line.

$y = \frac{3}{2}x - 10$ ← Rearrange to standard form.

▶ Continued on the next page

When $y = 0$, $0 = \frac{3}{2}x - 10 \Rightarrow x = \frac{20}{3}$ ← Substitute $x = 0$ and $y = 0$ to find coordinates of intercepts.

so P is $\left(\frac{20}{3}, 0\right)$.

When $x = 0$, $y = -10$ so Q is (0, −10).

Exercise 3.5

1. Find the value of k for which the line $y = 2kx + 7$ is a tangent to the curve $y = 3 + kx^2$.

2. A circle with centre C has equation $x^2 + y^2 - 6x + 4y + 8 = 0$.
   a) Express the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.
   b) Find the coordinates of C and the radius of the circle.

3. Find the set of values of k for which the line $y = kx - 4$ intersects the curve $y = x^2$ at two distinct points.

4. The line $y = 5 - kx$, where k is an integer, is a tangent to the curve $y = 2k - x^2$.
   a) Find the possible values of k.
   When $k = 2$, the line $y = 5 - kx$ is a tangent to the curve $y = 2k - x^2$ at point A.
   b) Find the coordinates of A.

5. Determine the shortest distance from the point $A(1, 3)$ to the circle with equation $x^2 + y^2 - 10x - 12y + 45 = 0$.

6. A curve has equation $y = 2x^2 + kx - 1$ and a line has equation $x + y + k = 0$, where k is a constant.
   a) State the value of k for which the line is a tangent to the curve.
   b) For this value of k find the coordinates of the point where the line touches the curve.

7. The equation of a line is $y = x - k$, where k is a constant, and the equation of a curve is $x^2 + 2y = k$.
   a) When $k = 1$, the line intersects the curve at the points A and B. Find the coordinates of A and the coordinates of B.
   b) Find the value of k for which the line $y = x - k$ is a tangent to the curve $x^2 + 2y = k$.

8. A circle with centre C has equation $x^2 + y^2 - 8x - 6y - 20 = 0$.
   a) Find the coordinates of C and the radius of the circle.
   b) $A(10, 0)$ lies on the circle. Find the equation of the tangent to the circle at A.

9. Find the equation of the tangent to the circle $x^2 + y^2 - 12x + 26 = 0$ at the point $P(3, 1)$.
   Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

10. A line has equation $y = 2kx - 9$ and a curve has equation $y = x^2 - kx$, where k is a constant.
    a) Find the two values of k for which the line is a tangent to the curve.
    b) For each value of k, find the coordinates of the point where the line is a tangent to the curve, and find the equation of the line that joins these two points.

11. $P(2, 1)$ is a point on the circumference of the circle $x^2 + y^2 - 10x + 2y + 13 = 0$. PQ is a diameter of the circle. Find the equation of the line through P and Q.

12. Show that the line with equation $y = 2x - 3$ cannot be a tangent to the curve with equation $y = x^2 - 3x + 5$.

13. A line has equation $y = 2kx - 7$ and a curve has equation $y = x^2 + kx - 3$, where k is a positive integer.
    a) Find the value of k for which the line is a tangent to the curve.
    b) For this value of k, find the coordinates of the point where the line touches the curve.

Exercise 3.5 page 65

1. $k = -4$
2. a) $(x-3)^2 + (y+2)^2 = 5$
   b) Centre $(3, -2)$, radius $\sqrt{5}$
3. $k < -4$, $k > 4$
4. a) $k = 2$ or $k = -10$ b) $(1, 3)$
5. 1
6. a) $k = 3$ b) $(-1, -2)$
7. a) $(-3, -4)$ and $(1, 0)$ b) $k = -\frac{1}{3}$
8. a) $C(4, 3)$, radius $3\sqrt{5}$ b) $y = 2x - 20$
9. $3x - y - 8 = 0$
10. a) $k = 2$ or $k = -2$
    b) $(3, 3)$ and $(-3, 3)$; $y = 3$
11. $2x + 3y - 7 = 0$
12. Proof

Practice Questions

Exercise #1

1 A line joining a vertex of a triangle to the mid-point of the opposite side is called a median. Find the length of the median AM in the triangle $A(-1,1)$, $B(0,3)$, $C(4,7)$.

2 A triangle has vertices $A(-2,1)$, $B(3,-4)$ and $C(5,7)$.

(a) Find the coordinates of M, the mid-point of AB, and N, the mid-point of AC.

(b) Show that MN is parallel to BC.

3 The points $A(2,1)$, $B(2,7)$ and $C(-4,-1)$ form a triangle. M is the mid-point of AB and N is the mid-point of AC.

(a) Find the lengths of MN and BC. (b) Show that $BC = 2MN$.

4 The vertices of a quadrilateral $ABCD$ are $A(1,1)$, $B(7,3)$, $C(9,-7)$ and $D(-3,-3)$. The points P, Q, R and S are the mid-points of AB, BC, CD and DA respectively.

(a) Find the gradient of each side of $PQRS$. (b) What type of quadrilateral is $PQRS$?

5 Find the equations of the lines joining the following pairs of poin.

(a) $(1,4)$ and $(3,10)$ (b) $(4,5)$ and $(-2,-7)$

(c) $(3,2)$ and $(0,4)$ (d) $(3,7)$ and $(3,12)$

6 Find the equation of the line through $(-2,1)$ parallel to $y = \frac{1}{2}x - 3$.

7 Find the equation of the line through $(4,-3)$ parallel to $y + 2x = 7$.

8 Find the equation of the line through $(1,2)$ parallel to the line joining $(3,-1)$ and $(-5,2)$.

9 Find the equation of the line through $(3,9)$ parallel to the line joining $(-3,2)$ and $(2,-3)$.

10 Find the equation of the line through $(1,7)$ parallel to the x-axis.

11 Find the equation of the line through the point $(-2,5)$ which is perpendicular to the line $y = 3x + 1$. Find also the point of intersection of the two lines.

12 Find the equation of the line through the point $(1,1)$ which is perpendicular to the line $2x - 3y = 12$. Find also the point of intersection of the two lines.

13 A line through a vertex of a triangle which is perpendicular to the opposite side is called an altitude. Find the equation of the altitude through the vertex A of the triangle ABC where A is the point $(2,3)$, B is $(1,-7)$ and C is $(4,-1)$.

Answers:

1 5

2 (a) M is $\left(\frac{1}{2}, -1\frac{1}{2}\right)$, N is $\left(1\frac{1}{2}, 4\right)$

3 (a) $MN = 5, BC = 10$

4 (a) Gradients PQ and SR both -1, QR and PS both $\frac{3}{5}$

(b) Parallelogram

5 (a) $y = 3x + 1$ (b) $y = 2x - 3$

(c) $2x + 3y - 12 = 0$ (d) $x - 3 = 0$

6 $y = \frac{1}{2}x + 2$

7 $y + 2x = 5$

8 $3x + 8y = 19$

9 $x + y = 12$

10 $y = 7$

11 $x + 3y = 13, (1,4)$

$3x + 2y = 5, (3, -2)$

$x + 2y = 8$

Exercise #2

1 The straight line $5y + 2x = 1$ meets the curve $xy + 24 = 0$ at the points A and B. Find the length of AB, correct to one decimal place. [6]

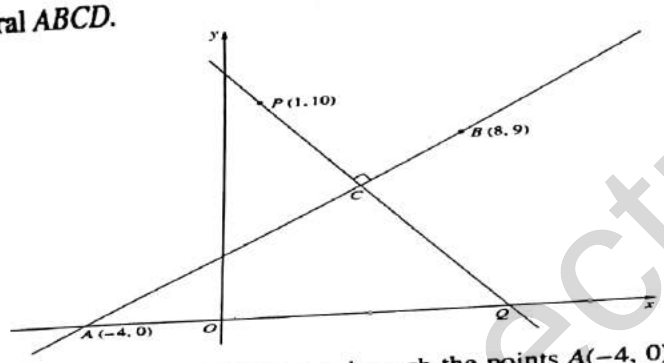
2 The line $y + 4x = 23$ intersects the curve $xy + x = 20$ at two points, A and B. Find the equation of the perpendicular bisector of the line AB. [6]

3 The straight line $3x = 2y + 18$ intersects the curve $2x^2 - 23x + 2y + 50 = 0$ at the points A and B. Given that A lies below the x-axis and that the point P lies on AB such that $AP : PB = 1 : 2$, find the coordinates of P. [6]

4 Find the coordinates of the points where the straight line $y = 2x - 3$ intersects the curve $x^2 + y^2 + xy + x = 30$. [5]

5 The points $A(-2, 2)$, $B(4, 4)$ and $C(5, 2)$ are the vertices of a triangle. The perpendicular bisector of AB and the line through A parallel to BC intersect at the point D. Find the area of the quadrilateral $ABCD$.

6



The diagram shows the line AB passing through the points $A(-4, 0)$ and $B(8, 9)$. The line through the point $P(1, 10)$, perpendicular to AB, meets AB at C and the x-axis at Q. Find

(i) the coordinates of C and of Q, [7]

(ii) the area of triangle ACQ. [2]

Answers

1 $\rightarrow$ (8,-3) and (-7.5,3.2)

$d = \sqrt{(15.5^2+6.2^2)} = 16.7$

2 $\rightarrow$ (5,3) and (1,19)
$m = -4$ Perpendicular = ¼
Mid-point = (3,11)
$\rightarrow$ $4y = x + 41$

3 (2, −6) and (8, 3)

4 $P(4, -3)$
(3,3) and (−1,−5)

5 $x = 8, y = -18$
method for area
77

Answers

1 (i) y-step ÷ x-step = 2
$\rightarrow m = 1$

$C(-1, 6) \rightarrow D(5, 12)$

2 Equation $y - 3 = -3(x - 2)$

(ii) If $x = 0$, $y = 9$, $B(0, 9)$
Vector move $D(4, -3)$
Area = 40

3 $A = (1, -1)$

$x = -\frac{1}{2}$, $y = 3\frac{1}{2}$

4 Equation is $y - 3 = \frac{-1}{5}(x - 2)$

5 Solution of sim eqns $\rightarrow a = 6, b = 3$
Equation $MB : y - 4 = -2(x - 1)$
When $y = 0, x = 3$ or $B = (3, 0)$

(iii) $D = (-1, 8)$
$AD = \sqrt{40}$ or 6.32

6 $\rightarrow (6, 1)$ and $(2, 3)$

Midpoint $M(4, 2)$
$m = -\frac{1}{2}$
Perpendicular $m = 2$
$\rightarrow y - 2 = 2(x - 4)$
$k < -\sqrt{48}$ and $k > \sqrt{48}$

7 (i) $(3\frac{1}{2}, 2)$

$x - 2y + 4 = 0$

8 $y = 2x - 2$

$(0, -2)$ and $\left(\frac{8}{5}, \frac{6}{5}\right)$

9 $y - 1 = -\frac{1}{2}(x - 3)$

$AC = 13$

Past Paper Questions

Co-ordinate Geometry

1 The line L_1 has equation $2x + y = 8$. The line L_2 passes through the point A (7, 4) and is perpendicular to L_1.

(i) Find the equation of L_2. [4]

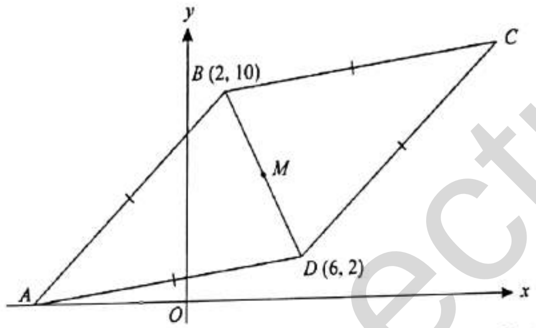
(ii) Given that the lines L_1 and L_2 intersect at the point B, find the length of AB. [4]

2 The curve $y = 9 - \frac{6}{x}$ and the line $y + x = 8$ intersect at two points. Find

(i) the coordinates of the two points, [4]

(ii) the equation of the perpendicular bisector of the line joining the two points. [4]

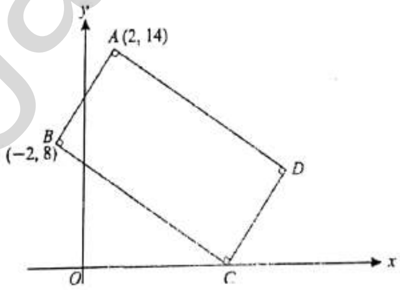
3



The diagram shows a rhombus $ABCD$. The points B and D have coordinates (2, 10) and (6, 2) respectively, and A lies on the x-axis. The mid-point of BD is M. Find, by calculation, the coordinates of each of M, A and C. [6]

4 The curve $y^2 = 12x$ intersects the line $3y = 4x + 6$ at two points. Find the distance between the two points. [6]

5

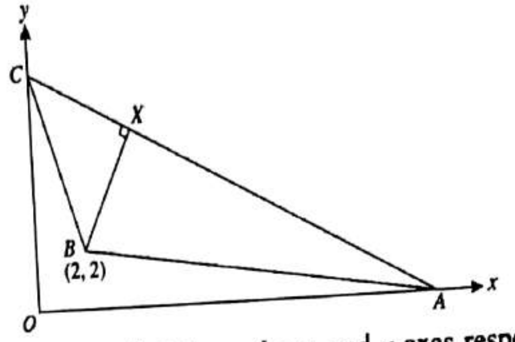


The diagram shows a rectangle $ABCD$. The point A is (2, 14), B is (−2, 8) and C lies on the x-axis. Find

(i) the equation of BC, [4]

(ii) the coordinates of C and D. [3]

6



In the diagram, the points A and C lie on the x- and y-axes respectively and the equation of AC is $2y + x = 16$. The point B has coordinates $(2, 2)$. The perpendicular from B to AC meets AC at the point X.

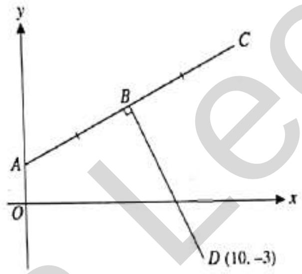
(i) Find the coordinates of X. [4]

The point D is such that the quadrilateral $ABCD$ has AC as a line of symmetry.

(ii) Find the coordinates of D. [2]

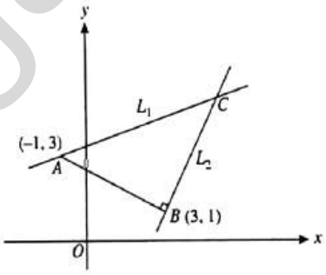
(iii) Find, correct to 1 decimal place, the perimeter of $ABCD$. [3]

7



The diagram shows points A, B and C lying on the line $2y = x + 4$. The point A lies on the y-axis and $AB = BC$. The line from $D(10, -3)$ to B is perpendicular to AC. Calculate the coordinates of B and C. [7]

8



In the diagram, A is the point $(-1, 3)$ and B is the point $(3, 1)$. The line L_1 passes through A and is parallel to OB. The line L_2 passes through B and is perpendicular to AB. The lines L_1 and L_2 meet at C. Find the coordinates of C. [6]

9 The line L_1 passes through the points $A(2, 5)$ and $B(10, 9)$. The line L_2 is parallel to L_1 and passes through the origin. The point C lies on L_2 such that AC is perpendicular to L_2. Find

(i) the coordinates of C, [5]

(ii) the distance AC. [2]

10 The point A has coordinates $(-1, -5)$ and the point B has coordinates $(7, 1)$. The perpendicular bisector of AB meets the x-axis at C and the y-axis at D. Calculate the length of CD. [6]

11 The point R is the reflection of the point $(-1, 3)$ in the line $3y + 2x = 33$. Find by calculation the coordinates of R. [7]

12 Find the coordinates of the point at which the perpendicular bisector of the line joining $(2, 7)$ to $(10, 3)$ meets the x-axis. [5]

13 The point C lies on the perpendicular bisector of the line joining the points $A(4, 6)$ and $B(10, 2)$. C also lies on the line parallel to AB through $(3, 11)$.

(i) Find the equation of the perpendicular bisector of AB. [4]

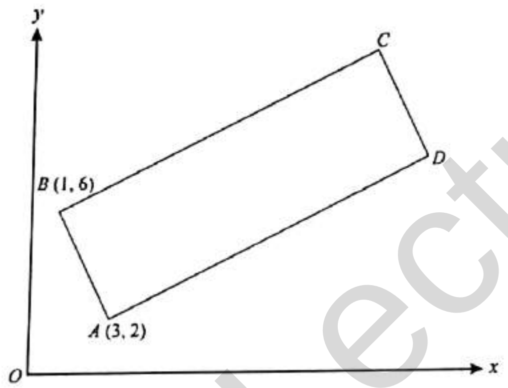
(ii) Calculate the coordinates of C. [3]

14 Three points have coordinates $A(0, 7)$, $B(8, 3)$ and $C(3k, k)$. Find the value of the constant k for which

(i) C lies on the line that passes through A and B, [4]

(ii) C lies on the perpendicular bisector of AB. [4]

15



The diagram shows a rectangle $ABCD$, where A is $(3, 2)$ and B is $(1, 6)$.

(i) Find the equation of BC. [4]

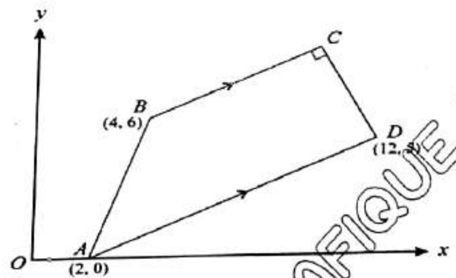
Given that the equation of AC is $y = x - 1$, find

(ii) the coordinates of C, [2]

(iii) the perimeter of the rectangle $ABCD$. [3]

16 Find the coordinates of the points of intersection of the line $y + 2x = 11$ and the curve $xy = 12$. [4]

17



The diagram shows a trapezium $ABCD$ in which BC is parallel to AD and angle $BCD = 90°$. The coordinates of A, B and D are $(2, 0)$, $(4, 6)$ and $(12, 5)$ respectively.

(i) Find the equations of BC and CD. [5]

(ii) Calculate the coordinates of C. [2]

18 The equation of a curve is $y = x^2 - 4x + 7$ and the equation of a line is $y + 3x = 9$. The curve and the line intersect at the points A and B.

(i) The mid-point of AB is M. Show that the coordinates of M are $\left(\frac{1}{2}, 7\frac{1}{2}\right)$. [4]

(ii) Find the coordinates of the point Q on the curve at which the tangent is parallel to the line $y + 3x = 9$. [3]

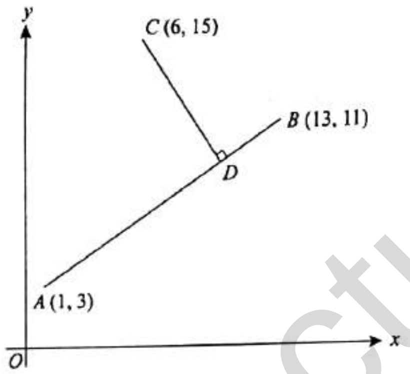
(iii) Find the distance MQ. [1]

19 Three points have coordinates $A(2, 6)$, $B(8, 10)$ and $C(6, 0)$. The perpendicular bisector of AB meets the line BC at D. Find

(i) the equation of the perpendicular bisector of AB in the form $ax + by = c$, [4]

(ii) the coordinates of D. [4]

20

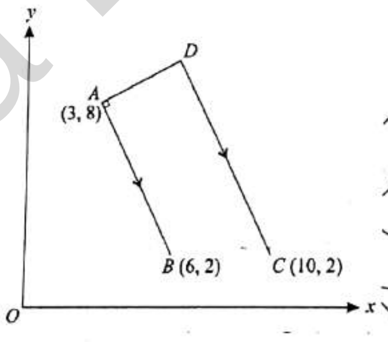


The three points $A(1, 3)$, $B(13, 11)$ and $C(6, 15)$ are shown in the diagram. The perpendicular from C to AB meets AB at the point D. Find

(i) the equation of CD, [3]

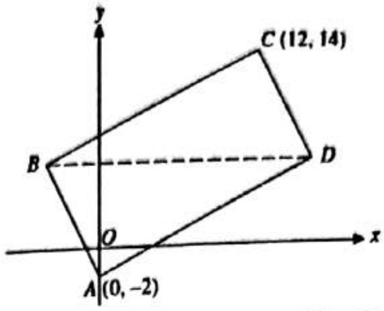
(ii) the coordinates of D. [4]

21



The three points $A(3, 8)$, $B(6, 2)$ and $C(10, 2)$ are shown in the diagram. The point D is such that the line DA is perpendicular to AB and DC is parallel to AB. Calculate the coordinates of D. [7]

22



The diagram shows a rectangle $ABCD$. The point A is $(0, -2)$ and C is $(12, 14)$. The diagonal BD is parallel to the x-axis.

(i) Explain why the y-coordinate of D is 6. [1]

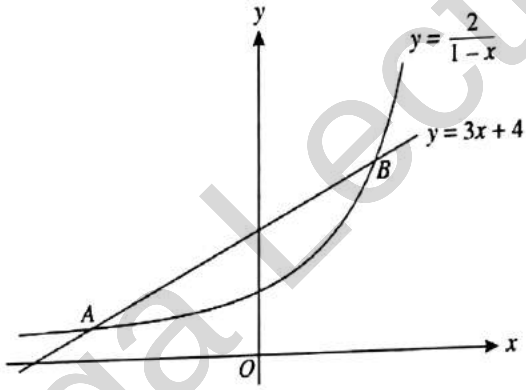
The x-coordinate of D is h.

(ii) Express the gradients of AD and CD in terms of h. [3]

(iii) Calculate the x-coordinates of D and B. [4]

(iv) Calculate the area of the rectangle $ABCD$. [3]

23

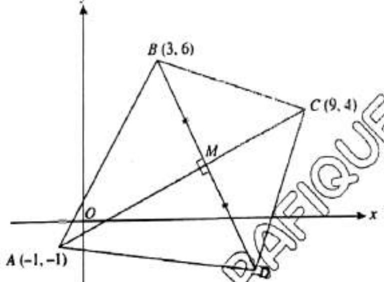


The diagram shows part of the curve $y = \dfrac{2}{1-x}$ and the line $y = 3x + 4$. The curve and the line meet at points A and B.

(i) Find the coordinates of A and B. [4]

(ii) Find the length of the line AB and the coordinates of the mid-point of AB. [3]

24



The diagram shows a quadrilateral $ABCD$ in which the point A is $(-1, -1)$, the point B is $(3, 6)$ and the point C is $(9, 4)$. The diagonals AC and BD intersect at M. Angle $BMA = 90°$ and $BM = MD$. Calculate

(i) the coordinates of M and D, [7]

(ii) the ratio $AM : MC$. [2]

Answers

1 (i) $2y-x-1=0$ (ii) : 4.47

2 (i) (−3, 11) (2, 6) (ii) $y-x=9$

3 : (16, 12)

4 (3, 6) $(\frac{3}{4}, 3)$: 3.75 units

5 (i) $2x+3y=20$ (ii) (10, 0) $D(14, 6)$

6 (i) (4, 6) (ii) $D(6, 10)$ (iii) 40.9 units

7 B are (6, 5) C are (12, 8)

8 coordinates of C are (5, 5)

9 (i) $(\frac{18}{5}, \frac{9}{5})$ (ii) $\frac{8\sqrt{5}}{5}$ units

10 C are $(\frac{3}{2}, 0)$ D are (0, 2) $CD=2.5$ units

11 coordinates of $R=(7, 15)$

12 (3.5, 0)

13 (i) $2y=3x-13$ (ii) (9, 7)

14 (i) $k=2.8$ (ii) $k=0.6$

15 (i) $2y-x=11$ (ii) $C(13, 12)$ (iii) 35.8 units

16 ∴ coordinates are: (4, 3) and $\left(\frac{3}{2}, 8\right)$

17 (i) $y+2x=29$ (ii) (10, 9)

18 (ii) $\left(\frac{1}{2}, \frac{21}{4}\right)$ (iii) : $2\frac{1}{4}$ units

19 (i) $3x+2y=31$ (ii) (7, 5)

20 (i) $3x+2y=48$ (ii) (10, 9)

21 (6.2, 9.6)

22 (i) 6 (ii) $\frac{8}{12-h}$

(iii) B is −4, D is 16. (iv) 160 unit²

23 (i) A are (−1, 1) B are $\left(\frac{2}{3}, 6\right)$

(ii) 5.27 units. $\left(-\frac{1}{6}, \frac{7}{2}\right)$

24 (i) $M_{(5, 2)}$ D are (7, −2) (ii) 3:2

25 (i) $x+2y=27$ (ii) (13, 7)

26 (12, 14)

27 (i) $3y=x+16$ (ii) (6.5, 7.5) (iii) 15 units

28 (i) $k=-7$ and 9 (ii) (−2, 0)

29 $a=6$, $b=8$

SERIES (A.P AND G.P)

Arithmetic Progression (A.P)

If difference of any two consecutive terms of a sequence is constant, then it the sequence is called arithemetic progression. The difference is usually denoted by 'd' and the first term is denoted by 'a'. general term formula of 'A.P' is

$a_n = \alpha + (n-1) d$

If 'Sn' denotes sum of n terms in A.P, then

$S_n = \frac{n}{2}[2a+(n-1)d]$. If last term of sequence is given, which is denoted by 'l' then

$$S_n = \frac{n}{2}[a+l]$$

Geometric progression (G.P):

If ratio of any two consecutive terms in a sequence is constant, then the sequence is called geometric progression (G.P). Its ratio is called common ratio and is denoted by 'r'. If 'a' is the first term of G.P, then its general term formula is $a_n = a\,r^{n-1}$

Sum of finite terms in G.P

$S_n = \frac{a(1-r^n)}{1-r}$ if $r<1$ (ii) $S_n = \frac{a(r^n-1)}{r-1}$ if $r>1$

Sum of infinite terms in G.P

$$S_\infty = \frac{a}{1-r} \text{ if } r<1$$

Convergence series

If the sum of all infinite terms of a function is equal to the original function, then it is called convergence series OR if infinite term is nearly equal to zero and S_∞ exist to original function, then it is denoted as convergence function.

Divergence series

If sum of infinite terms in given series is equal to infinite values, then it is called divergence function i.e. $S_\infty=\infty$

Use of general term formula and sum of terms (Sn)

General term (a_n) formula is used to find out single term OR to find value of unknown variables if value of any specific term is given in A.P and G.P.

Sum of terms (Sn) formula is used to find sum of terms or to find value of unknown variables when sum of specific terms is given. To find value of unknown variables, make two linear equations according to given conditions and solve them simultaneously.

Comparison of common difference (d) OR common ratio (r)

To find unknown value in A.P or G.P, when the sequence is given in variable terms, first find two times value of d or r from any two pairs of consecutive terms in sequence and put them equal and simplify.
21

Exp: If a, a + 4d a + 14 d are in G.P then find value of r.

Sol: $r = \frac{a+4d}{a}$ or $r = \frac{a+14d}{a+4d}$

By comparison $\frac{a+4d}{d} = \frac{a+14d}{a+4d}$

Value of 'r' in $r = 2\frac{1}{2}$

Case of %

In case of %, G.P is used to find value of r for increased amount, change given % into decimal form and add it in 1 and for decreased amount, change given % into decimal form and subtract it from 1. For value of specific year, apply 'a_n' formula and for total amount apply Sn formula in G.P. if no increase or decrease is given with %, then given % in decimal form is taken as value of r.

. Techniques relevant to problem solving:

(i) Forming an equation relating three terms of a sequence.

If x, y and z are consecutive terms of an AP, then we have $y-x=z-y$

If x, y and z are consecutive terms of a GP, then we have $\frac{z}{y}=\frac{y}{x}$

(ii) Recognising that any uninterrupted part of a sequence will retain the main characteristics of its parent sequence:

Example: A descending geometric series has first term a and the common ratio r is positive. The sum of the first 5 terms is twice the sum of terms from the 6^{th} to 15^{th} inclusive.

While it is not erroneous to state that $S_5=2(S_{15}-S_5)\Rightarrow \frac{a(r^5-1)}{r-1}=2\left[\frac{a(r^{15}-1)}{r-1}-\frac{a(r^5-1)}{r-1}\right]$,

it would be far more elegant to use the understanding that the 6^{th} to 15^{th} (10 terms altogether) terms of the GP can be construed as a separate GP series with first term $=ar^{6-1}=ar^5$ but with the same common ratio r as the parent series. Hence, we have

$$\frac{a(r^5-1)}{r-1}=2\left[\frac{ar^5(r^{10}-1)}{r-1}\right],$$

which would render future developments of any solution a tad more efficient.

(iii) The bank interest problem (and its parallel storylines):

Example: At the end of a month, a customer owes a bank \$1500. In the middle of the month, the customer pays \$$x$ to the bank where x<1000, and at the end of the month the bank adds interest at a rate of 4% of the remaining amount still owed. This process continues every month until the money owed is repaid in full.

(i) Find the value of x for which the customer still owes \$1500 at the start of every month.

(ii) Find the value of x for which the whole amount owed is paid off exactly after the second payment.

(iii) Show that the value of x for which the whole amount owed is paid off exactly after the $(n+1)$th payment is given by

$$x=\frac{1500r^n(r-1)}{r^{n+1}-1}, \text{ where } r=1.04$$

SOLUTIONS :

(i) $1500=(1500-x)(1.04) \Rightarrow x=\$57.69$ (shown)

(ii) After 1st payment of $\$x$, the amount owed= $(1500-x)(1.04)$

Therefore, $(1500-x)(1.04)=x \Rightarrow x=\$764.71$ (shown)

(iii) After the second payment of $\$x$, amount owed at beginning of 3rd month is

$[(1500-x)1.04-x](1.04)=1500(1.04)^2-1.04^2x-1.04x$

After the second payment of $x, amount owed at beginning of 4th month is

$[1500(1.04)^2-1.04^2x-1.04x-x](1.04)$

$=1500(1.04)^3-1.04^3x-1.04^2x-1.04x$

After *nth* payment of $\$x$, the amount still owed at the beginning of the $(n+1)$th month

$=1500(1.04)^n-1.04^nx-1.04^{n-1}x-\ldots\ldots\ldots-1.04^2x-1.04x$

$=1500r^n-x(r+r^2+\ldots\ldots+r^n)$ where $r=1.04$

At the $(n+1)$th payment,

$x=1500r^n-x(r+r^2+\ldots\ldots+r^n)$

$x(1+r+r^2+\ldots..+r^n)=1500r^n$

$\Rightarrow x(\frac{r^{n+1}-1}{r-1})=1500r^n \Rightarrow x=\frac{1500r^n(r-1)}{r^{n+1}-1}$ (shown)

Example

A student reading a 426-page book finds that he reads faster as he gets into the subject. He reads 19 pages on the first day, and his rate of reading then goes up by 3 pages each day. How long does he take to finish the book?

You are given that $a = 19$, $d = 3$ and $S = 426$. Since $S = \frac{1}{2}n(2a + (n-1)d)$,

$$426 = \tfrac{1}{2}n(38 + (n-1)3),$$

$$852 = n(3n + 35),$$

$$3n^2 + 35n - 852 = 0.$$

Using the quadratic formula,

$$n = \frac{-35 \pm \sqrt{35^2 - 4 \times 3 \times (-852)}}{2 \times 3} = \frac{-35 \pm 107}{6}.$$

Since n must be positive, $n = \frac{-35 + 107}{6} = \frac{72}{6} = 12$. He will finish the book in 12 days.

Exercise #1

1 State the first term and the common difference and then find the 7th and the 15th terms of the following APs:
(a) 2, 6, 10, ...
(b) −3, 0, 3 ...
(c) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, ...$
(d) 1.7, 1.4, 1.1, ...

2 For the AP −2, 1, 4, ... state (a) the 8th term, (b) the 12th term. (c) Find a formula for the nth term.

3 Find (a) the 9th, (b) the 20th, terms of the AP 17, 13, 9, ...
(c) Which term of this AP will be −35 ?
(d) Find a formula for the nth term.

4 (a) Find a formula for the nth term of the AP −10, −7, −4, ...
(b) If $2x - 3$ is the arithmetic mean of $x^2 - 4$ and $5x - 8$ find the values of x.

5 The 8th term of the AP a, $a + 4$, $a + 8$, ... is 33. Find (a) the value of a, (b) the 15th term.

6 If the first term of an AP is 5 and the 9th term is 25, find the common difference.

7 Find (a) the 7th term and (b) a formula for the nth term of the AP $\frac{1}{2}, \frac{5}{6}, 1\frac{1}{6}, ...$

8 The 4th term of an AP is 13 and the 10th term is 31. Find the 20th term.

9 The 5th term of an AP is $5\frac{1}{2}$ and the 9th term is $8\frac{1}{2}$. Find the 17th term and a formula for the nth term.

10 The 4th term of an AP is 5 and the 11th term is 26. Find
(a) the first term and the common difference,
(b) a formula for the nth term.

11 A ball rolls down a slope so that it travels 4 cm in the lst second, 7 cm in the 2nd, 10 cm in the 3rd and so on. How far does it travel in
(a) the 7th,
(b) the nth, second?

12 The nth term (T_n) of an AP is given by $T_n = 3 - 5n$.
(a) State (i) the 8th term, (ii) the 21st term.
(b) What is the first term?
(c) What is the common difference?

13 The nth term (T_n) of an AP is given by $T_n = \frac{1}{2}(4n - 3)$.
(a) State (i) the 5th term, (ii) the 10th term.
(b) Find the common difference.

14 If the 5th term of an AP is 10 and the 10th term is 5, find the first term and the common difference.

15 If the 12th term of an AP is double the 5th term find the common difference, given that the first term is 7.

16 The 16th term of an AP is three times the 5th term. If the 12th term is 20 more than the 7th term, find the first term and the common difference.

17 In an AP the 9th term is −4 times the 4th term and the sum of the 5th and 7th terms is 9. Find the first term and the common difference.

18 In an AP the 22nd term is 4 times the 5th term while the 12th term is 12 more than the 8th term. Find the first term and the common difference.

19 Find a formula for the nth term of the AP 3, $2\frac{1}{2}$, 2, ...

20 The sum of 3 consecutive terms of an AP is 6 and their product is -42. Find these terms.

21 p, q and r are three consecutive terms of an AP. Express p and r in terms of q and d, where d is the common difference. If the sum of the terms is 21 and $p = 6r$, find p, q and r.

22 Which of the following sequences is an AP?

(a) $\frac{1}{8}, \frac{11}{24}, \frac{19}{24}, \ldots$ (b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

(c) $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \ldots$ (d) $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$

23 The 9th term of an AP is 3 times the 5th term.

(a) Find a relation between a and d.

(b) Prove that the 8th term is 5 times the 4th term.

24 If $x + 1$, $2x - 1$ and $x + 5$ are three consecutive terms of an AP, find the value of x.

25 $x + 2$, $x + 3$ and $2x^2 + 1$ are three consecutive terms of an AP, find the possible values of x.

Answers:

1 (a) 2, 4; 26, 58 (b) -3, 3; 15, 39 (c) $\frac{1}{3}$, $\frac{1}{6}$; $1\frac{1}{3}$, $2\frac{2}{3}$ (d) 1.7, -0.3; -0.1, -2.5
2 (a) 19 (b) 31 (c) $3n - 5$ 3 (a) -15 (b) -59 (c) 14th (d) $21 - 4n$
4 (a) $3n - 13$ (b) 2, -3 5 (a) 5 (b) 61 6 $2\frac{1}{2}$ 7 (a) $2\frac{1}{2}$ (b) $\frac{1}{6}(2n + 1)$ 8 61
9 $14\frac{1}{2}$; $\frac{1}{4}(3n + 7)$ 10 (a) -4, 3 (b) $3n - 7$ 11 (a) 22 cm (b) $3n + 1$ cm
12 (a)(i) -37 (ii) 102 (b) -2 (c) -5 13 (a)(i) $8\frac{1}{2}$ (ii) $18\frac{1}{2}$ (b) 2 14 14, -1
15 $2\frac{1}{3}$ 16 6, 4 17 -18, $4\frac{1}{2}$ 18 5, 3 19 $\frac{1}{2}(7 - n)$ 20 -3, 2, 7 or 7, 2, -3
21 $q - d$, $q + d$; 12, 7, 2 22 (a) and (d) 23 (a) $a + 2d = 0$ 24 4 25 -1 or $1\frac{1}{2}$

Exercise #2

1 The ninth term of an arithmetic progression is 22 and the sum of the first 4 terms is 49.

(i) Find the first term of the progression and the common difference. [4]

The nth term of the progression is 46.

(ii) Find the value of n. [2]

2 The first term of a geometric progression is 12 and the second term is -6. Find

(i) the tenth term of the progression, [3]

(ii) the sum to infinity. [2]

3 (a) A geometric progression has a third term of 20 and a sum to infinity which is three times the first term. Find the first term. [4]

(b) An arithmetic progression is such that the eighth term is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms. [4]

4 (a) The first two terms of an arithmetic progression are 1 and $\cos^2 x$ respectively. Show that the sum of the first ten terms can be expressed in the form $a - b\sin^2 x$, where a and b are constants to be found. [3]

(b) The first two terms of a geometric progression are 1 and $\frac{1}{3}\tan^2\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(i) Find the set of values of θ for which the progression is convergent. [2]

(ii) Find the exact value of the sum to infinity when $\theta = \frac{1}{6}\pi$. [2]

5 The first term of an arithmetic progression is 12 and the sum of the first 9 terms is 135.

(i) Find the common difference of the progression. [2]

The first term, the ninth term and the nth term of this arithmetic progression are the first term, the second term and the third term respectively of a geometric progression.

(ii) Find the common ratio of the geometric progression and the value of n. [5]

6 The third term of a geometric progression is -108 and the sixth term is 32. Find

(i) the common ratio, [3]

(ii) the first term, [1]

(iii) the sum to infinity. [2]

7 (a) The fifth term of an arithmetic progression is 18 and the sum of the first 5 terms is 75. Find the first term and the common difference. [4]

(b) The first term of a geometric progression is 16 and the fourth term is $\frac{27}{4}$. Find the sum to infinity of the progression. [3]

8 **(a)** A geometric progression has first term 100 and sum to infinity 2000. Find the second term. [3]

(b) An arithmetic progression has third term 90 and fifth term 80.

(i) Find the first term and the common difference. [2]

(ii) Find the value of m given that the sum of the first m terms is equal to the sum of the first $(m + 1)$ terms. [2]

(iii) Find the value of n given that the sum of the first n terms is zero. [2]

9 **(a)** The sixth term of an arithmetic progression is 23 and the sum of the first ten terms is 200. Find the seventh term. [4]

(b) A geometric progression has first term 1 and common ratio r. A second geometric progression has first term 4 and common ratio $\frac{1}{4}r$. The two progressions have the same sum to infinity, S. Find the values of r and S. [3]

10 The first and second terms of a progression are 4 and 8 respectively. Find the sum of the first 10 terms given that the progression is

(i) an arithmetic progression, [2]

(ii) a geometric progression. [2]

11 The first term of an arithmetic progression is 61 and the second term is 57. The sum of the first n terms is n. Find the value of the positive integer n. [4]

12 The first term of a geometric progression is $5\frac{1}{3}$ and the fourth term is $2\frac{1}{4}$. Find

(i) the common ratio, [3]

(ii) the sum to infinity. [2]

13 **(a)** In an arithmetic progression the sum of the first ten terms is 400 and the sum of the next ten terms is 1000. Find the common difference and the first term. [5]

(b) A geometric progression has first term a, common ratio r and sum to infinity 6. A second geometric progression has first term $2a$, common ratio r^2 and sum to infinity 7. Find the values of a and r. [5]

14 **(a)** In a geometric progression, the sum to infinity is equal to eight times the first term. Find the common ratio. [2]

(b) In an arithmetic progression, the fifth term is 197 and the sum of the first ten terms is 2040. Find the common difference. [4]

Answers

1 → $d = 1.5, a = 10$

→ $n = 25$

2 $ar^6 = \frac{-3}{128}$

→ 8

3 →$(r = ⅔)$ $a = 45$

$S_8 = 4(2a + 7d) = 32d$ or $64a$

$S_4 = 2(2a + 3d) = 8d$ or $16a$

4 $S_{10} = 10 - 45\sin^2 x$

$(0 <)\theta < \frac{\pi}{3}$

5 $S_\infty = \frac{9}{8}$ or 1.125

→ $d = ¾$ → $n = 21$

6 $r = \left(-\frac{2}{3}\right)$, (ii) $a = -243$

r−145.8

7 →$a = 12$, $d = 1½$

Sum to infinity = 64

8 $ar = 95$

$d = -5, a = 100$

$m = 20$

$n = 41$

9 29

$r = \frac{4}{5}$ oe $S = 5$

10 220 ;

4092

11 $n = 31$

12 $r = \frac{3}{4}$ or 0.75

$21\frac{1}{3}$ or 21.3)

13 $d = 6$ $a = 13$

$a = \frac{12}{7}$ or 1.71(4)

$r = \frac{5}{7}$ or 0.714

14 $r = \frac{7}{8}$

$d = 14$

Past Paper Questions

A P & G P

1 In an arithmetic progression, the 1st term is −10, the 15th term is 11 and the last term is 41. Find the sum of all the terms in the progression. [5]

2 A geometric progression has first term 64 and sum to infinity 256. Find

(i) the common ratio, [2]

(ii) the sum of the first ten terms. [2]

3 A geometric progression has 6 terms. The first term is 192 and the common ratio is 1.5. An arithmetic progression has 21 terms and common difference 1.5. Given that the sum of all the terms in the geometric progression is equal to the sum of all the terms in the arithmetic progression, find the first term and the last term of the arithmetic progression. [6]

4 Each year a company gives a grant to a charity. The amount given each year increases by 5% of its value in the preceding year. The grant in 2001 was $5000. Find

(i) the grant given in 2011, [3]

(ii) the total amount of money given to the charity during the years 2001 to 2011 inclusive. [2]

5 The second term of a geometric progression is 3 and the sum to infinity is 12.

(i) Find the first term of the progression. [4]

An arithmetic progression has the same first and second terms as the geometric progression.

(ii) Find the sum of the first 20 terms of the arithmetic progression. [3]

6 The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression, [2]

(ii) the sum to infinity of the progression. [2]

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression. [3]

7 (a) Find the sum to infinity of the geometric progression with first three terms 0.5, 0.5^3 and 0.5^5. [3]

(b) The first two terms in an arithmetic progression are 5 and 9. The last term in the progression is the only term which is greater than 200. Find the sum of all the terms in the progression. [4]

8 (a) Find the sum of all the multiples of 5 between 100 and 300 inclusive. [3]

(b) A geometric progression has a common ratio of $-\frac{2}{3}$ and the sum of the first 3 terms is 35. Find

(i) the first term of the progression, [3]

(ii) the sum to infinity. [2]

9 **(a)** A circle is divided into 6 sectors in such a way that the angles of the sectors are in arithmetic progression. The angle of the largest sector is 4 times the angle of the smallest sector. Given that the radius of the circle is 5 cm, find the perimeter of the smallest sector. [6]

(b) The first, second and third terms of a geometric progression are $2k+3$, $k+6$ and k, respectively. Given that all the terms of the geometric progression are positive, calculate

(i) the value of the constant k, [3]

(ii) the sum to infinity of the progression. [2]

10 **(a)** In an arithmetic progression, the sum of the first n terms, denoted by S_n, is given by

$$S_n = n^2 + 8n.$$

Find the first term and the common difference. [3]

(b) In a geometric progression, the second term is 9 less than the first term. The sum of the second and third terms is 30. Given that all the terms of the progression are positive, find the first term. [5]

11

(a) The first and last terms of an arithmetic progression are 12 and 48 respectively. The sum of the first four terms is 57. Find the number of terms in the progression. [4]

(b) The third term of a geometric progression is four times the first term. The sum of the first six terms is k times the first term. Find the possible values of k. [4]

12 The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 9th and 21st terms respectively of an arithmetic progression. The 1st term of each progression is 8 and the common ratio of the geometric progression is r, where $r \neq 1$. Find

(i) the value of r, [4]

(ii) the 4th term of each progression. [3]

13 **(a)** The first, second and last terms in an arithmetic progression are 56, 53 and -22 respectively. Find the sum of all the terms in the progression. [4]

(b) The first, second and third terms of a geometric progression are $2k+6$, $2k$ and $k+2$ respectively, where k is a positive constant.

(i) Find the value of k. [3]

(ii) Find the sum to infinity of the progression. [2]

14 A water tank holds 2000 litres when full. A small hole in the base is gradually getting bigger so that each day a greater amount of water is lost.

(i) On the first day after filling, 10 litres of water are lost and this increases by 2 litres each day.

(a) How many litres will be lost on the 30th day after filling? [2]

(b) The tank becomes empty during the nth day after filling. Find the value of n. [3]

(ii) Assume instead that 10 litres of water are lost on the first day and that the amount of water lost increases by 10% on each succeeding day. Find what percentage of the original 2000 litres is left in the tank at the end of the 30th day after filling. [4]

15 A geometric progression, for which the common ratio is positive, has a second term of 18 and a fourth term of 8. Find

(i) the first term and the common ratio of the progression, [3]

(ii) the sum to infinity of the progression. [2]

16 (a) A debt of \$3726 is repaid by weekly payments which are in arithmetic progression. The first payment is \$60 and the debt is fully repaid after 48 weeks. Find the third payment. [3]

(b) Find the sum to infinity of the geometric progression whose first term is 6 and whose second term is 4. [3]

17 Find

(i) the sum of the first ten terms of the geometric progression 81, 54, 36, ..., [3]

(ii) the sum of all the terms in the arithmetic progression 180, 175, 170, ..., 25. [3]

18 A small trading company made a profit of \$250 000 in the year 2000. The company considered two different plans, plan A and plan B, for increasing its profits.

Under plan A, the annual profit would increase each year by 5% of its value in the preceding year. Find, for plan A,

(i) the profit for the year 2008, [3]

(ii) the total profit for the 10 years 2000 to 2009 inclusive. [2]

Under plan B, the annual profit would increase each year by a constant amount \$$D$.

(iii) Find the value of D for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for both plans. [3]

19 (a) Find the sum of all the integers between 100 and 400 that are divisible by 7. [4]

(b) The first three terms in a geometric progression are 144, x and 64 respectively, where x is positive. Find

(i) the value of x,

(ii) the sum to infinity of the progression. [5]

20 The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term. [1]

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$. [3]

(iii) Find the common ratio of the geometric progression. [2]

21 The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has n terms and the sum of all the terms is 90. Find the value of n. [4]

Answers

1 $542.5

2 (i) $\frac{3}{4}$ (ii) 242

3 last term = 205

4 (i) $8140 (3 sf) (ii) $71034

5 (i) first term = 6 (ii) −450

6 (i) $r=\frac{2}{3}$ (ii) 243 (iii) 270

7 (a) $\frac{2}{3}$ (b) 5150

8 (a) 8200 (b) (i) $a=45$ (ii) 27

9 (a) 12.1 cm (b) (i) $k=12$ (ii) 81

10 (a) 2 (b) first term = 27

11 (a) $n=25$ (b) 63, −21

12 (i) $r=\frac{3}{2}$ (ii) $9\frac{1}{2}$

13 (a) 459 (b) (i) $k=6$ (ii) 54

14 (i) (a) 68 litres (b) 41th day. (ii) 17.75%

15 (i) $a=27$ (ii) $S_{\infty}=81$

16 (a) $ 61.50 (b) $S_{\infty}=18$

17 (i) 239 (ii) 3280

18 (i) 369000 (ii) $3140000 (iii) 14300

19 (a) 10836 (b) (i) $x=96$ (ii) 432

20 (i) $T_5=a+4d$ $T_{15}=a+14d$ (iii) $2\frac{1}{2}$

21 $n=8$

22 (i) $a=117$ (ii) $a=128$

23 (a) $m=47$

24 (a) (i) 3 (ii) 57 (iii) 570

(b (i) $6520 (ii) $56800

25 (a) (i) first term = 32 (ii) 128 (b) $n=18$

26 (a) 3 hours 15 minutes (b) 216

27 (a) -2 (b) 8

28 (a) (i) 130 km (ii) the cyclist will finish the event on 20th May

(b) (i) $a=16$ (ii) 32

QUADRATICS

Syllabus:

- Carry out the process of completing the square for a quadratic polynomial ax^2+bx+c, and use this form, e.g. to locate the vertex of the graph of $y = ax^2+bx+c$ or to sketch the graph;
- Find the discriminant of a quadratic polynomial ax^2+bx+c and use the discriminant, e.g. to determine the number of real roots of the equation $ax^2+bx+c=0$:
- Solve quadratic equations, and linear and quadratic inequalities, in one unknown;
- Solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic;
- Recognize and solve equations in x which are quadratic in some function of x, e.g. $x^4 - 5x^2+4=0$

QUADRATICS EQUATION

An equation with highest power of variable '2' is called quadratic equation. Standard form of quadratic eq is $ax^2+bx+c=0$ where 'a' is co efficient of x^2, 'b' is co-efficient of x with sign and 'c' is independent of x term.

Graph of quadratic

The graph of quadratic is with minimum or maximum turning point, depends upon the value of 'a' (co-efficient of x^2). if a>0 (+ive), then turning point or (stationary) point will have minimum value. If a<0 (-ive), then turning point has maximum value.

Sketch of quadratic graph

Express $y = ax^2+bx+c$ as $y = a(x-h)^2 + k$ by completing square method. In either case (min or max) the turning point is (h, k) and the curve is symmetrical about x = h the extreme value (min or mx) is k. this value occurs when the square term $(x-h)^2 = 0$ i.e. when x =h.

If $ax^2+bx+c=0$ is factorised to $a(x-\alpha)(x-\beta)$, then graphs are:

The coordinate of turning point can also be found directly by the formula $x = \frac{-b}{2a}$, $y = c - \frac{b^2}{4a}$

Roots of a quadratic equation

Roots of equation means values of x of points of intersection of the line or curve. Real roots means, the values of x whose square is +ive. To find roots or real roots of equation simplify given equation and apply quadratic formula.

$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$, and discard the values, whose square values is -ive, for real roots.

Discriminant

In quadratic formula, the value b^2-4ac is called he discriminant of the quadratic expression ax^2+bx+c, as it allows to discriminate among the possible types of roots. It also tells the position of the curve $y = ax^2+bx+c$ relative to the x-axis,

Nature of roots

i) If b^2-4ac>0, then equation has two distinct real roots and the curve cuts the x-axis two point. If b^2-4ac≥0, then roots are real.

ii) If b^2-4ac =0, Then equation has equal real roots or repeated roots and the x-axis is tangent to the curve (i.e. meet the curve at only one point).

iii) If b^2-4ac<0, then equation has no real roots and the curve does not intersect the x-axis and y = ax^2+bx+c is either always +ive or always –ive or the line and curve have no common points.

❖ To show that eq has no real roots, we have to show that b^2-4ac is –ive or b^2-4ac<0, for real roots show that b^2-4ac is +ive or b^2-4ac > 0 and for exactly one real roots put b^2-4ac = 0

Point of intersection of line and curve

❖ The graph of quadratic equation is a curve while graph of linear equation is a straight line. To find points of intersection of line and curve, make subject to any variable of linear equation and substitute that value of variable in eq. of curve. Solve it and find values of x and y, which are coordinates of points of intersection of line and curve. For nature of roots apply discriminant formula after substitution of variable from linear equation to the quadratic equation in simplified form.

No point of intersection of graph with x-axis

❖ To show the graph of quadratic equation is above or below the x-axis, find y co ordinate of turning point by formula $y = c - \frac{b^2}{4a}$. If it is +ive and co-efficient of x^2 is also +ive, then the turning point is minimum and the curve lie above the x-axis if y coordinate of turning point (stationary point) is –ive and the co-efficient of x^2(a) is –ive, then turning point is maximum and the curve lie below the x-axis.

Minimum or maximum value

❖ To find minimum or maximum value of quadratic function means find coordinate of turning point i.e. $y = c - \frac{b^2}{4a}$.

❖ Acute angle between two straight lines, when eq. of lines or their gradients (m) are given, if θ is angle between lines then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Quadratic inequality

An inequalitiy with a quadratic expression in one variable on one side and zero on the other side, is called quadratic inequalty, if quadratic inequality is factoriseable to (x-α) (x-β). Where α and β are real numbers, then it may be solved by using he corresponding quadratic curve.

1) If (x-α) (x-β)>0, then x < α or x > β. The range of values of x lie outside the roots of the curve.
Always give your answer in set notation when the question requires to find the set of values i.e. {m:m≤-4 or m≥4, mεR}If (x-α) (x-β) <0, then α<x<β .

The range of value of x will lie inside the roots of the curve. Where α is the smaller root and β is the larger root.

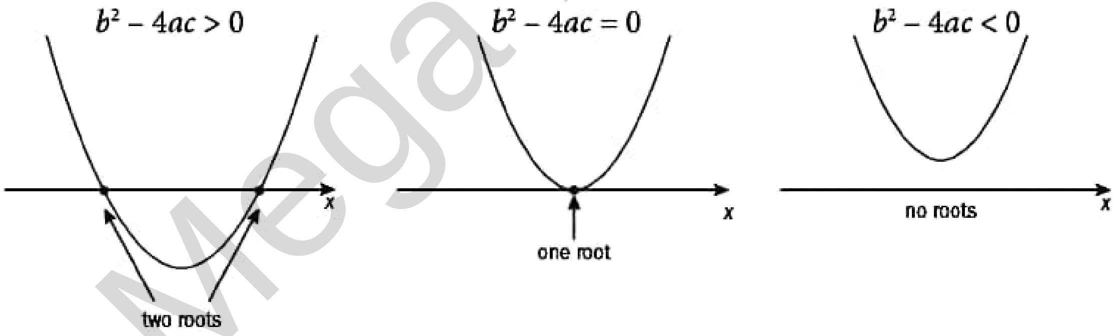
{If linear inequality is in modulus form, then to find its range first find turning point by putting modulus term equal to zero and find x, then put that value of x is given inequality and find y. also put given other values of x in given inequality and find y. then draw it on paper, its graph will be v shape graph and find its range}

If inequality holds variables on both sides. Then solve it by taking square on both sides and simplify

We sometimes call the solutions of a quadratic equation the roots of the equation. This also tells us where the quadratic graph crosses the x-axis. When we have an equation of the form $ax^2 + bx + c = 0$, we can tell the nature of the roots by looking at the discriminant of the quadratic equation.

The discriminant is the value of $b^2 - 4ac$ in $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac > 0$ there are two distinct real roots.
If $b^2 - 4ac = 0$ there are equal roots (one repeated root).
If $b^2 - 4ac < 0$ there are no real roots.



Practice Questions

Exercise #1

1. Use the discriminant to determine the nature of the roots of the following quadratic equations. When the roots are real, find these roots:
   (a) $x^2 - 2x - 5 = 0$ (b) $4x^2 + 4x + 1 = 0$ (c) $3x^2 = 8x + 3$
   (d) $x(1 - 3x) = 2$ (e) $(2x - 1)(x + 1) = 2$ (f) $2x(x - 3) = 3 - 4x$
2. The equation $x^2 + k = 6x$ has two distinct real roots. Find the range of values of k.
3. If the equation $2x(p - x) = 3$ has real and equal roots, find the exact values of p.
4. Find the value(s) or range of values of p for which the equation
   (a) $px^2 - 6x + p = 0$ has equal real roots,
   (b) $2x^2 - 4x + 3 = p$ has real roots,
   (c) $3x^2 = 2x + p - 1$ has distinct real roots,
   (d) $p(x + 1)(x - 3) = x - 4p - 2$ has no real roots.
5. Find the range of values of m such that the roots of the equation $mx^2 + 2 = x(x + 3)$ are not real.
6. What is the least value k can take if the roots of the equation $x^2 - 2kx + k^2 = 3 + x$ are real?
7. Find the range of values of k for which the roots of the equation $x^2 + (2k + 1)x + k^2 - 3 = 0$ are not real. State the range of values of k for which the
8. Find the range of values of k for which the equation $x^2 - 2kx + k^2 - 2k = 6$ has real roots. Find the roots in terms of k.
9. Show that the roots of the equation $2x^2 + p = 2(x - 1)$ are not real if $p > -\frac{3}{2}$.
10. Given that the equation $px^2 + 3px + p + q = 0$, where $p \neq 0$, has two equal real roots, find q in terms of p.

Answers

1. (a) real and distinct, $1 \pm \sqrt{6}$ (b) real and equal, $-\frac{1}{2}, -\frac{1}{2}$ (c) real and distinct, $-\frac{1}{3}, 3$
   (d) not real (e) real and distinct, 1, −1.5 (f) real and distinct, $\frac{1}{2}(1 \pm \sqrt{7})$
2. $k < 9$
3. $\pm\sqrt{6}$
4. (a) ± 3 (b) $p \geq 1$ (c) $p > \frac{2}{3}$ (d) $p > \frac{1}{4}$
5. $m > \frac{17}{8}$
6. $-\frac{13}{4}$
7. $k < -\frac{13}{4}$; $k \geq -\frac{13}{4}$
8. $k \geq -3$, $k \pm \sqrt{2k + 6}$
10. $q = \frac{5}{4}p$

Exercise #2

1. Find the solution set of each of the following quadratic inequalities.
   (a) $x^2 > 9$
   (b) $x(x - 2) < 3$
   (c) $x^2 + 5x - 6 < 0$
   (d) $2x^2 - 7x + 3 \geqslant 0$
   (e) $x^2 > 4x + 12$
   (f) $4x(x + 1) \leqslant 3$
   (g) $(1 - x)^2 \geqslant 17 - 2x$
   (h) $(x + 2)(x + 3) \leqslant x + 6$
   (i) $(x - 1)(5x + 4) > 2(x - 1)$
   (j) $(x + 2)^2 < x(4 - x) + 40$
2. Find the set of values of x for which $2x^2 - 4x - 3$ is greater than x.
3. Sketch the curve $y = x^2 - 4x - 6$, indicating the exact values of the x-intercepts. Hence solve the inequality $x(x - 4) > 6$.
4. There is no real value of x for which $mx^2 + 8x + m = 6$. Find m.
5. The equation $2x^2 + 4x + 2 = p(x + 3)$ has two distinct real roots. Find the range of values of p.
6. If the equation $(p + 2)x^2 - 12x + 2(p - 1) = 0$ has real and distinct roots, find the set of values of p.
7. Find the set of values of m for the equation $2x^2 + 4x + 2 + m(x + 2) = 0$ to have real roots.
8. The roots of the equation $3kx^2 + (k - 5)x = 5x^2 + 2$ are not real. Find the set of values of k.
9. If the equation $px^2 - p + 10 = 2(p + 2)x$ has real solution(s), show that p cannot lie between 1 and 2.
10. Given that the line $y = c - 3x$ does not intersect the curve $xy = 3$, find the range of values of c.

Answers

1. (a) $\{x : x < -3 \text{ or } x > 3, x \in \mathbb{R}\}$
   (b) $\{x : -1 < x < 3, x \in \mathbb{R}\}$
   (c) $\{x : -6 < x < 1, x \in \mathbb{R}\}$
   (d) $\left\{x : x \leqslant \frac{1}{2} \text{ or } x \geqslant 3, x \in \mathbb{R}\right\}$
   (e) $\{x : x < -2 \text{ or } x > 6, x \in \mathbb{R}\}$
   (f) $\left\{x : -\frac{3}{2} \leqslant x \leqslant \frac{1}{2}, x \in \mathbb{R}\right\}$
   (g) $\{x : x \leqslant -4 \text{ or } x \geqslant 4, x \in \mathbb{R}\}$
   (h) $\{x : -4 \leqslant x \leqslant 0, x \in \mathbb{R}\}$
   (i) $\left\{x : x < -\frac{2}{5} \text{ or } x > 1, x \in \mathbb{R}\right\}$
   (j) $\{x : -\sqrt{18} < x < \sqrt{18}, x \in \mathbb{R}\}$

2. $\left\{x : x < -\frac{1}{2} \text{ or } x > 3, x \in \mathbb{R}\right\}$ 3. $x < 2 - \sqrt{10}$ or $x > 2 + \sqrt{10}$ 4. $m < -2$ or $m > 8$

5. $p < -16$ or $p > 0$ 6. $\{p : -5 < p < 4, p \in \mathbb{R}\}$ 7. $\{m : m \leqslant 0 \text{ or } m \geqslant 8, m \in \mathbb{R}\}$

8. $\{k : -15 < k < 1, k \in \mathbb{R}\}$ 10. $-6 < c < 6$

Past Paper Questions

1 Find the value of the constant c for which the line $y = 2x + c$ is a tangent to the curve $y^2 = 4x$. [4]

2 Find the set of values of k for which the line $y = kx - 4$ intersects the curve $y = x^2 - 2x$ at two distinct points. [4]

3 The equation $x^2 + px + q = 0$, where p and q are constants, has roots -3 and 5.

(i) Find the values of p and q. [2]

(ii) Using these values of p and q, find the value of the constant r for which the equation $x^2 + px + q + r = 0$ has equal roots. [3]

4 The straight line $y = mx + 14$ is a tangent to the curve $y = \frac{12}{x} + 2$ at the point P. Find the value of the constant m and the coordinates of P. [5]

5 Determine the set of values of the constant k for which the line $y = 4x + k$ does not intersect the curve $y = x^2$. [3]

6 A curve has equation $y = kx^2 + 1$ and a line has equation $y = kx$, where k is a non-zero constant.

(i) Find the set of values of k for which the curve and the line have no common points. [3]

(ii) State the value of k for which the line is a tangent to the curve and, for this case, find the coordinates of the point where the line touches the curve. [4]

7 The equation of a curve is $y^2 + 2x = 13$ and the equation of a line is $2y + x = k$, where k is a constant.

(i) In the case where $k = 8$, find the coordinates of the points of intersection of the line and the curve. [4]

(ii) Find the value of k for which the line is a tangent to the curve. [3]

8 The line $y = \frac{x}{k} + k$, where k is a constant, is a tangent to the curve $4y = x^2$ at the point P. Find

(i) the value of k, [3]

(ii) the coordinates of P. [3]

9 A curve has equation $y = 2x^2 - 6x + 5$.

(i) Find the set of values of x for which $y > 13$. [3]

(ii) Find the value of the constant k for which the line $y = 2x + k$ is a tangent to the curve. [3]

10 Find the values of k for which the line $y = kx - 2$ meets the curve $y^2 = 4x - x^2$. [4]

11 The line $y = 3x + k$ is a tangent to the curve $x^2 + xy + 16 = 0$.

(i) Find the possible values of k. [3]

(ii) For each of these values of k, find the coordinates of the point of contact of the tangent with the curve. [2]

Answers

1 $c = \frac{1}{2}$

2 $k < -6$, $k > 2$

3 (i) $p = -2$, $q = -15$ (ii) $r = 16$

4 $m = -3$ $P(2, 8)$

5 $k < -4$

6 (i) $0 < k < 4$ (ii) $k = 4$ $\left(\frac{1}{2}, 2\right)$

7 (i) $(2, 3)$ $(6, 1)$ (ii) $k = 8.5$

8 (i) $k = -1$ (ii) $(-2, 1)$

9 (i) $x < -1$, $x > 4$ (ii) $k = -3$

10 $k \geq 0$

11 (i) $k = \pm 16$ (ii) $(-2, 10)$ and $(2, -10)$

FUNCTIONS

Syllabus:

- Understand the terms function, domain, range, one, one function, inverse function and composition of functions;
- Identify the range of a given function in simple cases, and find the composition of two given functions;
- Determine whether or not a given function is one-one, and find the inverse of a one -one function in simple cases;
- Illustrate in graphical terms the relation between a one-one function and its inverse.

FUNCTION

A relation between two sets is called function if all elements of first set are connected uniquely. The element of first set is called domain and its corresponding element in 2nd set is called range or image.

Graphically if any vertical line intersect the graph of function exactly at one point, then it is called graph of a function.

One – one function

If each element of 2nd set is the image of only one element of first set, then the function is called a one – one function. Graphically, if any horizontal line intersect the graph of function exactly at one point, then it is called graph of one-one function.

Composite function

A relation between more than two sets (satisfying the all conditions of function) is called composite function. To solve composite function, start solution by inner most part to substitute its value in the given relation.

Inverse function

By inter changing domain and range of given function, the resulting function is inverse function. Graphically one-one functions and inverse function are the reflection of each other in y = x. if any horizontal line intersect the graph of function exactly at one point, then the function is called one-one functions. A quadratic function is one – one function till its turning point.

Graph of inverse function

To draw the graph of inverse function, find x and y co ordinates of given function, and its turning pint, its x – intercept and y – intercept if possible. Then inter-change all these x, y coordinates and draw it, which is the required graph of inverse function.

Expression of inverse function

To find expression of inverse function put given function equal to y and make x as a subject, then replace y by x in answer, the answer is expression of inverse function.

Domain and range

To find domain and range of inverse function, find domain and range of given simple function and interchange it for inverse function i.e. domain of inverse function = range of given simple function. And range of inverse function = domain of given function.

To find range of function find three values of given function. First find values of function at given values of x and their y – coordinate of turning point i.e. Y = c- $\frac{b^2}{4a}$. In range, with co ordinate of turning point always put ≤ or ≥ sign according to minimum or maximum value and other sign of inequality will be same as the sign of corresponding value of x. then write smallest value of function < range < largest value of function. If no value of x is given.

To find domain of inverse function, when domain of given function is x>any value, then put given value of x in given function and put ∞ as other value of x and find range and then replace that range as a domain of inverse function.

To change one function into other form of function, first reduce the given function in such a way that the domain of required function can be replaced by the domain of other function in its simple function form.

Exp: If f (x) = x^2-4x+7, g (x) = x-2 for x >2

The function h is such that f = hg for h: x>0

Obtain an expression for h(x)

Sol: f (x) = $(x-2)^2$+3, f = hg

h (8(x)) = $(x-2)^2$+3

h(g(x)=$(g(x))^2$+3

h(x)=x^2+3

Example

Express $3-4x-2x^2$ in completed square form.

Start by taking out the factor -2 from the terms which involve x:

$$3-4x-2x^2=3-2(x^2+2x).$$

Dealing with the term inside the bracket, $x^2+2x=(x+1)^2-1$,

so $$3-4x-2x^2=3-2(x^2+2x)=3-2\{(x+1)^2-1\}$$
$$=3-2(x+1)^2+2=5-2(x+1)^2.$$

Example

Express $x^2-8x+12$ in completed square form. Use your result to find the range of the function $f(x)=x^2-8x+12$, which is defined for all real values of x.

$$x^2-8x+12=(x-4)^2-4.$$

As $(x-4)^2\geqslant 0$ for all values of x,

$$x^2-8x+12=(x-4)^2-4\geqslant -4, \text{ so } f(x)\geqslant -4.$$

Writing $f(x)$ as y, the range is $y\geqslant -4$.

If f is a one–one function, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.

Example

Express $3 - 4x - 2x^2$ in completed square form.

Start by taking out the factor -2 from the terms which involve x:

$$3 - 4x - 2x^2 = 3 - 2(x^2 + 2x).$$

Dealing with the term inside the bracket, $x^2 + 2x = (x+1)^2 - 1$,

so $$3 - 4x - 2x^2 = 3 - 2(x^2 + 2x) = 3 - 2\{(x+1)^2 - 1\}$$
$$= 3 - 2(x+1)^2 + 2 = 5 - 2(x+1)^2.$$

Example

Express $x^2 - 8x + 12$ in completed square form. Use your result to find the range of the function $f(x) = x^2 - 8x + 12$, which is defined for all real values of x.

$$x^2 - 8x + 12 = (x-4)^2 - 4.$$

As $(x-4)^2 \geqslant 0$ for all values of x,

$$x^2 - 8x + 12 = (x-4)^2 - 4 \geqslant -4, \text{ so } f(x) \geqslant -4.$$

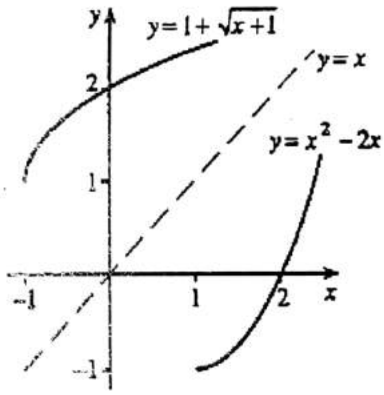
Writing $f(x)$ as y, the range is $y \geqslant -4$.

Example

For the function in Example 11.7.2, draw the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Example 11.7.2 showed that
$f^{-1}(x) = 1 + \sqrt{x+1}$, $x \in \mathbb{R}$, $x \geqslant -1$.

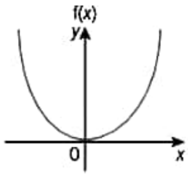
Fig. 11.10 shows the graphs of
$y = f(x) = x^2 - 2x$ for $x \geqslant 1$ and
$y = f^{-1}(x) = 1 + \sqrt{x+1}$ for $x \geqslant -1$. You can see that these graphs are reflections of each other in the line $y = x$.

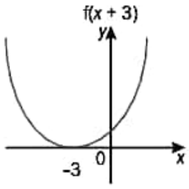


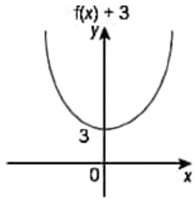
2.4 Transformations: translations

You can transform the graph of a function by moving it horizontally or vertically.
This transformation is called a translation.

> f($x + a$) is a horizontal translation of $-a$.
> f(x) + a is a vertical translation of a.







Note: We can write $y = \text{f}(x)$, $y = \text{f}(x + 3)$ and $y = \text{f}(x) + 3$ for these graphs.

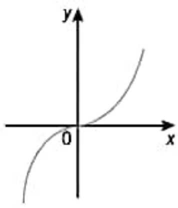
Example 10

$\text{f}(x) = x^3$ $\qquad$ $\text{g}(x) = \frac{1}{a}$

Sketch the graphs of the following functions, marking on each sketch where the curve cuts the axes and state the equations of any asymptotes:

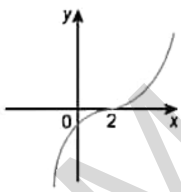
a) $\text{f}(x - 2)$ $\qquad$ **b)** $\text{g}(x) - 2$

a) The graph of $\text{f}(x) = x^3$ is



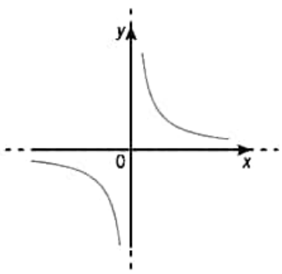
First sketch the graph of f(x), marking where it crosses the axes.

So the graph of $\text{f}(x - 2)$ is



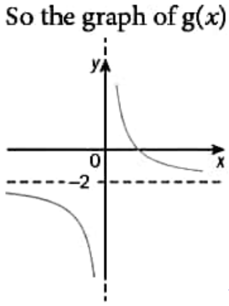
Translate the graph 2 units to the **right**, so it cuts the x-axis at 2, i.e. a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

b) The graph of $\text{g}(x) = \frac{1}{a}$ is



Sketch the graph of g(x), showing the asymptotes with dotted lines.

So the graph of $\text{g}(x) - 2$ is



Translate the graph and the asymptotes 2 units **down**, i.e. a translation by the vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

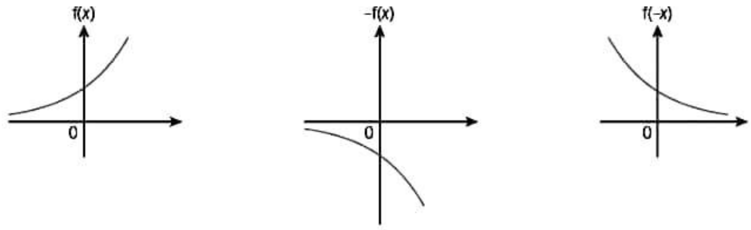
Write these equations on or below the graph.

The equations of the asymptotes are $x = 0$ and $y = -2$.

2.5 Transformations: reflections

You can transform the graph of a function by reflecting the graph in one of the axes.

> $-f(x)$ is a reflection in the x-axis.
> $f(-x)$ is a reflection in the y-axis.



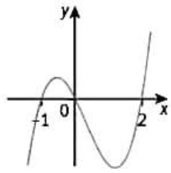
Example 11

$f(x) = x(x + 1)(x - 2)$

Sketch the graphs of the following functions

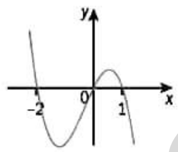
a) $1 + f(-x)$ **b)** $-f(x + 3)$.

a) The graph of $f(x) = x(x + 1)(x - 2)$ is



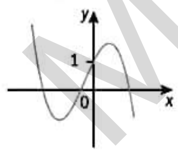
Sketch the graph of $f(x)$, marking where it cuts the axes.

The graph of $f(-x)$ is



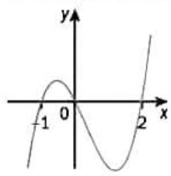
First reflect the graph in the y-axis.

The graph of $1 + f(-x)$ is



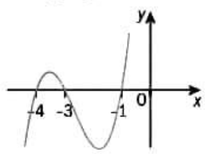
Then translate the graph 1 unit up and show where it cuts the y-axis.

b) The graph of $f(x) = x(x + 1)(x - 2)$ is



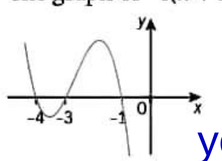
Sketch the graph of $f(x)$, marking where it crosses the x-axis.

The graph of $f(x + 3)$ is



First translate the graph 3 units to the left, i.e. translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

The graph of $-f(x + 3)$ is



Then reflect the graph in the x-axis.

Practice Questions

Exercise #1

1 Given $f: x \mapsto \frac{4}{x+5}$, where $x \in \mathbb{R}$ and $x \neq -5$, find the values of
(a) $f(-1)$, (b) $f(-4)$, (c) $f(3)$.

Given $g: x \mapsto \frac{4}{x} + 5$, where $x \in \mathbb{R}$ and $x \neq 0$, find the values of
2 (a) $g(-1)$, (b) $g(-4)$, (c) $g(3)$.

3 Find the natural domain and corresponding range of each of the following functions.

(a) $f: x \mapsto x^2$
(b) $f: x \mapsto \cos x^{\circ}$
(c) $f: x \mapsto \sqrt{x-3}$
(d) $f: x \mapsto x^2 + 5$
(e) $f: x \mapsto \frac{1}{\sqrt{x}}$
(f) $f: x \mapsto x(4-x)$
(g) $f: x \mapsto \sqrt{x(4-x)}$
(h) $f: x \mapsto x^2 + 4x + 10$
(i) $f: x \mapsto \left(1-\sqrt{x-3}\right)^2$

4 Given that $f: x \mapsto 2x+1$ and $g: x \mapsto 3x-5$, where $x \in \mathbb{R}$, find the value of the following.

(a) $gf(1)$
(b) $gf(-2)$
(c) $fg(0)$
(d) $fg(7)$
(e) $ff(5)$
(f) $ff(-5)$
(g) $gg(4)$
(h) $gg\left(2\frac{2}{9}\right)$

5 Given that $f: x \mapsto ax+b$ and that $ff: x \mapsto 9x-28$, find the possible values of a and b.

6 For $f: x \mapsto ax+b$, $f(2)=19$ and $ff(0)=55$. Find the possible values of a and b.

7 The functions $f: x \mapsto 4x+1$ and $g: x \mapsto ax+b$ are such that $fg = gf$ for all real values of x. Show that $a = 3b+1$.

Given the function $f: x \mapsto x-6, x \in \mathbb{R}$, find the values of
8 (a) $f^{-1}(4)$, (b) $f^{-1}(1)$, (c) $f^{-1}(-3)$, (d) $ff^{-1}(5)$, (e) $f^{-1}f(-4)$.

9 find the inverse of each of the following functions.

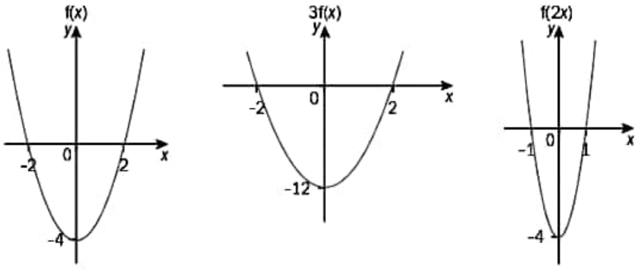
(a) $f: x \mapsto 6x+5, x \in \mathbb{R}$
(b) $f: x \mapsto \frac{x+4}{5}, x \in \mathbb{R}$
(c) $f: x \mapsto 4-2x, x \in \mathbb{R}$
(d) $f: x \mapsto \frac{2x+7}{3}, x \in \mathbb{R}$
(e) $f: x \mapsto 2x^3+5, x \in \mathbb{R}$
(f) $f: x \mapsto \frac{1}{x}+4, x \in \mathbb{R}$ and $x \neq 0$
(g) $f: x \mapsto \frac{5}{x-1}, x \in \mathbb{R}$ and $x \neq 1$
(h) $f: x \mapsto (x+2)^2+7, x \in \mathbb{R}$ and $x \geqslant -2$
(i) $f: x \mapsto (2x-3)^2-5, x \in \mathbb{R}$ and $x \geqslant \frac{3}{2}$
(j) $f: x \mapsto x^2-6x, x \in \mathbb{R}$ and $x \geqslant 3$

You can transform the graph of a function by **stretching** (or compressing) the graph horizontally or vertically.

> $af(x)$ is a stretch with factor a in the y-direction.
> $f(ax)$ is a stretch with factor $\frac{1}{a}$ in the x-direction.

Note: $af(x)$ means **multiply** all the y-values by a while the x-values stay the same.

$f(ax)$ means **divide** all the x-values by a while the y-values stay the same.

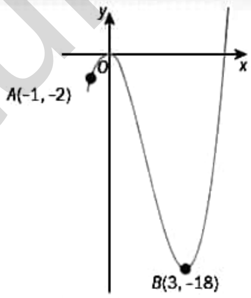


Example 12

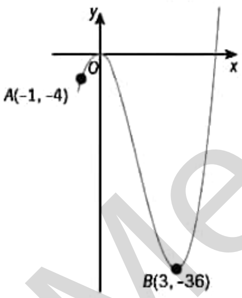
The diagram shows a sketch of the curve $f(x)$, which passes through the origin, O, and the points $A(-1, -2)$ and $B(3, -18)$.
Sketch the graphs of

a) $-2f(x)$ **b)** $f(-3x)$

In each case, mark the new position of the points O, A and B, writing down their coordinates.

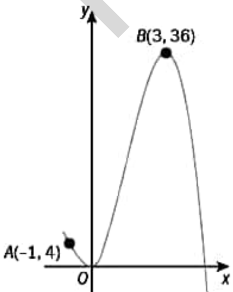


a) The graph of $2f(x)$ is



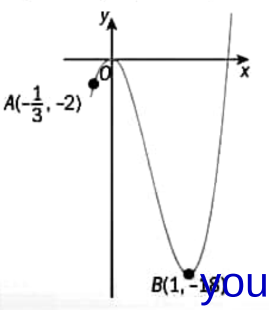
Apply a stretch with factor 2 in the y-direction, i.e. keep the x-values the same, double the y-values.

The graph of $-2f(x)$ is



Then reflect the graph in the x-axis.

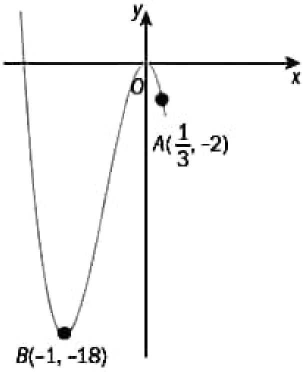
b) The graph of $f(3x)$ is



Apply a stretch with factor $\frac{1}{3}$ in the x-direction, i.e. divide the x-values by 3, keep the y-values the same.

▶ Continued on the next page

The graph of f(−3x) is



Then reflect the graph in the y-axis

Exercise 2.6

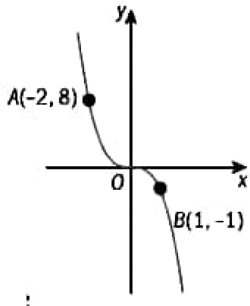
1. **a)** Sketch the graph of $y = f(x)$ where $f(x) = 16 - x^2$.

b) On separate diagrams, sketch the graphs of

i) $y = f(2x)$ **ii)** $y = 4f(x)$ **iii)** $y = f(-4x)$.

Mark on each sketch the coordinates of the points where the curve cuts the axes.

2. The diagram shows a sketch of the curve f(x), which passes through the origin, O, and the points A(−2, 8) and B(1, −1).

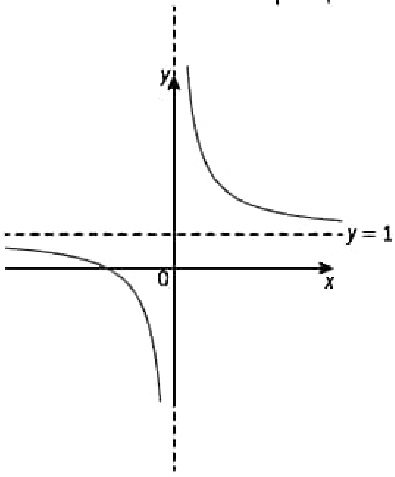


On separate diagrams, sketch the graphs of

a) $y = 4f(x)$ **b)** $y = f(-\frac{1}{2}x)$ **c)** $y = -2f(x + 1)$ **d)** $y = 1 + f(-2x)$.

Mark the new position of the points O, A and B on each transformation, stating their coordinates.

3. The diagram shows a sketch of the curve f(x).



The curve has a horizontal asymptote with equation $y = 1$ and a vertical asymptote with equation $x = 0$.

On separate diagrams, sketch the graphs of

a) $y = f(2x)$ **b)** $y = -3f(x)$ **c)** $y = 2f(x - 3)$

d) $y = 4 - f(\frac{1}{2}x)$ **e)** $y = 3 + 4f(x)$ **f)** $y = 4f(-\frac{1}{2}x)$.

Mark on each sketch the equations of the asymptotes and, if possible, any points where the curve cuts the axes.

4. The graph of $y = f(x)$ is transformed to the graph of $y = -f(2x)$.

Describe fully the two single transformations that have been combined to give the resulting transformation.

5. The graph of $y = f(x)$ is transformed to the graph of $y = 4 + 3f(x - 2)$.

Describe fully the three single transformations that have been combined to give the resulting transformation.

Answers:

1 (a) 1 (b) 4 (c) $\frac{1}{2}$

2 (a) 1 (b) 4 (c) $6\frac{1}{3}$

3 (a) $\mathbb{R}, f(x) \geqslant 0$ (b) $\mathbb{R}, -1 \leqslant f(x) \leqslant 1$
(c) $x \geqslant 3, f(x) \geqslant 0$ (d) $\mathbb{R}, f(x) \geqslant 5$
(e) $x > 0, f(x) > 0$ (f) $\mathbb{R}, f(x) \leqslant 4$
(g) $0 \leqslant x \leqslant 4, 0 \leqslant f(x) \leqslant 2$
(h) $\mathbb{R}, f(x) \geqslant 6$ (i) $x \geqslant 3, f(x) \geqslant 0$

4 (a) 4 (b) −14 (c) −9 (d) 33
(e) 23 (f) −17 (g) 16 (h) 0

5 $a = 3, b = -7$ or $a = -3, b = 14$

6 $a = 4, b = 11$ or $a = 4\frac{1}{2}, b = 10$

8 (a) 10 (b) 7 (c) 3
(d) 5 (e) −4

9 (a) $y \mapsto \frac{1}{6}(y-5)$ (b) $y \mapsto 5y-4$
(c) $y \mapsto \frac{1}{2}(4-y)$ (d) $y \mapsto \frac{1}{2}(3y-7)$
(e) $y \mapsto \sqrt[3]{\frac{1}{2}(y-5)}$ (f) $y \mapsto \frac{1}{y-4}, y \neq 4$
(g) $y \mapsto \frac{5}{y}+1, y \neq 0$
(h) $y \mapsto \sqrt{y-7}-2, y \geqslant 7$
(i) $y \mapsto \frac{1}{2}\left(3+\sqrt{y+5}\right), y \geqslant -5$
(j) $y \mapsto 3+\sqrt{y+9}, y \geqslant -9$

10 (a) $f(x) > -1$
(b) $x \mapsto 2+\sqrt{x+1}, x > -1; f^{-1}(x) > 2$

11 (a) $f(x) > 3$
(b) $x \mapsto 2+(x-3)^2, x > 3; f^{-1}(x) > 2$

12 $k = -1$
(a) $f(x) \geqslant 5$
(b) $x \mapsto -1-\sqrt{x-5}, x \geqslant 5; f^{-1}(x) \leqslant -1$

Functions

1 The function f is defined by $f : x \mapsto ax + b$, for $x \in \mathbb{R}$, where a and b are constants. It is given that $f(2) = 1$ and $f(5) = 7$.

(i) Find the values of a and b. [2]

(ii) Solve the equation $ff(x) = 0$. [3]

2 The functions f and g are defined as follows:

$$f : x \mapsto x^2 - 2x, \quad x \in \mathbb{R},$$
$$g : x \mapsto 2x + 3, \quad x \in \mathbb{R}.$$

(i) Find the set of values of x for which $f(x) > 15$. [3]

(ii) Find the range of f and state, with a reason, whether f has an inverse. [4]

(iii) Show that the equation $gf(x) = 0$ has no real solutions. [3]

(iv) Sketch, in a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, making clear the relationship between the graphs. [2]

3 Functions f and g are defined by

$$f : x \mapsto k - x \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant},$$
$$g : x \mapsto \frac{9}{x+2} \quad \text{for } x \in \mathbb{R}, \ x \neq -2.$$

(i) Find the values of k for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases. [6]

(ii) Solve the equation $fg(x) = 5$ when $k = 6$. [3]

(iii) Express $g^{-1}(x)$ in terms of x. [2]

4 The function f is defined by $f : x \mapsto 2x^2 - 12x + 13$ for $0 \leqslant x \leqslant A$, where A is a constant.

(i) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) State the value of A for which the graph of $y = f(x)$ has a line of symmetry. [1]

(iii) When A has this value, find the range of f. [2]

The function g is defined by $g : x \mapsto 2x^2 - 12x + 13$ for $x \geqslant 4$.

(iv) Explain why g has an inverse. [1]

(v) Obtain an expression, in terms of x, for $g^{-1}(x)$. [3]

5 The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 4x - 2x^2,$$
$$g : x \mapsto 5x + 3.$$

(i) Find the range of f. [2]

(ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots. [3]

6 The function f is defined by $f : x \mapsto \frac{x+3}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(i) Show that $ff(x) = x$. [3]

(ii) Hence, or otherwise, obtain an expression for $f^{-1}(x)$. [2]

7 Functions f and g are defined by

$$f : x \mapsto 2x + 5 \quad \text{for } x \in \mathbb{R},$$
$$g : x \mapsto \frac{8}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3.$$

(i) Obtain expressions, in terms of x, for $f^{-1}(x)$ and $g^{-1}(x)$, stating the value of x for which $g^{-1}(x)$ is not defined. [4]

(ii) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram, making clear the relationship between the two graphs. [3]

(iii) Given that the equation $fg(x) = 5 - kx$, where k is a constant, has no solutions, find the set of possible values of k. [5]

8 A function f is defined by $f(x) = \frac{5}{1-3x}$, for $x \geqslant 1$.

(i) Find an expression for $f'(x)$. [2]

(ii) Determine, with a reason, whether f is an increasing function, a decreasing function or neither. [1]

(iii) Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}. [5]

9 Functions f and g are defined by

$$f : x \mapsto 10 - 3x, \quad x \in \mathbb{R},$$
$$g : x \mapsto \frac{10}{3-2x}, \quad x \in \mathbb{R},\ x \neq \frac{3}{2}.$$

Solve the equation $ff(x) = gf(2)$. [3]

10 Functions f and g are defined by

$$f : x \mapsto 2x - 3, \quad x \in \mathbb{R},$$
$$g : x \mapsto x^2 + 4x, \quad x \in \mathbb{R}.$$

[2]

(i) Solve the equation $ff(x) = 11$. [2]

(ii) Find the range of g. [3]

(iii) Find the set of values of x for which $g(x) > 12$. [3]

(iv) Find the value of the constant p for which the equation $gf(x) = p$ has two equal roots.

Function h is defined by $h : x \mapsto x^2 + 4x$ for $x \geqslant k$, and it is given that h has an inverse. [1]

(v) State the smallest possible value of k. [4]

(vi) Find an expression for $h^{-1}(x)$.

11 The function f is defined by $f : x \mapsto 2x^2 - 6x + 5$ for $x \in \mathbb{R}$. [3]

(i) Find the set of values of p for which the equation $f(x) = p$ has no real roots.

The function g is defined by $g : x \mapsto 2x^2 - 6x + 5$ for $0 \leqslant x \leqslant 4$. [3]

(ii) Express $g(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [2]

(iii) Find the range of g.

The function h is defined by $h : x \mapsto 2x^2 - 6x + 5$ for $k \leqslant x \leqslant 4$, where k is a constant. [1]

(iv) State the smallest value of k for which h has an inverse. [3]

(v) For this value of k, find an expression for $h^{-1}(x)$. [3]

12 (i) Express $2x^2 + 8x - 10$ in the form $a(x + b)^2 + c$.

(ii) For the curve $y = 2x^2 + 8x - 10$, state the least value of y and the corresponding value of x. [2]

(iii) Find the set of values of x for which $y \geqslant 14$. [3]

Given that $f : x \mapsto 2x^2 + 8x - 10$ for the domain $x \geqslant k$,

(iv) find the least value of k for which f is one-one, [1]

(v) express $f^{-1}(x)$ in terms of x in this case. [3]

13 Functions f and g are defined by

$$f : x \mapsto 2x - 5, \quad x \in \mathbb{R},$$
$$g : x \mapsto \frac{4}{2-x}, \quad x \in \mathbb{R}, \ x \neq 2.$$

(i) Find the value of x for which $fg(x) = 7$. [3]

(ii) Express each of $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x. [3]

(iii) Show that the equation $f^{-1}(x) = g^{-1}(x)$ has no real roots. [3]

(iv) Sketch, on a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between these two graphs. [3]

14 The function f is defined by $f : x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) State the range of f. [1]

(iii) Explain why f does not have an inverse. [1]

The function g is defined by $g : x \mapsto 2x^2 - 8x + 11$ for $x \leqslant A$, where A is a constant.

(iv) State the largest value of A for which g has an inverse. [1]

(v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}. [4]

15 The function f is defined by

$$f(x) = x^2 - 4x + 7 \text{ for } x > 2.$$

(i) Express $f(x)$ in the form $(x-a)^2 + b$ and hence state the range of f. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

The function g is defined by

$$g(x) = x - 2 \text{ for } x > 2.$$

The function h is such that $f = hg$ and the domain of h is $x > 0$.

(iii) Obtain an expression for $h(x)$. [1]

16 The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 3x + a,$$
$$g : x \mapsto b - 2x,$$

where a and b are constants. Given that $ff(2) = 10$ and $g^{-1}(2) = 3$, find

(i) the values of a and b, [4]

(ii) an expression for $fg(x)$. [2]

17 The function f is defined by $f : x \mapsto x^2 - 3x$ for $x \in \mathbb{R}$.

(i) Find the set of values of x for which $f(x) > 4$. [3]

(ii) Express $f(x)$ in the form $(x - a)^2 - b$, stating the values of a and b. [2]

(iii) Write down the range of f. [1]

(iv) State, with a reason, whether f has an inverse. [1]

The function g is defined by $g : x \mapsto x - 3\sqrt{x}$ for $x \geqslant 0$.

(v) Solve the equation $g(x) = 10$. [3]

18 A function f is such that $f(x) = \sqrt{\left(\frac{x+3}{2}\right)} + 1$, for $x \geqslant -3$. Find

(i) $f^{-1}(x)$ in the form $ax^2 + bx + c$, where a, b and c are constants, [3]

(ii) the domain of f^{-1}. [1]

19 Functions f and g are defined by

$$f : x \mapsto 3x + 2, \quad x \in \mathbb{R},$$
$$g : x \mapsto 4x - 12, \quad x \in \mathbb{R}.$$

Solve the equation $f^{-1}(x) = gf(x)$. [4]

20 The functions f and g are defined by

$$f(x) = \frac{4}{x} - 2 \quad \text{for } x > 0,$$

$$g(x) = \frac{4}{5x + 2} \quad \text{for } x \geqslant 0.$$

(i) Find and simplify an expression for $fg(x)$ and state the range of fg. [3]

(ii) Find an expression for $g^{-1}(x)$ and find the domain of g^{-1}. [5]

Answers

1 (i) $a = 2, b = -3$; (ii) 2.25 .

Range of y is $f(x) \geq -1$

2

No inverse since not 1 : 1 (

[or gf(x)=0 → f(x)=–3/2.

Either inverse as mirror image in y=x
or $y = g^{-1}(x) = \frac{1}{2}(x-3)$ drawn

3

$k = -8$, root is -5.

$fg(x) = 5 \rightarrow g(x) = 1 \rightarrow x = 7$]

$g^{-1}(x) = \frac{9}{x} - 2$ or $\frac{9-2x}{x}$

4 (i) $2x^2 - 12x + 13 = 2(x-3)^2 - 5$

(ii) Symmetrical about $x = 3$. $A = 6$.

(iii) One limit is -5
Other limit is 13

(iv) Inverse since 1:1 (4 > 3).

(v) Makes x the subject of the equation
Order of operations correct

$\sqrt{x+5}$

5 (i) Turning point at $x = 1$. (ii) $\rightarrow k = 13$
Range is $\leqslant 2$.

6 (i) $f^{-1}(x) = \frac{x+3}{2x-1}$

7 (i) $f^{-1}(x) = \frac{x-5}{2}$ $g^{-1}(x) = \frac{8}{x} + 3$, $x \in \mathbb{R}$, $x \neq 0$ (iii) $0 < k < \frac{64}{9}$

8 (i) $\frac{15}{(1-3x)^2}$ (ii) f(x) is an increasing function. (iii) $f^{-1}(x) = \frac{x-5}{3x}$
domain of $f^{-1}(x)$ is: $-2.5 \leq x < 0$ range of $f^{-1}(x)$ is: $f^{-1}(x) \geq 1$

9 $x = 2$ 10 (i) $x = 5$ (ii) $g(x) \geq -4$ (iii) $x < -6$, $x > 2$

(iv) $p = -4$ (v) $k = -2$ (vi) $h^{-1}(x) = \sqrt{x+4} - 2$

11 (i) $p < \frac{1}{2}$ (ii) $2\left(x - \frac{3}{2}\right)^2 + \frac{1}{2}$ (iii) $\frac{1}{2} \leq g(x) \leq 13$

(iv) $k = \frac{3}{2}$ (v) $h^{-1}(x) = \frac{3 + \sqrt{2x-1}}{2}$

12 (i) $2(x+2)^2 - 18$ (ii) $x = -2$ (iii) $x \leq -6$, $x \geq 2$ (iv) $k = -2$
$f^{-1}(x) = -2 + \sqrt{\frac{x+18}{2}}$

13 (i) $x = \frac{4}{3}$ (ii) $f^{-1}(x) = \frac{x+5}{2}$ (iii) $-31 < 0$

(iv) $f^{-1}(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$.

14 (i) $2(x-2)^2 + 3$ (ii) $f(x) \geq 3$ (iii) f is not a one-one function.

(iv) 2 (v) $g^{-1}(x) = 2 - \sqrt{\frac{x-3}{2}}$ (vi) $g^{-1}(x) \leq 2$

17 (i) $x < -1$ and $x > 4$ (ii) $a = \frac{3}{2}$, $b = \frac{9}{4}$ (iii) $f(x) \geq -\frac{9}{4}$

(iv) f is not a 1–1 function. (iv) $x = 25$

15 (i) $f(x) \geq 3$ (ii) $f^{-1}(x) = 2 + \sqrt{x-3}$ $x \geq 3$ (iii) $h(x) = (x)^2 + 3$

16 (i) $a = -2$ $b = 8$ (ii) $22 - 6x$

18 (i) $2x^2 - 4x - 1$ (ii) $f(x) \geq 1$, $x \geq 1$

19 $x = \frac{2}{7}$

20 (i) $fg(x) = 5x$ (ii) $g^{-1}(x) = (4-2x)/5x$
Range of fg is $y \geqslant 0$ $0 < x \leqslant 2$

Binomial Expansion

n Factorial

(1) n factorial	$n! = n(n-1)(n-2)(n-3)\ldots\frac{3\times2\times1}{\textit{stop at } 1}$

0! is defined to be equal to 1.

This function is available in most scientific calculators.

- ${}^nC_r$ or $\binom{n}{r}$

${}^nC_r$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

r indicates the positions of each individual term in the expansion:

(1st term, $r=0$)	${}^nC_0 = \binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{n!}{n!} = 1$
(2nd term, $r=1$)	${}^nC_1 = \binom{n}{1} = \frac{n}{1} = n$
(3rd term, $r=2$)	${}^nC_2 = \binom{n}{2} = \frac{n(n-1)}{2\times1} = \frac{n(n-1)}{2}$
(4th term, $r=3$)	${}^nC_3 = \binom{n}{3} = \frac{n(n-1)(n-2)}{3\times2\times1} = \frac{n(n-1)(n-2)}{6}$
⋮	⋮
($n+1$th term, $r=n$)	${}^nC_n = \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n!} = 1$

An important concept in binomial theorem:
there is a total of ($n+1$) terms for any expansion $(a+b)^n$

If n = 0, r can only be 0, (the expansion has only 1 term)
If n = 1, r can only be 0 or 1. (the expansion has 2 terms)
If n = 2, r can only be 0, 1 or 2 (a total of 3 terms)
If n = 3, r can only be 0, 1,2 or 3 (a total of 4 terms)

If n = k, r can only be 0, 1, 2, 3 … k. (a total of k + 1 terms)

BINOMIAL EXPANSION

- The formula for the expansion of binomials raised to exponential powers:

Binomial Theorem =	$(a+b)^n$ ${}^nC_r\, a^n b^0 + {}^nC_r\, a^{n-1} b^1 + {}^nC_r\, a^{n-2} b^2 + \ldots + \underbrace{{}^nC_r a^{n-r} b^r}_{\text{general term}} = + \cdots + \; {}^nC_r a^0 b^n$

$$(a+b)^n$$

$$= \underbrace{\binom{n}{0} a^n b^0}_{1^{st}\,term} + \underbrace{\binom{n}{1} a^{n-1} b}_{2nd\,term} + \underbrace{\binom{n}{2} a^{n-2} b^2}_{3^{rd}\,term} + \cdots + = \underbrace{\binom{n}{r} a^{n-r} b^r}_{\substack{general\\ term\\ (r+1^{th}\,term)}} = \underbrace{\binom{n}{n} a^0 b^{0n}}_{\substack{last\,term\\ = b^n}}$$

- The powers of a are in descending order while the powers of b are in ascending order. The sum or the powers of a and b in any term is equal to the power of the binomial.

In the above formula, we make (a + b) as representative of all binomials. At this level, we are only concerned with n as a positive integer.

When the power is n, there are altogether n + 1 terms.

The general term represents all the terms in the expansion (you only need to remember it to apply binomial theorem.

[Q] Given that the expansion of $(2+x)^2(1-ax)^7$ in ascending powers of x is $4 - 80 + bx^2$. Calculate the value of a and of b.

SOLUTION

$$1-ax)^7 = 1 + {}^7C_r(-ax) + {}^7C_2(-ax)^2 \ldots\ldots$$
$$= 1 - 7ax + 21a^2x^2$$
$$(2+x)^2(1-ax)^7 = (4+4x+x^2)(1-7ax+21a^2x^2+\cdots..)$$
$$= 4 + 4x - 28ax + x^2 - 28ax^2 84a^2x^2 + \cdots..$$
$$= 4 + (4-28a)x + (1 - 28a + 84a^2)x^2 + \cdots.$$

Since $(2+x)^2(1-ax)^7 is\ 4 - 80x + bx^2$

$\Rightarrow$ $(4-28a) = -80$
$-28a = -84$
$\therefore a = 3 \quad (Ans)$
$b = (1 - 28a + 84a^2)$
$= 1 - 28(3) + 84\,(3)^2$
$= 673 \quad (Ans)$

[Q] Write down the expansion of $(1+2x)(1-x)^8$ as far as the term in x^4. Hence estimate

SOLUTION

$$(1-x)^8 = 1 + 8(-x) + {}^8C_3(-x)^3 + {}^8C_4(-x)^4 + \cdots$$
$$= 1 - 8x + 28x^2 + 56\,(-x)^3 + 70(x^4) + \ldots$$
$$= 1 - 8x + 28x^2 + 56x^3 + 70x^4 + \cdots$$
$$(1+2x)(1-x)^8 = (1+2x)(1 - 8x + 28x^2 - 56x^2 + 70x^4 + \cdots)$$
$$= 1 - 8x + 28x^2 - 56x^3 + 70x^4 + 2x - 16x^2 + 56x^2 - 11x^4 + \cdots$$
$$= 1 - 6x + 12x^2 - 42x^4 + \cdots.$$

Let $x = 0.$,

$\Rightarrow \quad 1.4 \times 0.8^8 \approx 1 - 6(0.2)^2 - 42(0.2)^4$

≈ 0.2128 (*Ans*)

GENERAL TERM FORMULAE

Finding a specific term in the expansion

- If we want to find any specific term in the expansion, apply general term formulae. The general term represents any term in the binomial expansion

General Term Formulae	The general term, or $(r+1)^{th}$ *term*, in $(a+b)^n$ is $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

- The general term is $(r+1)^{th}$ term since r = 0 gives the first term (denoted by T_1), r = 1 gives the second term (T_2) r = 2 gives the third term (T_3) and so on.

[Q] Evaluate the term which is independent of x is the expansion of $\left(x^3 - \frac{1}{2x^2}\right)^{10}$

SOLUTION

$$T_{r+1} = {}^{10}C_r (x^3)^{10} \left(-\frac{1}{2x^2}\right)^r$$

$$= {}^{10}C_r x^{30-3r} \left(-\frac{1}{2}\right)^r \left(\frac{1}{x^{2r}}\right)$$

$$= {}^{10}C_r \left(-\frac{1}{2}\right)^r (x^{30-5r})$$

$\Rightarrow \quad x^{30-5r} = x^0$

$\Rightarrow \quad 30 - 5r = 0$

$r = 6$

$$\therefore T_{6+1} \text{ gives } 7^{th} \text{ term} = {}^{10}C_6 \left(-\frac{1}{2}\right)^6$$

$$= \frac{105}{32}$$

$$= 3\frac{9}{32}$$

The term independent of x is the 7th term and the value is $3\frac{9}{32}$ **(Ans)**

Use the general term formulae to regroup x. Then solve for r by equating the index of x to the required index. $r+1$ gives the particular term.

To understand this, if we apply binomial theorem to find the 7th term, you should see that this is the particular term that has x being cancelled out:

$$T_7 = {}^{10}C_6 (x^3)^4 \left(\frac{1}{2x^2}\right)^6 = {}^{10}C_6 x^{12} \left(-\frac{1}{2^6 x^{12}}\right) = {}^{16}C_6 \left(-\frac{1}{2^6}\right)$$

[Q] (a) Write down the fourth term in the binomial expansion of the function $\left(px + \frac{q}{x}\right)^n$ in its simplest form.

(b) Find the value of n if the fourth term is independent of x.

(c) Write this value of n, calculate the values of p and q, given that the fourth term is equal to 160 and p – q = 1, p, q ϵZ^+.

SOLUTION

(a) $\left(px + \frac{q}{x}\right)^n$

$$T_4 = {}^{n}C_3 (px)^{n-3} (qx^{-1})^3$$

$$= {}^{n}C_3 \cdot p^{n-3} \cdot x^{n-3-3}$$

$$= {}^{n}C_3 \cdot p^{n-3} \cdot q^3 \cdot x^{n-3} \quad \textbf{(Ans)}$$

b) $x^{n-6} = x^0$

$n - 6 = 0$

$n = 6\ (Ans)$

c) $T_4 = {}^6C_3 \cdot p^3 \cdot q^3$

$= 20p^3q^3$

d) $20p^3q^3 = 160$

$(pq)^3 = 8$

$pq = 2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$

$p - q = 1$

$p = 1 + q \ldots\ldots\ldots\ldots\ldots\ldots (2)$

Sub (2) into (1):

$$(1+q)(q = 2)$$
$$q + q^2 = 2$$
$$q + q^2 = 2$$
$$q^2 + q - 2 = 0$$
$$(q-1)(q+2) = 0$$
$$q = 1 \text{ or } -2\ (rej) \quad (Ans)$$

Sub q = 1 into (2)

$p = 1 + 1$

$= 2\ (Ans)$

Example

Use the Binomial Theorem to expand $(1+b)^5$. Hence deduce the expansion of
(a) $(1-b)^5$, **(b)** $(1+2x)^5$.

$$(1+b)^5 = 1 + {}^5C_1 b + {}^5C_2 b^2 + {}^5C_3 b^3 + {}^5C_4 b^4 + {}^5C_5 b^5$$
$$= 1 + 5b + 10b^2 + 10b^3 + 5b^4 + b^5$$

(a) $(1-b)^5 = [1+(-b)]^5$
$$= 1 + 5(-b) + 10(-b)^2 + 10(-b)^3 + 5(-b)^4 + (-b)^5$$
$$= 1 - 5b + 10b^2 - 10b^3 + 5b^4 - b^5$$

(b) Let $b = 2x$.
$$(1+2x)^5 = 1 + 5(2x) + 10(2x)^2 + 10(2x)^3 + 5(2x)^4 + (2x)^5$$
$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

Example

Find, in ascending powers of x, the first 4 terms in the binomial expansion of **(a)** $(1+4x)^6$, **(b)** $(1-3x)^7$.

(a) $(1+4x)^6 = 1 + {}^6C_1(4x) + {}^6C_2(4x)^2 + {}^6C_3(4x)^3 + \dots$
$$= 1 + 6(4x) + 15(16x^2) + 20(64x^3) + \dots$$
$$= 1 + 24x + 240x^2 + 1280x^3 + \dots$$

(b) $(1-3x)^7 = [1+(-3x)]^7$
$$= 1 + {}^7C_1(-3x) + {}^7C_2(-3x)^2 + {}^7C_3(-3x)^3 + \dots$$
$$= 1 + 7(-3x) + 21(9x^2) + 35(-27x^3) + \dots$$
$$= 1 - 21x + 189x^2 - 945x^3 + \dots$$

Example

Find the terms in x^2 and x^5 in the expansion of $\left(1-\frac{x}{2}\right)^{12}$. Hence find the coefficient of x^5 in the expansion of $(3+2x^3)\left(1-\frac{x}{2}\right)^{12}$.

For $\left(1-\frac{x}{2}\right)^{12} = \left[1+\left(-\frac{x}{2}\right)\right]^{12}$, the term in x^r is $T_{r+1} = {}^{12}C_r\left(-\frac{x}{2}\right)^r$.

For x^2, $T_3 = {}^{12}C_2\left(-\frac{x}{2}\right)^2 = 66\left(\frac{x^2}{4}\right) = \frac{33}{2}x^2$

For x^5, $T_6 = {}^{12}C_5\left(-\frac{x}{2}\right)^5 = 792\left(-\frac{x^5}{32}\right) = -\frac{99}{4}x^5$

$$\therefore\ (3+2x^3)\left(1-\frac{x}{2}\right)^{12} = (3+2x^3)\left(\dots + \frac{33}{2}x^2 - \frac{99}{4}x^5 + \dots\right)$$
$$= \dots + 3\left(-\frac{99}{4}x^5\right) + 2x^3\left(\frac{33}{2}x^2\right) + \dots$$
$$= \dots - \frac{165}{4}x^5 + \dots$$

$\therefore$ coefficient of x^5 is $-\frac{165}{4}$.

Example

Find, in descending powers of x, the first 4 terms in the binomial expansion of

(a) $(2x-3)^5$, (b) $\left(x+\frac{1}{x^2}\right)^6$.

(a) $(2x-3)^5$

$= [2x+(-3)]^5$

$= (2x)^5 + {}^5C_1(2x)^4(-3) + {}^5C_2(2x)^3(-3)^2 + {}^5C_3(2x)^2(-3)^3 + \ldots$

$= 32x^5 + 5(16x^4)(-3) + 10(8x^3)9 + 10(4x^2)(-27) + \ldots$

$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + \ldots$

(b) $\left(x+\frac{1}{x^2}\right)^6 = x^6 + {}^6C_1x^5\left(\frac{1}{x^2}\right) + {}^6C_2x^4\left(\frac{1}{x^2}\right)^2 + {}^6C_3x^3\left(\frac{1}{x^2}\right)^3 + \ldots$

$= x^6 + 6x^3 + 15 + \frac{20}{x^3} + \ldots$

Example

Given that $\left(p-\frac{1}{2}x\right)^6 = r - 96x + sx^2 + \ldots$, find p, r and s.

$\left(p-\frac{1}{2}x\right)^6 = \left[p+\left(-\frac{1}{2}x\right)\right]^6$

$= p^6 + {}^6C_1p^5\left(-\frac{1}{2}x\right) + {}^6C_2p^4\left(-\frac{1}{2}x\right)^2 + \ldots$

$= p^6 - 3p^5x + \frac{15}{4}p^4x^2 + \ldots$

$\therefore \quad p^6 - 3p^5x + \frac{15}{4}p^4x^2 = r - 96x + sx^2$

Comparing coefficients:

x: $-3p^5 = -96 \Rightarrow p^5 = 32$

$\Rightarrow p = 2$

x^0: $r = p^6 = 64$

x^2: $s = \frac{15}{4}p^4 = 60$

Example

In each of the following expansions, find the indicated term.

(a) $\left(2+\frac{x^2}{2}\right)^{11}$, 7th term (b) $\left(2x-\frac{1}{x^2}\right)^{12}$, term in $\frac{1}{x^{15}}$

(a) For $\left(2+\frac{x^2}{2}\right)^{11}$, $T_{r+1} = {}^{11}C_r2^{11-r}\left(\frac{x^2}{2}\right)^r$

$\therefore \; T_7 = {}^{11}C_6 2^5\left(\frac{x^2}{2}\right)^6$

$= 462(2^5)\left(\frac{1}{2}\right)^6(x^2)^6$

$= 231x^{12}$

(b) For $\left(2x-\frac{1}{x^2}\right)^{12}$, $T_{r+1} = {}^{12}C_r(2x)^{12-r}(-x^{-2})^r$

Power of x in $T_{r+1} = 12 - r - 2r$

$= 12 - 3r$

For $\frac{1}{x^{15}}$ (i.e. x^{-15}), $12 - 3r = -15$

$r = 9$

$\therefore$ the term in $\frac{1}{x^{15}} = {}^{12}C_9(2x)^3(-x^{-2})^9$

$= -\frac{1760}{x^{15}}$

Practice Questions

Exercise #1

1. Expand each of the following:
   (a) $(1-2x)^4$ (b) $(1+3x)^5$ (c) $(1-ax)^6$

2. Show that $(1+\sqrt{x})^5-(1-\sqrt{x})^5 = 10\sqrt{x}+20x\sqrt{x}+2x^2\sqrt{x}$. Hence deduce the value of $(1+\sqrt{2})^5-(1-\sqrt{2})^5$.

3. Find the first 4 terms, in ascending powers of x, in the following expansions:
   (a) $(1+x)^{10}$ (b) $(1-x)^{12}$ (c) $(1-2x)^8$
   (d) $(1+2x)^9$ (e) $(1-3x)^8$ (f) $(1+x^2)^9$
   (g) $(1-2x^2)^7$ (h) $\left(1-\frac{1}{2}x^3\right)^{16}$ (i) $\left(1+\frac{2x}{y}\right)^8$

4. Find the non-zero values of a and b for which $(2x-a)^3 = 8x^3-bx^2+\frac{3}{2}bx-a^3$.

5. Find the first 3 terms, in ascending powers of x, in the expansion of $(1-2x)^9$ and of $(2+x)^5$. Hence expand $(1-2x)^9(2+x)^5$ up to the terms in x^2.

6. Find the indicated term in each of the following expansions.
   (a) $(2+x)^{10}$, 7th term (b) $(3x-2)^9$, 4th term
   (c) $(y-2x)^{10}$, 5th term (d) $\left(x+\frac{1}{2x^2}\right)^{12}$, middle term

7. In the expansion of $\left(x^3-\frac{2}{x^2}\right)^{10}$, find
   (a) the term in x^{10}, (b) the coefficient of $\frac{1}{x^5}$, (c) the constant term.

8. Expand $\left(\frac{1}{2}-2x\right)^5$ up to the term in x^3. If the coefficient of x^2 in the expansion of $(1+ax+3x^2)\left(\frac{1}{2}-2x\right)^5$ is $\frac{13}{2}$, find the coefficient of x^3.

9. In the expansion of $(1+x)(a-bx)^{12}$, where $ab \neq 0$, the coefficient of x^8 is zero. Find in its simplest form the value of the ratio $\frac{a}{b}$.

Answers

1. **(a)** $1 - 8x + 24x^2 - 32x^3 + 16x^4$ **(b)** $1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5$
(c) $1 - 6ax + 15a^2x^2 - 20a^3x^3 + 15a^4x^4 - 6a^5x^5 + a^6x^6$

2. $58\sqrt{2}$

3. **(a)** $1 + 10x + 45x^2 + 120x^3 + \ldots$ **(b)** $1 - 12x + 66x^2 - 220x^3 + \ldots$
(c) $1 - 16x + 112x^2 - 448x^3 + \ldots$ **(d)** $1 + 18x + 144x^2 + 672x^3 + \ldots$
(e) $1 - 24x + 252x^2 - 1512x^3 + \ldots$ **(f)** $1 + 9x^2 + 36x^4 + 84x^6 + \ldots$
(g) $1 - 14x^2 + 84x^4 - 280x^6 + \ldots$ **(h)** $1 - 8x^3 + 30x^6 - 70x^9 + \ldots$
(i) $1 + \frac{16x}{y} + \frac{112x^2}{y^2} + \frac{448x^3}{y^3} + \ldots$

4. $a = 3,\ b = 36$

5. $1 - 18x + 144x^2 + \ldots$; $32 + 80x + 80x^2 + \ldots$; $32 - 496x + 3248x^2 + \ldots$

6. **(a)** $3360x^6$ **(b)** $-489\ 888x^6$ **(c)** $3360x^4y^6$ **(d)** $14\frac{7}{16}x^{-6}$

7. **(a)** $3360x^{10}$ **(b)** $-15\ 360$ **(c)** $13\ 440$

8. $\frac{1}{32} - \frac{5}{8}x + 5x^2 - 20x^3 + \ldots$; $-33\frac{1}{8}$

9. $\frac{5}{8}$

Past Paper Questions

Binomial Expansion

1 Find the value of the coefficient of $\frac{1}{x}$ in the expansion of $\left(2x - \frac{1}{x}\right)^5$. [3]

2 Find the coefficient of x^3 in the expansion of

(i) $(1+2x)^6$, [3]

(ii) $(1-3x)(1+2x)^6$. [3]

3 (i) Find the first 3 terms in the expansion of $(2-x)^6$ in ascending powers of x. [3]

(ii) Find the value of k for which there is no term in x^2 in the expansion of $(1+kx)(2-x)^6$. [2]

4 The first three terms in the expansion of $(2+ax)^n$, in ascending powers of x, are $32-40x+bx^2$. Find the values of the constants n, a and b. [5]

5 (i) Find the first 3 terms in the expansion, in ascending powers of x, of $(2+x^2)^5$. [3]

(ii) Hence find the coefficient of x^4 in the expansion of $(1+x^2)^2(2+x^2)^5$. [3]

6 (i) Find the first 3 terms in the expansion of $(2+3x)^5$ in ascending powers of x. [3]

(ii) Hence find the value of the constant a for which there is no term in x^2 in the expansion of $(1+ax)(2+3x)^5$. [2]

7 (i) Find the first 3 terms in the expansion of $(1+ax)^5$ in ascending powers of x. [2]

(ii) Given that there is no term in x in the expansion of $(1-2x)(1+ax)^5$, find the value of the constant a. [2]

(iii) For this value of a, find the coefficient of x^2 in the expansion of $(1-2x)(1+ax)^5$. [3]

8 (i) Find the terms in x^2 and x^3 in the expansion of $\left(1-\frac{3}{2}x\right)^6$. [3]

(ii) Given that there is no term in x^3 in the expansion of $(k+2x)\left(1-\frac{3}{2}x\right)^6$, find the value of the constant k. [2]

9 The coefficient of x^3 in the expansion of $(a+x)^5+(2-x)^6$ is 90. Find the value of the positive constant a. [5]

10 Find the coefficient of x^2 in the expansion of

(i) $\left(2x-\frac{1}{2x}\right)^6$, [2]

(ii) $(1+x^2)\left(2x-\frac{1}{2x}\right)^6$. [3]

11 Find the coefficient of x^2 in the expansion of $(1+x^2)\left(\frac{x}{2}-\frac{4}{x}\right)^6$. [5]

12 (i) Find the coefficients of x^2 and x^3 in the expansion of $(2-x)^6$. [3]

(ii) Find the coefficient of x^3 in the expansion of $(3x+1)(2-x)^6$. [2]

13 Find the term that is independent of x in the expansion of

(i) $\left(x-\frac{2}{x}\right)^6$, [2]

(ii) $\left(2+\frac{3}{x^2}\right)\left(x-\frac{2}{x}\right)^6$. [4]

14 Find the value of the term which is independent of x in the expansion of $\left(x+\frac{3}{x}\right)^4$. [3]

15 Find the coefficient of x in the expansion of $\left(3x-\frac{2}{x}\right)^5$. [4]

16 Find the coefficient of x^2 in the expansion of $\left(x+\frac{2}{x}\right)^6$. [3]

17 (i) Find the first three terms in the expansion of $(2+u)^5$ in ascending powers of u. [3]

(ii) Use the substitution $u = x + x^2$ in your answer to part (i) to find the coefficient of x^2 in the expansion of $(2+x+x^2)^5$. [2]

18 Find the value of the coefficient of x^2 in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^6$. [3]

19 (i) Find, in terms of the non-zero constant k, the first 4 terms in the expansion of $(k+x)^8$ in ascending powers of x. [3]

(ii) Given that the coefficients of x^2 and x^3 in this expansion are equal, find the value of k. [2]

20 (i) Find the first 3 terms in the expansion, in ascending powers of x, of $(1-2x^2)^8$. [2]

(ii) Find the coefficient of x^4 in the expansion of $(2-x^2)(1-2x^2)^8$. [2]

21 (i) Find the first 3 terms in the expansion of $(2-y)^5$ in ascending powers of y. [2]

(ii) Use the result in part (i) to find the coefficient of x^2 in the expansion of $\left(2-(2x-x^2)\right)^5$. [3]

22 In the expansion of $\left(x^2-\frac{a}{x}\right)^7$, the coefficient of x^5 is -280. Find the value of the constant a. [3]

23

(i) Find the first 3 terms, in ascending powers of x, in the expansion of $(1+x)^5$. [2]

The coefficient of x^2 in the expansion of $\left(1+(px+x^2)\right)^5$ is 95.

(ii) Use the answer to part (i) to find the value of the positive constant p. [3]

24 In the expansion of $(x+2k)^7$, where k is a non-zero constant, the coefficients of x^4 and x^5 are equal. Find the value of k. [4]

25 In the expansion of $(3-2x)\left(1+\frac{x}{2}\right)^n$, the coefficient of x is 7. Find the value of the constant n and hence find the coefficient of x^2. [6]

Answers

1 coefficient of $\frac{1}{x} = -40$

2 (i) coefficient of $x^3 = 160$ (ii) coefficient of $x^3 = -20$

3 (i) $64 - 192x + 240x^2$ (ii) $k = \frac{5}{4}$

4 $n = 5$ $a = -\frac{1}{2}$ $b = 20$

5 (i) $32 + 80x^2 + 80x^4$ (ii) coefficient of $x^4 = 272$

6 (i) $32 + 240x + 720x^2$ (ii) $a = -3$

7 (i) $1 + 5ax + 10a^2x^2$ (ii) $a = \frac{2}{5}$ (iii) -2.4

8 (i) $-\frac{135}{2}x^3$ (ii) $k = 1$

9 $a = 5$

10 (i) co-efficient of $x^2 = 60$ (ii) coefficient of $x^2 = 40$

11 coefficient of $x^2 = -145$

12 (i) 240 and -160 (ii) coefficient of $x^3 = 560$

13 (i) -160 (ii) term independent of $x = -140$

14 54

15 coefficient of $x = 1080$

16 coefficient of $x^2 = 60$

17 (i) $32 + 80u + 80u^2$ (ii) coefficient of $x^2 = 160$

18 coefficient of $x^2 = \frac{15}{4}$

19 (i) $k^8 + 8k^7x + 28k^6x^2 + 56k^5x^3$ (ii) $k = 2$

20 (i) $1 - 16x^2 + 112x^4$ (ii) coefficient of $x^4 = 240$

21 (i) $32 - 80y + 80y^2$ (ii) coefficient of $x^2 = 400$

22 $a = 2$

23 (i) $1 + 5x + 10x^2$ (ii) $p = 3$

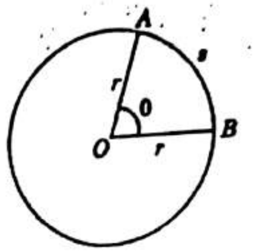
24 $k = \frac{3}{10}$

25 $n = 6$

CIRCULAR MEASURE

Syllabus:

- Understand the definition of a radian and use the relationship between range decrees;
- Use the formulae s = rθ and A = in solving problems concerning the arc length and sector area of a circle.



RADIAN MEASURE

The measure of angle subtended at the centre of a circle by as arc equal in length to the radius is called radian. The size of the angle is the ratio of the arc length to the length of the radius. Radian has no unit of measurement.

Relation between radian and degree measures

If θ is the angle of semi circle with radius 'r' then arc length S = $\frac{1}{2} \times 2\pi r = \pi r$ and in radian θ =

$\frac{s}{r} = \frac{\pi r}{r}$ $\theta = \pi$ radian

180º = π radian

To change given angle into radian multiply it by π/180 and to change given angle into degree, multiply it by 180/π.

Arc length is denoted by 's'. s = r θ where θ must be in radian.

Area of sector is denoted by A : A = $\frac{1}{2}$ r² θ OR A = $\frac{1}{2} rS$

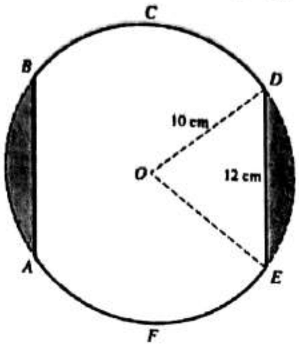
Perimeter of shaded region = sum of all outer boundaries of shaded region.

Shaded area = area of whole figure – unshaded area OR shaded area = sum of areas of all shaded figures.

Practice Questions

Exercise # 1

1



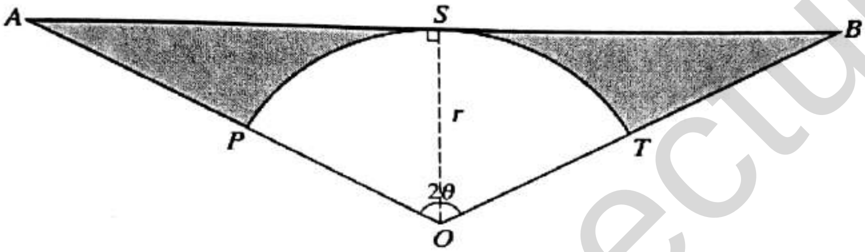
The diagram shows a metal plate $ABCDEF$ which has been made by removing the two shaded regions from a circle of radius 10 cm and centre O. The parallel edges AB and ED are both of length 12 cm.

(i) Show that angle DOE is 1.287 radians, correct to 4 significant figures. [2]

(ii) Find the perimeter of the metal plate. [3]

(iii) Find the area of the metal plate. [3]

2

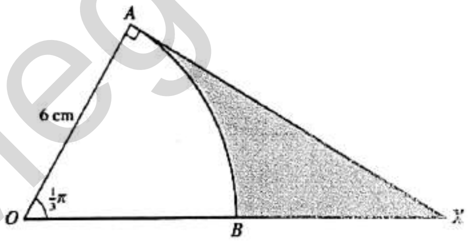


In the diagram, OAB is an isosceles triangle with $OA = OB$ and angle $AOB = 2\theta$ radians. Arc PST has centre O and radius r, and the line ASB is a tangent to the arc PST at S.

(i) Find the total area of the shaded regions in terms of r and θ. [4]

(ii) In the case where $\theta = \frac{1}{3}\pi$ and $r = 6$, find the total perimeter of the shaded regions, leaving your answer in terms of $\sqrt{3}$ and π. [5]

3



In the diagram, AB is an arc of a circle, centre O and radius 6 cm, and angle $AOB = \frac{1}{3}\pi$ radians. The line AX is a tangent to the circle at A, and OBX is a straight line.

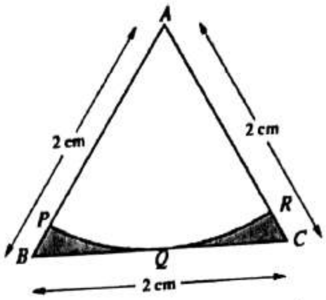
(i) Show that the exact length of AX is $6\sqrt{3}$ cm. [1]

Find, in terms of π and $\sqrt{3}$,

(ii) the area of the shaded region, [3]

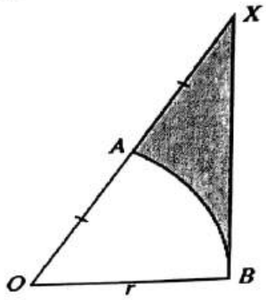
(iii) the perimeter of the shaded region. [4]

4



In the diagram, ABC is an equilateral triangle of side 2 cm. The mid-point of BC is Q. An arc of a circle with centre A touches BC at Q, and meets AB at P and AC at R. Find the total area of the shaded regions, giving your answer in terms of π and $\sqrt{3}$. [5]

5



In the diagram, AB is an arc of a circle with centre O and radius r. The line XB is a tangent to the circle at B and A is the mid-point of OX.

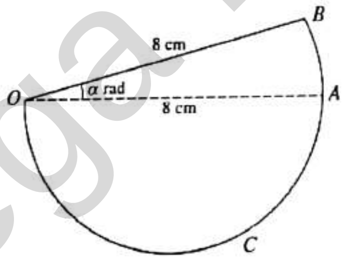
(i) Show that angle $AOB = \frac{1}{3}\pi$ radians. [2]

Express each of the following in terms of r, π and $\sqrt{3}$:

(ii) the perimeter of the shaded region, [3]

(iii) the area of the shaded region. [2]

6

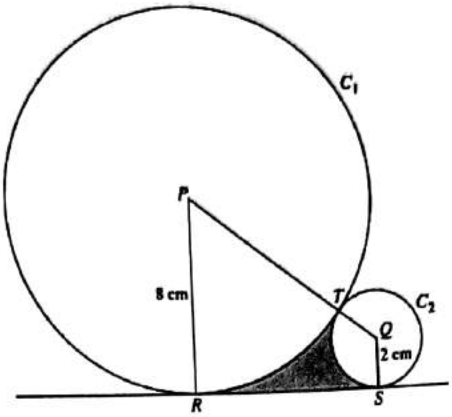


In the diagram, OAB is a sector of a circle with centre O and radius 8 cm. Angle BOA is α radians. OAC is a semicircle with diameter OA. The area of the semicircle OAC is twice the area of the sector OAB.

(i) Find α in terms of π. [3]

(ii) Find the perimeter of the complete figure in terms of π. [2]

7



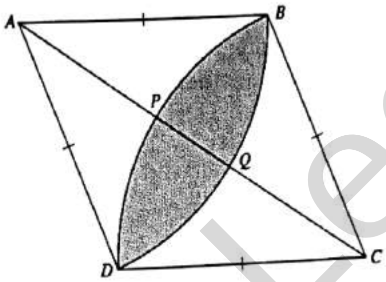
The diagram shows two circles, C_1 and C_2, touching at the point T. Circle C_1 has centre P and radius 8 cm; circle C_2 has centre Q and radius 2 cm. Points R and S lie on C_1 and C_2 respectively, and RS is a tangent to both circles.

(i) Show that $RS = 8$ cm. [2]

(ii) Find angle RPQ in radians correct to 4 significant figures. [2]

(iii) Find the area of the shaded region. [4]

8

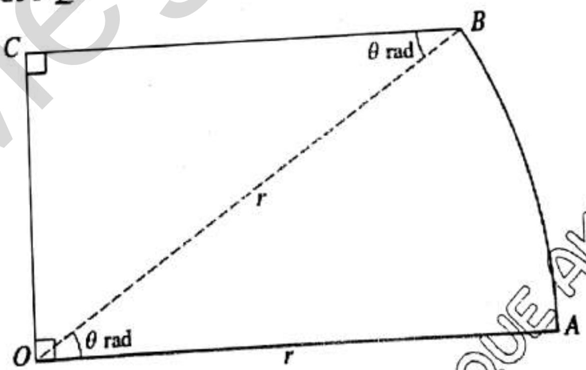


The diagram shows a rhombus $ABCD$. Points P and Q lie on the diagonal AC such that BPD is an arc of a circle with centre C and BQD is an arc of a circle with centre A. Each side of the rhombus has length 5 cm and angle $BAD = 1.2$ radians.

(i) Find the area of the shaded region $BPDQ$. [4]

(ii) Find the length of PQ. [4]

9

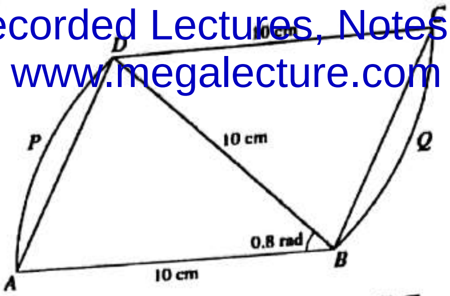


The diagram represents a metal plate $OABC$, consisting of a sector OAB of a circle with centre O and radius r, together with a triangle OCB which is right-angled at C. Angle $AOB = \theta$ radians and OC is perpendicular to OA.

(i) Find an expression in terms of r and θ for the perimeter of the plate. [3]

(ii) For the case where $r = 10$ and $\theta = \frac{1}{5}\pi$, find the area of the plate. [3]

10



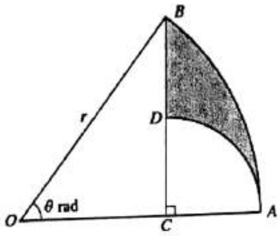
In the diagram, $ABCD$ is a parallelogram with $AB = BD = DC = 10$ cm and angle $ABD = 0.8$ radians. APD and BQC are arcs of circles with centres B and D respectively.

(i) Find the area of the parallelogram $ABCD$. [2]

(ii) Find the area of the complete figure $ABQCDP$. [2]

(iii) Find the perimeter of the complete figure $ABQCDP$. [2]

11

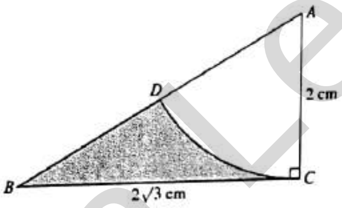


The diagram shows a sector OAB of a circle with centre O and radius r. Angle AOB is θ radians. The point C on OA is such that BC is perpendicular to OA. The point D is on BC and the circular arc AD has centre C.

(i) Find AC in terms of r and θ. [1]

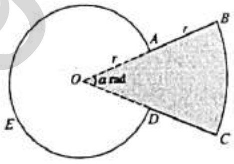
(ii) Find the perimeter of the shaded region ABD when $\theta = \frac{1}{3}\pi$ and $r = 4$, giving your answer as an exact value. [6]

12



In the diagram, D lies on the side AB of triangle ABC and CD is an arc of a circle with centre A and radius 2 cm. The line BC is of length $2\sqrt{3}$ cm and is perpendicular to AC. Find the area of the shaded region BDC, giving your answer in terms of π and $\sqrt{3}$. [4]

13



The diagram shows a metal plate made by fixing together two pieces, $OABCD$ (shaded) and $OAED$ (unshaded). The piece $OABCD$ is a minor sector of a circle with centre O and radius $2r$. The piece $OAED$ is a major sector of a circle with centre O and radius r. Angle AOD is α radians. Simplifying your answers where possible, find, in terms of α, π and r,

(i) the perimeter of the metal plate, [3]

(ii) the area of the metal plate. [3]

It is now given that the shaded and unshaded pieces are equal in area.

(iii) Find α in terms of π. [2]

Answers

1
$\rightarrow 61.1$
$\rightarrow 281$ or 282

2 Shaded area $= r^2(\tan\theta - \theta)$
Perimeter $= 12 + 12\sqrt{3} + 4\pi$

3 Area shaded $= 18\sqrt{3} - 6\pi$
Perimeter $= 6\sqrt{3} + 2\pi + 6$

4 Shaded region $= \sqrt{3} - \frac{\pi}{2}$

5 $\theta = \frac{1}{3}\pi$.
$P = r + (\frac{1}{3} r\pi + r\sqrt{3})$
(iii) Area $= \frac{1}{2}r^2\sqrt{3} - \frac{1}{6}r^2\pi$

6 $\alpha = \frac{\pi}{8}$ $\quad 8 + 5\pi$

7 $RS = 8$ cm.
angle $RPQ = 0.9273$ radians
$\rightarrow 5.90 \text{ cm}^2$

8
6.70
1.75

9 Total area $= 55.2$

10 Total area $= 80$, 36

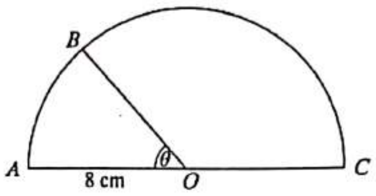
11 (i) $AC = r - r\cos\theta$ Perimeter $= \frac{7\pi}{3} + 2\sqrt{3} - 2$

12 Shaded area $= 2\sqrt{3} - \frac{2\pi}{3}$

13 $2\pi r + r\alpha + 2r$
$\frac{3r^2\alpha}{2} + \pi r^2$
$\alpha = \frac{2}{5}\pi$

Past Paper Questions

1



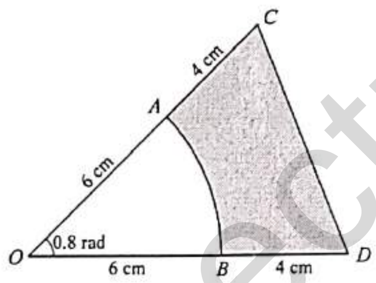
The diagram shows a semicircle ABC with centre O and radius 8 cm. Angle $AOB = \theta$ radians.

(i) In the case where $\theta = 1$, calculate the area of the sector BOC. [3]

(ii) Find the value of θ for which the perimeter of sector AOB is one half of the perimeter of sector BOC. [3]

(iii) In the case where $\theta = \frac{1}{3}\pi$, show that the exact length of the perimeter of triangle ABC is $(24 + 8\sqrt{3})$ cm. [3]

2

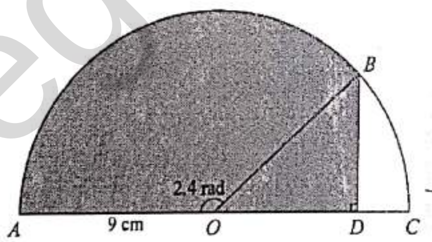


In the diagram, OCD is an isosceles triangle with $OC = OD = 10$ cm and angle $COD = 0.8$ radians. The points A and B, on OC and OD respectively, are joined by an arc of a circle with centre O and radius 6 cm. Find

(i) the area of the shaded region, [3]

(ii) the perimeter of the shaded region. [4]

3



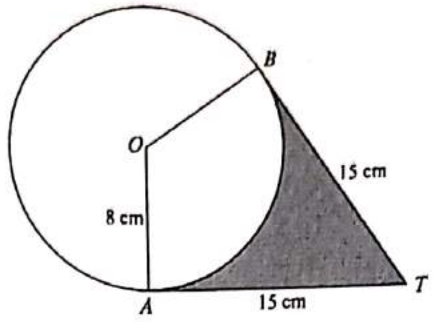
In the diagram, ABC is a semicircle, centre O and radius 9 cm. The line BD is perpendicular to the diameter AC and angle $AOB = 2.4$ radians.

(i) Show that $BD = 6.08$ cm, correct to 3 significant figures. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [3]

4

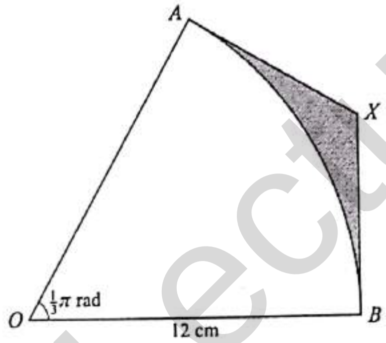


The diagram shows a circle with centre O and radius 8 cm. Points A and B lie on the circle. The tangents at A and B meet at the point T, and $AT = BT = 15$ cm.

(i) Show that angle AOB is 2.16 radians, correct to 3 significant figures. [3]

(ii) Find the perimeter of the shaded region. [2]

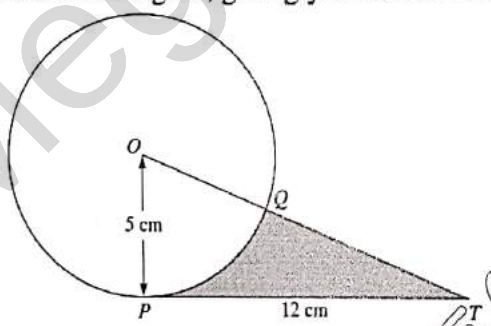
(iii) Find the area of the shaded region. [3]



In the diagram, OAB is a sector of a circle with centre O and radius 12 cm. The lines AX and BX are tangents to the circle at A and B respectively. Angle $AOB = \frac{1}{3}\pi$ radians.

(i) Find the exact length of AX, giving your answer in terms of $\sqrt{3}$. [2]

(ii) Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$. [3]

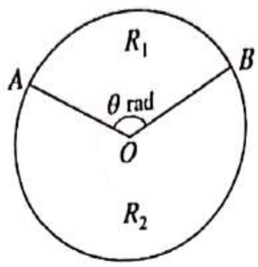


The diagram shows a circle with centre O and radius 5 cm. The point P lies on the circle, PT is a tangent to the circle and $PT = 12$ cm. The line OT cuts the circle at the point Q.

(i) Find the perimeter of the shaded region. [4]

(ii) Find the area of the shaded region. [3]

7

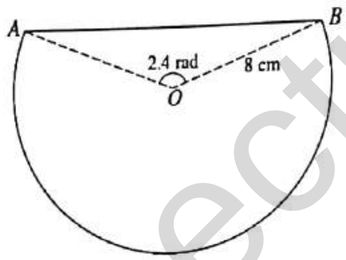


The diagram shows a circle with centre O. The circle is divided into two regions, R_1 and R_2, by the radii OA and OB, where angle $AOB = \theta$ radians. The perimeter of the region R_1 is equal to the length of the major arc AB.

(i) Show that $\theta = \pi - 1$. [3]

(ii) Given that the area of region R_1 is $30\,\text{cm}^2$, find the area of region R_2, correct to 3 significant figures. [4]

8



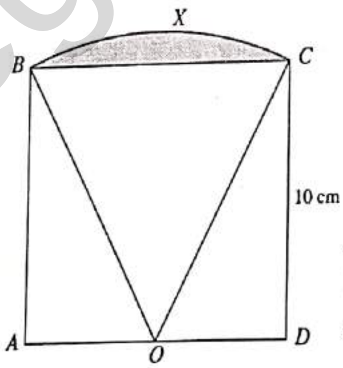
The diagram shows a metal plate made by removing a segment from a circle with centre O and radius 8 cm. The line AB is a chord of the circle and angle $AOB = 2.4$ radians. Find

(i) the length of AB, [2]

(ii) the perimeter of the plate, [3]

(iii) the area of the plate. [3]

9



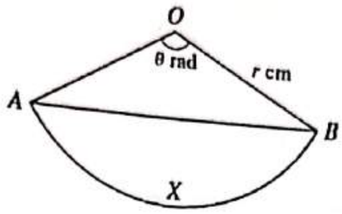
The diagram shows a square $ABCD$ of side 10 cm. The mid-point of AD is O and BXC is an arc of a circle with centre O.

(i) Show that angle BOC is 0.9273 radians, correct to 4 decimal places. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [2]

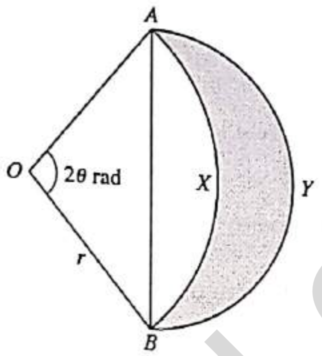
10



The diagram shows a sector of a circle with radius r cm and centre O. The chord AB divides the sector into a triangle AOB and a segment AXB. Angle AOB is θ radians.

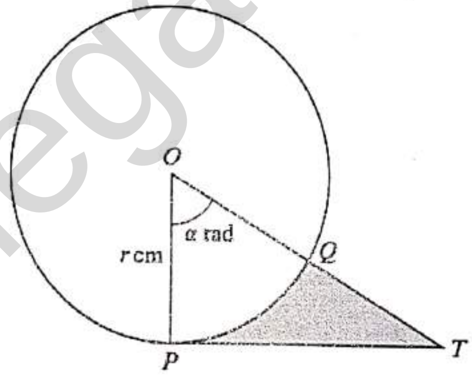
(i) In the case where the areas of the triangle AOB and the segment AXB are equal, find the value of the constant p for which $\theta = p \sin \theta$. [2]

11 (ii) In the case where $r = 8$ and $\theta = 2.4$, find the perimeter of the segment AXB. [3]



In the diagram, AYB is a semicircle with AB as diameter and $OAXB$ is a sector of a circle with centre O and radius r. Angle $AOB = 2\theta$ radians. Find an expression, in terms of r and θ, for the area of the

12 shaded region. [4]

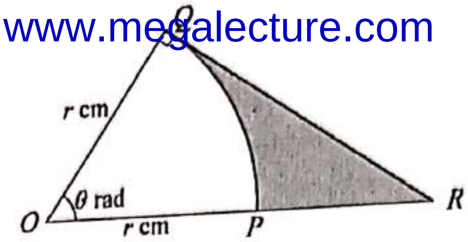


The diagram shows a circle with radius r cm and centre O. The line PT is the tangent to the circle at P and angle $POT = \alpha$ radians. The line OT meets the circle at Q.

(i) Express the perimeter of the shaded region PQT in terms of r and α. [3]

(ii) In the case where $\alpha = \frac{1}{3}\pi$ and $r = 10$, find the area of the shaded region correct to 2 significant figures. [3]

13

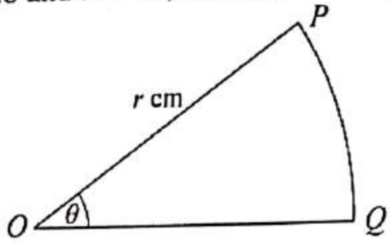


In the diagram, OPQ is a sector of a circle, centre O and radius r cm. Angle $QOP = \theta$ radians. The tangent to the circle at Q meets OP extended at R.

(i) Show that the area, $A\text{ cm}^2$, of the shaded region is given by $A = \frac{1}{2}r^2(\tan\theta - \theta)$. [2]

(ii) In the case where $\theta = 0.8$ and $r = 15$, evaluate the length of the perimeter of the shaded region. [4]

14



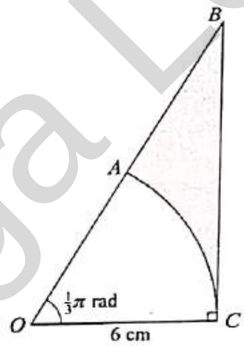
The diagram shows the sector OPQ of a circle with centre O and radius r cm. The angle POQ is θ radians and the perimeter of the sector is 20 cm.

(i) Show that $\theta = \dfrac{20}{r} - 2$. [2]

(ii) Hence express the area of the sector in terms of r. [2]

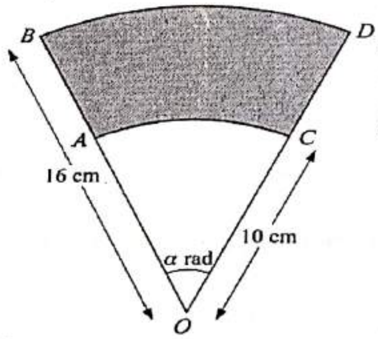
(iii) In the case where $r = 8$, find the length of the chord PQ. [3]

15



In the diagram, AC is an arc of a circle, centre O and radius 6 cm. The line BC is perpendicular to OC and OAB is a straight line. Angle $AOC = \frac{1}{3}\pi$ radians. Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$. [5]

16

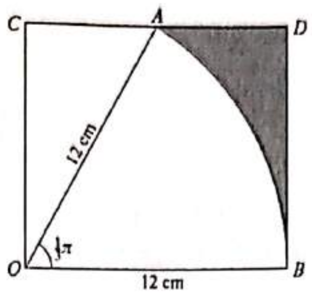


In the diagram, OAB and OCD are radii of a circle, centre O and radius 16 cm. Angle $AOC = \alpha$ radians. AC and BD are arcs of circles, centre O and radii 10 cm and 16 cm respectively.

(i) In the case where $\alpha = 0.8$, find the area of the shaded region. [2]

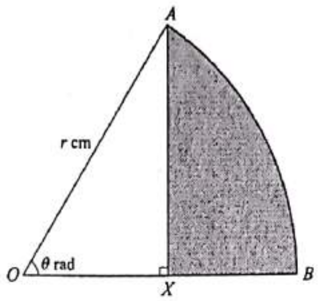
(ii) Find the value of α for which the perimeter of the shaded region is 28.9 cm. [3]

17



In the diagram, AOB is a sector of a circle with centre O and radius 12 cm. The point A lies on the side CD of the rectangle $OCDB$. Angle $AOB = \frac{1}{3}\pi$ radians. Express the area of the shaded region in the form $a(\sqrt{3}) - b\pi$, stating the values of the integers a and b. [6]

18



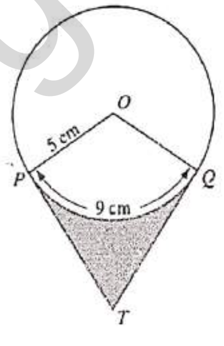
In the diagram, AB is an arc of a circle, centre O and radius r cm, and angle $AOB = \theta$ radians. The point X lies on OB and AX is perpendicular to OB.

(i) Show that the area, A cm^2, of the shaded region AXB is given by

$$A = \frac{1}{2}r^2(\theta - \sin\theta\cos\theta).$$ [3]

(ii) In the case where $r = 12$ and $\theta = \frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π. [4]

19



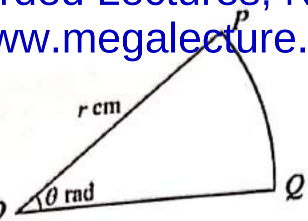
In the diagram, the circle has centre O and radius 5 cm. The points P and Q lie on the circle, and the arc length PQ is 9 cm. The tangents to the circle at P and Q meet at the point T. Calculate

(i) angle POQ in radians, [2]

(ii) the length of PT, [3]

(iii) the area of the shaded region. [3]

20



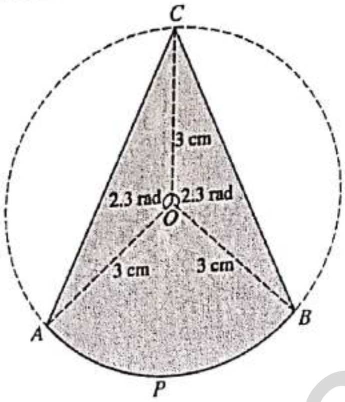
A piece of wire of length 50 cm is bent to form the perimeter of a sector POQ of a circle. The radius of the circle is r cm and the angle POQ is θ radians (see diagram).

(i) Express θ in terms of r and show that the area, A cm^2, of the sector is given by [4]

$$A = 25r - r^2.$$

(ii) Given that r can vary, find the stationary value of A and determine its nature. [4]

21

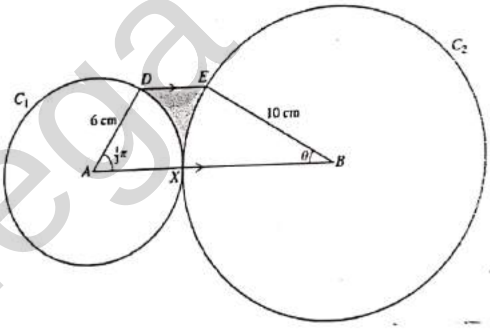


The diagram shows points A, C, B, P on the circumference of a circle with centre O and radius 3 cm. Angle AOC = angle BOC = 2.3 radians.

(i) Find angle AOB in radians, correct to 4 significant figures. [1]

(ii) Find the area of the shaded region $ACBP$, correct to 3 significant figures. [4]

22

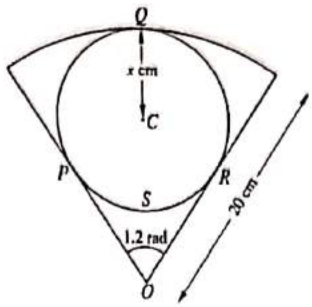


The diagram shows a circle C_1 touching a circle C_2 at a point X. Circle C_1 has centre A and radius 6 cm, and circle C_2 has centre B and radius 10 cm. Points D and E lie on C_1 and C_2 respectively and DE is parallel to AB. Angle $DAX = \frac{1}{3}\pi$ radians and angle $EBX = \theta$ radians.

(i) By considering the perpendicular distances of D and E from AB, show that the exact value of θ is $\sin^{-1}\left(\frac{3\sqrt{3}}{10}\right)$. [3]

(ii) Find the perimeter of the shaded region, correct to 4 significant figures. [5]

23



The diagram shows a sector of a circle with centre O and radius 20 cm. A circle with centre C and radius x cm lies within the sector and touches it at P, Q and R. Angle $POR = 1.2$ radians.

(i) Show that $x = 7.218$, correct to 3 decimal places. [4]

(ii) Find the total area of the three parts of the sector lying outside the circle with centre C. [2]

(iii) Find the perimeter of the region $OPSR$ bounded by the arc PSR and the lines OP and OR. [4]

24

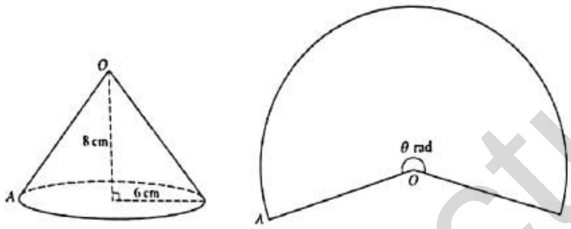


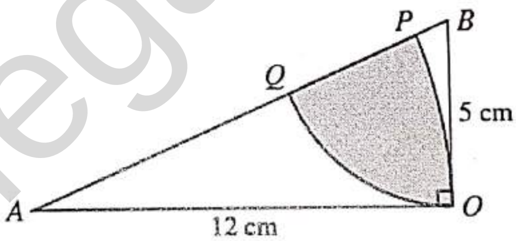
Fig. 1 Fig. 2

Fig. 1 shows a hollow cone with no base, made of paper. The radius of the cone is 6 cm and the height is 8 cm. The paper is cut from A to O and opened out to form the sector shown in Fig. 2. The circular bottom edge of the cone in Fig. 1 becomes the arc of the sector in Fig. 2. The angle of the sector is θ radians. Calculate

(i) the value of θ, [4]

(ii) the area of paper needed to make the cone. [2]

25



The diagram shows a triangle AOB in which OA is 12 cm, OB is 5 cm and angle AOB is a right angle. Point P lies on AB and OP is an arc of a circle with centre A. Point Q lies on AB and OQ is an arc of a circle with centre B.

(i) Show that angle BAO is 0.3948 radians, correct to 4 decimal places. [1]

(ii) Calculate the area of the shaded region. [5]

Answers

1 (i) $68.5\ cm^2$ (ii) : 0.381 radian

2 (i) : $21.5\ cm^2$ (ii) : 20.6 cm

3 (ii) : 43.3 cm (iii) : $117\ cm^2$

4 (ii) : 47.3 cm (iii) $50.9\ cm^2$

5 (i) $AX = \frac{12}{\sqrt{3}}$ (ii) $(48\sqrt{3} - 24\pi)$

6 (i) : 25.9 cm (ii) : $15.3\ cm^2$

7 (ii) : $58.0\ cm^2$

8 (i) 14.9 cm (ii) 46 cm (iii) $146\ cm^2$

9 (ii) $7.96\ cm^2$

10 (i) $p = 2$ (ii) 34.1 units

11 : $\frac{1}{2}r^2(\pi \sin^2\theta - 2\theta + \sin 2\theta)$

12 (i) $r\alpha + r\tan\alpha + \frac{r}{\cos\alpha} - r$ (ii) $34\ cm^2$

13 (ii) : 34.0cm

14 (ii) Area $= (10r - r^2)$ (iii) : 3.96 cm

15 (ii) : $6(3\sqrt{3} - \pi)\ cm^2$

16 (i) $62.4\ cm^2$ (ii) $\alpha = 0.65$ rad

17 $a = 54$, $b = 24$

18 (ii) $2(\pi + 9 - 3\sqrt{3})$ cm

19 (i) 1.8 radians (ii) $PT = 6.3$ cm (iii) $9\ cm^2$

20 (ii) $156.25\ cm^2$ ∴ A is a maximum

21 (i) : 1.683 radians (ii) : $14.3\ cm^2$

22 (ii) 16.20 cm.

23 (ii) $76.3\ cm^2$ (iii) : 35.1 cm

24 (i) $\theta = 1.2\pi$ radians. (ii) $60\pi\ cm^2$

25 (i) : 0.3948 radians (ii) $13.1\ cm^2$

TRIGONOMETRY

Syllabus:

- Sketch and use graphs of the sine, cosine and tangent functions (for angles of any size and using either degrees or radians):
- Use the exact values of the sine. Cosine and tangent of 30º. 45º 60º and related angles, e.g. cos 150º = $\frac{1}{2}\sqrt{3}$
- Use the notations sin x. cos x tan x to denote the principal values of the inverse trigonometric relations;
- Use the identities $\frac{\text{Sin}\theta}{\text{Cos}\theta} = \tan\theta$ and $\sin^2\theta + \text{Cos}^2\theta = 1$
- Find all the solutions of simple trigonometrical equations lying in a specified interval (general forms of solution are not included).

BASIC ANGLE

The +ive acute angle between angle ray and nearest x-axis is called basic angle. Its value always +ive either counted clock wise or anticlock wise. Calculated angle by calculator with out using sign of angle value is basic angle. The sign of angle value only indicates the position of angle ray in quadrant and should not be used in calculating angle.

General angle

The angle counted form +ive x-axis to angle ray is called general angle. If angle is counted anticlock wise, its sign is +ive. If it is counted clock wise, its sign is –ive. All required angles are considered as general angles. To find out all possible angles of any trigonometric ratio in given domain of function, first find its basic angle by calculator without using its sign. Then with the help of sign, draw angle rays in concerned quadrants and find all possible general angles which are the required angles.

Trigonometric ratios of complementary angles

Sin (90-θ) = Cos θ (2) Cos (90-θ) = Sin θ

(3) tan (90-θ) Cot θ

Trigonometric ratios of negative angles

For negative angles take sign of all trigonometric ratios in 4th quadrant.

(1) Cost (-θ) = Cos θ (2) Sin (-θ) = - sin θ (3) tan (-θ) = - tan θ

Solution of triangle

To solve right angled triangle, apply trigonometric ratios sine, cos or tan. To solve oblique triangle (with no right angle), apply sine rule or cosine rule. Sine rule : $\frac{a}{SinA} = \frac{b}{Sin\,B} = \frac{c}{SinC}$ Co sine rule: $a^2=b^2+c^2-2bc\,\text{Cos}\,A$

Sine rule is applicable, when two angles and one side of a triangle is given OR two sides and one of their opposite angle is given.

Cosine rule is applicable when two sides and their included angle is given OR all three sides are given.

- If given angle is in composite form i.e. (ax+b) where a and b. are any constants, then first change given domain of angle (x) into the doman of (ax+b) and then suppose composite angle equal to any simple angle like α or θ and find all its possible angles in given domain, then replace the value of substituted angle in general angle and find value of required angle.
- To find value of any trigonometric function if value of other trigonometric function is given, first draw angle ray of given trigonometric ratio in concerned quadrant and make right angled triangle with x-axis and given angle ray and find unknown side of triangle by Pythagoras theorem and then apply formula of required trigonometric ratio and find its value. Exp: if $x = \text{Sin}^{-1}\left(\frac{2}{5}\right)$, find Cos x and tan x when $90 < x < 180$.

Solution of trigonometric equation

- To find solution of trigonometric equation, if equation is in fraction or non linear, then first simplify the equation and change it into quadratic form by substituting the value of larger power, trigonometric ratio in term of smaller power trigonometric ratio in given function, solve it and find all possible general angles in given domain. For simplification, change tan, cot, sec and cosec trigonometric ratio into sine or cos if directly its value by formula is not possible. If domain of function is in degree, then write your answer in degree. If domain is in radian, then answer must be in radian form.

Conversion of trigonometric function

- To show that the given trigonomic equation can be written in quadratic form, first of all change given equation into required trigonometric ratio by substituting the value of given trigonometric ratio or by simplification. Then replace it in to quadratic function.

Exp: Show that $\text{Sin}^2\theta + 3\,\text{Sin}\,\theta\,\text{Cos}\,\theta = 4\,\text{Cos}^2\theta$ can be written as a quadratic equation in tan θ.

Sketching of trigonometric graphs:

In trigonometric equation y = a+ b sin c x where a,b and c are constants y = a is line of axis of curve, b = amplitude of function and 360/c is period function.

Line of axis of curve

The line parallel to x axis from which the curve move at equal length to upward and downward OR line of symmetry of the curve is called line of axis. It is obtained by putting value of trigonometric ratio equal to zero.

Amplitude of function

The distance of maximum or minimum point from the line of axis of curve is called amplitude. OR coefficient of trigonometric ratio without sign is amplitude of function.

Period of trigonometric function

The smallest +ive number which is added to the original angle and give the same value of the trigonometric function OR graphically the length of the interval over which the graph of function repeats itself is called period of function.

Range of function

The distance between maximum and minimum position of the curve along y – axis is called range of function numerically. To find range of trigonometric function, put minimum and maximum value of trigonometric ratio in given equation and simplify. Its answer is denoted by the range of the function.

Maximum or minimum value of function

To find maximum value of function, put maximum value of trigonometric ratio in given equation and to find minimum value, put minimum value of trigonometric ratio and simplify.

Asymptotes

The vertical or horizontal lines through the point, on which the given function is undefined are called asymptotes, the graph of the function moves parallel along the asymptotes.

Sketching of sine and cos graph

1. Draw co ordinate axis and mark angles at x-axis and their values on y-axis. If domain of function is given in radian than write angles in radian. Usually mark quadrant angles on x-axis and then take three equal intervals in each quadrant.
2. Draw line of axis of curve by putting value of trigonometric ratio equal to zero.
3. Put quadrant angles in given equation if co efficient of given angle is 1, simplify the equation and mark its answer along y – axis with respect to given angle. Also find x-intercept of the graph by putting y = 0 in given equation and mark it as point of intersection of curve and x-axis. If co efficient of angle is more than 1, then first divide quadrant angle by co-efficient of given angle and put answer in given equation and mark its value. Continue that process till first period of the function completed. Then repeat the all remaining graph by same pattern up to given domain of function.

Tangent and cotangent graph

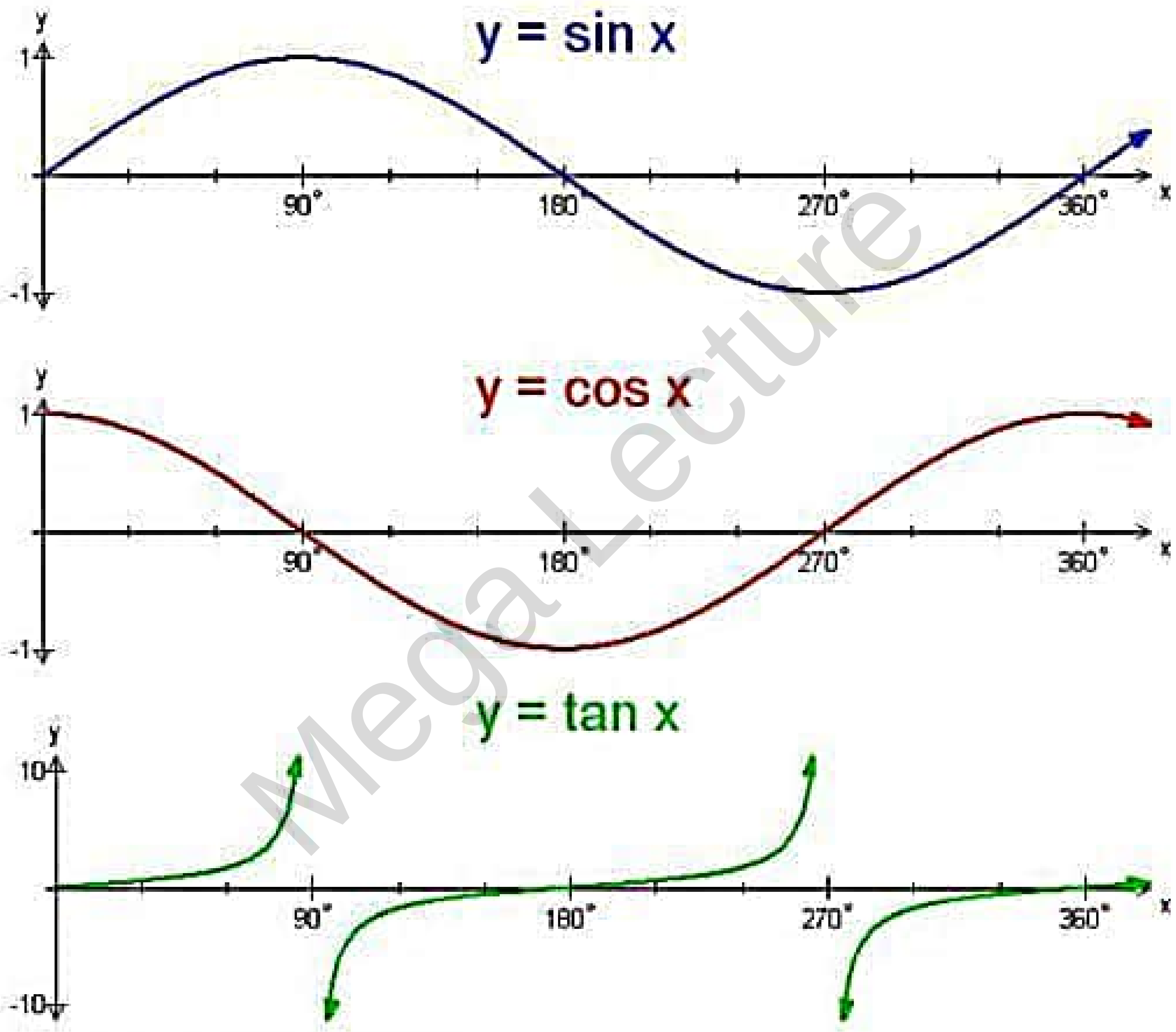
1. Draw co ordinate axis mark angles on x-axis and its values on –axis with the interval of co efficient of trigonometric ratio.
2. Draw line of axis and asymptotes (vertical dotted lines) though the angle on which function is undefined.
3. Usually put two angles of equal intravel from each quadrant and find value of given function and mark it on the y axis. Also find x-intercept of the curve by putting y = 0 in given function. Join all these points by smooth curve and draw parallel graph along asymptotes. Repeat the process up to given domain of function.
4. To find point of intersection of trigonometric graph and a line. Draw graph of each side of given equation by putting it equal to y or draw by given conditions. Then find co ordinate of point of intersection from graph.
5. For more accurate graph of sine and cos, first find period of one cycle by 360/c, divide the period into four equal parts and find their corresponding values of y from given equation and sketch them.
6. To prove trigonometric identity, if direct substitution of given trigonometric ratio is not possible, then first change, tan, cot, sec and cosc function into sine or cos and then simplify, some basic trigonometric identities are

i) $\sec\theta = \dfrac{1}{\cos\theta}$

ii) $\operatorname{Cosec}\theta = \dfrac{1}{\sin\theta}$

iii) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

iv) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$


y = sin x
y
1
-1
90°
180
270°
360°
x
y = cos x
y
1
-1
90
180°
270°
360°
x
y = tan x
y
10
-10
90°
180°
270°
360°
x

v) $\sin^2\theta + \cos^2\theta = 1$
vi) $1+\tan^2\theta=\sec^2\theta$
vii) $1+\cot^2\theta=\text{Cosec}^2\theta$

Practice Questions

Exercise #1

Prove the identities

1 $\dfrac{1+\cos x}{1+\sec x} \equiv \cos x$

2 $1-2\sin^2 x \equiv 2\cos^2 x - 1$

3 $1-\dfrac{\cos^2 x}{1+\sin x} \equiv \sin x$

4 $\cot x + \tan x \equiv \operatorname{cosec} x \sec x$

5 $\dfrac{\sin x}{\operatorname{cosec} x - \cot x} \equiv 1+\cos x$

6 $\sin^4 x - \cos^4 x \equiv \sin^2 x - \cos^2 x$

7 $\sin^4 x - \sin^2 x \equiv \cos^4 x - \cos^2 x$

8 $(1-\cos x)\left(1+\dfrac{1}{\cos x}\right) \equiv \sin x \tan x$

9 $\sin^2 x + \tan^2 x \sin^2 x \equiv \tan^2 x$

10 $\tan^2 x - \cot^2 x \equiv \sec^2 x - \operatorname{cosec}^2 x$

11 $\dfrac{1-\tan^2 x}{1+\tan^2 x} \equiv \cos^2 x - \sin^2 x$

12 $\dfrac{1}{\sin\theta+1} - \dfrac{1}{\sin\theta-1} \equiv 2\sec^2\theta$

13 $(1+\tan x - \sec x)(1+\cot x + \operatorname{cosec} x) \equiv 2$

14 $\dfrac{\sin x}{\sec x - 1} + \dfrac{\sin x}{\sec x + 1} \equiv 2\cot x$

15 $\dfrac{\sec\theta}{\sec\theta+\tan\theta} + \dfrac{\tan\theta}{\sec\theta-\tan\theta} \equiv 1+2\tan^2\theta$

16 $\dfrac{\cos A + \sec B}{\cos B + \sec A} \equiv \cos A \sec B$

17 $\dfrac{3-6\cos^2 x}{\sin x - \cos x} \equiv 3(\sin x + \cos x)$

18 $\dfrac{1+\cos A}{1-\cos A} - \dfrac{1-\cos A}{1+\cos A} \equiv 4\cot A \operatorname{cosec} A.$

19 $\dfrac{2-\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A + 2\cot A} \equiv \dfrac{\sin A - \cos A}{\sin A + \cos A}.$

20 $\cos^2 x + \cot^2 x \cos^2 x \equiv \cot^2 x.$

Exercise #2

1 Find all the angles between 0° and 360° which satisfy the equation

(a) $5 \cos x + 2 \sin x = 0$, (b) $3(\sin x - \cos x) = \cos x$.

2 Find all the angles between 0° and 360° inclusive which satisfy the following equations.

(a) $4 \sin x \cos x = \cos x$ (b) $2 \cos^2 x - \cos x = 1$

(c) $2 \tan x = 4 - \sec^2 x$ (d) $2 \sin x \cos x = \tan x$

3 Find all the angles between 0° and 360° which satisfy the following equations.

(a) $\cos 2x = 0.5$ (b) $\tan (x - 60°) = \frac{1}{\sqrt{3}}$

(c) $3 \sin 2x + 2 = 0$ (d) $\cos (2x - 40°) = 0.8$

(e) $\cot (2x + 10°) = -0.5$ (f) $\operatorname{cosec} (2x + 60°) = 4$

4 Find all the angles between 0° and 360° which satisfy the equation

(a) $\sin \frac{1}{2}x = \frac{1}{2}$, (b) $3 \cos y = 2 \sec y$, (c) $4 \tan z + \cot z = 5$.

5 Find all the angles between 0° and 360° which satisfy the equation

(a) $\tan (2x - 60°) = -1$, (b) $2 \sin y = \tan y$, (c) $\sec^2 z = 4 \sec z - 3$.

6 Solve the following equations for angles between 0° and 360° inclusive.

(a) $2 \cos 2x + 1 = \sqrt{2}$ (b) $3 \cot^2 x = \operatorname{cosec} x \cot x$

(c) $2 \cos^2 x + 3 \sin x = 3$ (d) $3 \sin x + 2 \tan x = 0$

(e) $|\sec (x - 50°)| = 3$ (f) $\sec^2 x + 2 \tan^2 x = 4$

(g) $2 \sin x \cos x + \cos^2 x = 1$ (h) $3 \cos (x + 40°) = 4 \sin (x + 40°)$

(i) $(\cos x - 2)(\cos x + 1) = \sin^2 x$ (j) $2 \sin x \tan x = 3$

(k) $\cot \left(\frac{x}{2} - 10°\right) = \frac{1}{2}$ (l) $2 \sin^2 x - 5 \sin x \cos x = 3 \cos^2 x$

7 Find the values of x, where $0° < x < 180°$ such that

(a) $2 \cos^2 x + \sin 20° = 1$, (b) $\sin (3x + 70°) = 0.2$,

(c) $8 \operatorname{cosec} 2x \cot 2x = 3$, (d) $|2 \cos x + 3 \sin x| = \sin x$.

8 Find all the angles between 0° and 360° which satisfy the following equations.

(a) $2 \tan 2x + \sec 40° = 1$ (b) $2 + \sin y \cos y = 2 \sin^2 y$

(c) $2 \cos^2 z - \sin z + 1 = 0$

9 Find all the angles between 0° and 360° inclusive which satisfy the equation.

(a) $3 \sin x - 2 \operatorname{cosec} x = 1$, (b) $\tan y = 2 \sin y \sin 40°$,

(c) $\cos (2x - 70°) = 0.5$.

10 Find all the angles between 0° and 360° which satisfy the equation

(a) $5 \cos^2 x - 8 \sin x \cos x = 0$,

(b) $5 \tan^2 y + 7 = 11 \sec y$,

(c) $1 + 2 \sin \left(\frac{3z}{2} + 75°\right) = 0$.

Answers

1. (a) 111.8°, 291.8° (b) 53.1°, 233.1°
2. (a) 14.5°, 90°, 165.5°, 270° (b) 0°, 120°, 240°, 360°
   (c) 45°, 108.4°, 225°, 288.4° (d) 0°, 45°, 135°, 180°, 225°, 315°, 360°
3. (a) 30°, 150°, 210°, 330° (b) 90°, 270°
   (c) 110.9°, 159.1°, 290.9°, 339.1° (d) 1.6°, 38.4°, 181.6°, 218.4°
   (e) 53.3°, 143.3°, 233.3°, 323.3° (f) 52.8°, 157.2°, 232.8°, 337.2°
4. (a) 60°, 300° (b) 35.3°, 144.7°, 215.3°, 324.7° (c) 14.0°, 45°, 194.0°, 225°
5. (a) 7.5°, 97.5°, 187.5°, 277.5° (b) 60°, 180°, 300° (c) 70.5°, 289.5°
6. (a) 39.0°, 141.0°, 219.0°, 321.0° (b) 70.5°, 90°, 270°, 289.5°
   (c) 30°, 90°, 150° (d) 0°, 131.8°, 180°, 228.2°, 360°
   (e) 120.5°, 159.5°, 300.5°, 339.5° (f) 45°, 135°, 225°, 315°
   (g) 0°, 63.4°, 180°, 243.4°, 360° (h) 176.9°, 356.9°
   (i) 180° (j) 60°, 300° (k) 146.9° (l) 71.6°, 153.4°, 251.6°, 333.4°
7. (a) 55°, 125° (b) 32.8°, 100.5°, 152.8° (c) 35.3°, 144.7° (d) 135°, 153.4°
8. (a) 85.7°, 175.7°, 265.7°, 355.7° (b) 90°, 116.6°, 270°, 296.6° (c) 90°
9. (a) 90°, 221.8°, 318.2° (b) 0°, 38.9°, 180°, 321.1°, 360°
   (c) 5°, 65°, 185°, 245°
10. (a) 32.0°, 90°, 212.0°, 270° (b) 60°, 300° (c) 90°, 170°, 330°

Exercise #3

1. Find the values of θ, where $0° \leqslant \theta \leqslant 360°$, such that
   (a) $(\sin\theta - 1)(\sin\theta + 1) = 0$,
   (b) $(\cos\theta - 1)(\cos\theta + 1) = 0$,
   (c) $\sin\theta(2\cos\theta - 3) = 0$,
   (d) $\tan\theta(2\cos\theta - 1) = 0$,
   (e) $\sin^2\theta(\tan\theta + 4) = 0$,
   (f) $(3\sin\theta - 1)(\tan\theta + 1) = 0$.

2. If $0° \leqslant \theta \leqslant 360°$, find the minimum and maximum values of
   (a) $7\sin\theta - 3$,
   (b) $5\cos\theta + 2$,
   (c) $4 - 3\sin\theta$.

3. Sketch, on separate diagrams, the following curves for the domain $0° \leqslant x \leqslant 360°$ and state the corresponding range of y.
   (a) $y = \sin x - 2$
   (b) $y = 5\cos x$
   (c) $y = 4\sin x - 2$
   (d) $y = 2\cos x + 1$
   (e) $y = 3(\cos x - 1)$
   (f) $y = |3\sin x - 2|$
   (g) $y = |5\tan x|$
   (h) $y = 2\tan x - 3$
   (i) $y = |3\tan x + 2|$

4. Sketch, on separate diagrams, the following graphs.
   (a) $y = 4\cos x - 3$ for $0° \leqslant x \leqslant 180°$
   (b) $y = 6\tan x$ for $0° \leqslant x \leqslant 180°$
   (c) $y = 3 + 2\sin x$ for $0° \leqslant x \leqslant 270°$
   (d) $y = \cos x - 1$ for $0° \leqslant x \leqslant 720°$
   (e) $y = 1 + 3\sin x$ for $-360° \leqslant x \leqslant 720°$

5. On the same diagram, sketch the graphs of $y = |2\sin x|$ and $y = |2\sin x| + 1$ for $0° \leqslant x \leqslant 360°$.

6. Sketch the graph of $y = 3|\cos x| - 2$ for $0° \leqslant x \leqslant 360°$.

7. On the same diagram, sketch the graphs of $y = 3\cos x$ and $y = 2\sin x - 1$ for the interval $0° \leqslant x \leqslant 360°$. State the number of solutions, in this interval, of the equation $3\cos x + 1 = 2\sin x$.

8. Use a graphical method to determine how many solutions there are to the equation $|2\tan x| = 1 + \sin x$ in the interval $0° \leqslant x \leqslant 360°$.

9. Sketch, on the same diagram, the graphs of $y = \tan x$ and $y = \cos x$, for the values of x from 0° and 360°. Hence state
   (a) the number of roots of the equation $\tan x = \cos x$ in the range 0° to 360°,
   (b) the range of values of x, between 0° and 270°, for which $\tan x$ and $\cos x$ are both increasing as x increases. *(C)*

Answers

1. **(a)** $90°, 270°$ **(b)** $0°, 180°, 360°$
(c) $0°, 180°, 360°$ **(d)** $0°, 60°, 180°, 300°, 360°$
(e) $0°, 104.0°, 180°, 284.0°, 360°$ **(f)** $19.5°, 135°, 160.5°, 315°$

2. **(a)** $-10, 4$ **(b)** $-3, 7$ **(c)** $1, 7$

3. **(a)** $\{y : -3 \leqslant y \leqslant -1, y \in \mathbb{R}\}$ **(b)** $\{y : -5 \leqslant y \leqslant 5, y \in \mathbb{R}\}$
(c) $\{y : -6 \leqslant y \leqslant 2, y \in \mathbb{R}\}$ **(d)** $\{y : -1 \leqslant y \leqslant 3, y \in \mathbb{R}\}$
(e) $\{y : -6 \leqslant y \leqslant 0, y \in \mathbb{R}\}$ **(f)** $\{y : 0 \leqslant y \leqslant 5, y \in \mathbb{R}\}$
(g) $\{y : y \geqslant 0, y \in \mathbb{R}\}$ **(h)** $\mathbb{R}$ **(i)** $\{y : y \geqslant 0, y \in \mathbb{R}\}$

7. 2 **8.** 4

9. **(a)** 2 **(b)** $180° < x < 270°$

Exercise #4

1 The acute angle x radians is such that $\tan x = k$, where k is a positive constant. Express, in terms of k,

(i) $\tan(\pi - x)$, [1]

(ii) $\tan(\frac{1}{2}\pi - x)$, [1]

(iii) $\sin x$. [2]

2 (i) Show that the equation

$$3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$$

can be written in the form $\tan x = -\frac{3}{4}$. [2]

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \leqslant x \leqslant 360°$. [2]

3 (i) Show that the equation $2\sin x \tan x + 3 = 0$ can be expressed as $2\cos^2 x - 3\cos x - 2 = 0$. [2]

(ii) Solve the equation $2\sin x \tan x + 3 = 0$ for $0° \leqslant x \leqslant 360°$. [3]

4 (i) Show that the equation $2\tan^2\theta \sin^2\theta = 1$ can be written in the form

$$2\sin^4\theta + \sin^2\theta - 1 = 0.$$ [2]

(ii) Hence solve the equation $2\tan^2\theta\sin^2\theta = 1$ for $0° \leqslant \theta \leqslant 360°$. [4]

5 (i) Prove the identity $\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2 \equiv \frac{1-\cos\theta}{1+\cos\theta}$. [3]

(ii) Hence solve the equation $\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2 = \frac{2}{5}$, for $0° \leqslant \theta \leqslant 360°$. [4]

6 Solve the equation $\sin 2x = 2\cos 2x$, for $0° \leqslant x \leqslant 180°$. [4]

7 (i) Prove the identity $\tan^2\theta - \sin^2\theta \equiv \tan^2\theta\sin^2\theta$. [3]

(ii) Use this result to explain why $\tan\theta > \sin\theta$ for $0° < \theta < 90°$. [1]

8 (i) Solve the equation $\sin 2x + 3\cos 2x = 0$ for $0° \leqslant x \leqslant 360°$. [5]

(ii) How many solutions has the equation $\sin 2x + 3\cos 2x = 0$ for $0° \leqslant x \leqslant 1080°$? [1]

9 (i) Show that $\frac{\sin\theta}{\sin\theta + \cos\theta} + \frac{\cos\theta}{\sin\theta - \cos\theta} \equiv \frac{1}{\sin^2\theta - \cos^2\theta}$. [3]

(ii) Hence solve the equation $\frac{\sin\theta}{\sin\theta + \cos\theta} + \frac{\cos\theta}{\sin\theta - \cos\theta} = 3$, for $0° \leqslant \theta \leqslant 360°$. [4]

10 (i) Prove the identity $\frac{\sin x \tan x}{1 - \cos x} \equiv 1 + \frac{1}{\cos x}$. [3]

(ii) Hence solve the equation $\frac{\sin x \tan x}{1 - \cos x} + 2 = 0$, for $0° \leqslant x \leqslant 360°$. [3]

11 A function f is defined by $f : x \mapsto 3 - 2\tan(\frac{1}{2}x)$ for $0 \leqslant x < \pi$.

(i) State the range of f. [1]

(ii) State the exact value of $f(\frac{2}{3}\pi)$. [1]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Obtain an expression, in terms of x, for $f^{-1}(x)$. [3]

12 Solve the equation $15\sin^2 x = 13 + \cos x$ for $0° \leqslant x \leqslant 180°$. [4]

13

(i) Sketch the curve $y = 2\sin x$ for $0 \leqslant x \leqslant 2\pi$. [1]

(ii) By adding a suitable straight line to your sketch, determine the number of real roots of the equation

$$2\pi \sin x = \pi - x.$$

State the equation of the straight line. [3]

14

(i) Sketch, on a single diagram, the graphs of $y = \cos 2\theta$ and $y = \frac{1}{2}$ for $0 \leqslant \theta \leqslant 2\pi$. [3]

(ii) Write down the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $0 \leqslant \theta \leqslant 2\pi$. [1]

(iii) Deduce the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $10\pi \leqslant \theta \leqslant 20\pi$. [1]

15 (i) Given that

$$3\sin^2 x - 8\cos x - 7 = 0,$$

show that, for real values of x,

$$\cos x = -\frac{2}{3}.$$ [3]

(ii) Hence solve the equation

$$3\sin^2(\theta + 70°) - 8\cos(\theta + 70°) - 7 = 0$$

for $0° \leqslant \theta \leqslant 180°$. [4]

16 (i) Solve the equation $2\cos^2\theta = 3\sin\theta$, for $0° \leqslant \theta \leqslant 360°$. [4]

(ii) The smallest positive solution of the equation $2\cos^2(n\theta) = 3\sin(n\theta)$, where n is a positive integer, is $10°$. State the value of n and hence find the largest solution of this equation in the interval $0° \leqslant \theta \leqslant 360°$. [3]

17 Solve the equation $7\cos x + 5 = 2\sin^2 x$, for $0° \leqslant x \leqslant 360°$. [4]

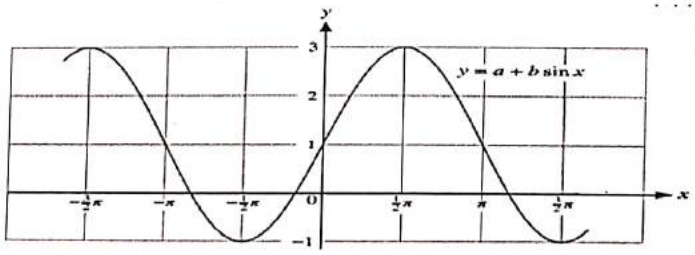
18 (i) Solve the equation $4\sin^2 x + 8\cos x - 7 = 0$ for $0° \leqslant x \leqslant 360°$. [4]

(ii) Hence find the solution of the equation $4\sin^2(\frac{1}{2}\theta) + 8\cos(\frac{1}{2}\theta) - 7 = 0$ for $0° \leqslant \theta \leqslant 360°$. [2]

19 (a) Find the possible values of x for which $\sin^{-1}(x^2 - 1) = \frac{1}{3}\pi$, giving your answers correct to 3 decimal places. [3]

(b) Solve the equation $\sin(2\theta + \frac{1}{3}\pi) = \frac{1}{2}$ for $0 \leqslant \theta \leqslant \pi$, giving θ in terms of π in your answers. [4]

20



The diagram shows part of the graph of $y = a + b\sin x$. State the values of the constants a and b. [2]

Answers

$\tan x = k$

1 (i) $\tan(\pi - x) = -k$

(ii) $\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{k}$

(iii) $\sin x = \frac{k}{\sqrt{1+k^2}}$

$\tan x = -\frac{3}{4}$

2 $x = 360 - 36.9 = 323.1°$

3 $x = 120°$ or $240°$

4 $\theta = 45°, 135°$
$\theta = 225°, 315°$

5 $\theta = 64.6°$ or $295.4°$

6 $x = 31.7$ or 121.7 (allow 122)

$x = 54.2°$ or $144.2°$

7 Also $234.2°$ and $324.2°$

8 (ii) 12 answers.

9 $\theta = 54.7°, 125.3°, 234.7°, 305.3°$

10 $x = 109.5°$ or $250.5°$

11 (i) Range of $f \leq 3$

(ii) $f(\frac{2}{3}\pi) = 3 - 2\sqrt{3}$

$f^{-1}(x) = 2\tan^{-1}\left(\frac{3-x}{2}\right)$

12 113(.6), 70.5

13 (ii) Required line $y = 1 - \frac{x}{\pi}$

Line through (0, 1), (π, 0) drawn

3 roots

14 Exactly 2 complete oscillations in $[0, 2\pi]$

Line $y = \frac{1}{2}$ correct

(ii) 4

(iii) 20

15 (ii) $\cos(\theta + 70) = -\frac{2}{3}$, $\theta = 61.8$

$\theta + 70 = 131.8$ (or 228.2)

$\theta = 158.2$

16 $\theta = 30°$ or $150°$

$\theta = 290°$

17 $x = 120°, 240°$

18 $x = 60°$ or $300°$

$\theta = 17.2°$ (0.300 rad)

19 $x = \pm 1.366$

$\theta = \frac{\pi}{4}, \frac{11\pi}{12}$

20 $a = 1,\quad b = 2$

Past Paper Questions

Trigonometry

1 Find all the values of x in the interval $0° \leqslant x \leqslant 180°$ which satisfy the equation $\sin 3x + 2\cos 3x = 0$. [4]

2 (i) Show that the equation $\sin^2\theta + 3\sin\theta\cos\theta = 4\cos^2\theta$ can be written as a quadratic equation in $\tan\theta$. [2]

(ii) Hence, or otherwise, solve the equation in part (i) for $0° \leqslant \theta \leqslant 180°$. [3]

3 (i) Show that the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$ can be expressed as $\tan\theta = 3$. [2]

(ii) Hence solve the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$, for $0° \leqslant \theta \leqslant 360°$. [2]

4 Solve the equation

$$\sin 2x + 3\cos 2x = 0,$$

for $0° \leqslant x \leqslant 180°$. [4]

5 Prove the identity $\dfrac{1-\tan^2 x}{1+\tan^2 x} \equiv 1 - 2\sin^2 x$. [4]

6 In the triangle ABC, $AB = 12$ cm, angle $BAC = 60°$ and angle $ACB = 45°$. Find the exact length of BC. [3]

7 (i) Show that the equation $2\tan^2\theta\cos\theta = 3$ can be written in the form $2\cos^2\theta + 3\cos\theta - 2 = 0$. [2]

(ii) Hence solve the equation $2\tan^2\theta\cos\theta = 3$, for $0° \leqslant \theta \leqslant 360°$. [3]

8 Prove the identity $\dfrac{\sin x}{1-\sin x} - \dfrac{\sin x}{1+\sin x} \equiv 2\tan^2 x$. [3]

9 (i) Show that the equation

$$3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$$

can be written in the form $\tan x = -\frac{3}{4}$. [2]

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \leqslant x \leqslant 360°$. [2]

10 (i) Prove the identity $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} \equiv 1 + \dfrac{1}{\sin\theta}$. [3]

(ii) Hence solve the equation $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} = 4$, for $0° \leqslant \theta \leqslant 360°$. [3]

11 (i) Prove the identity $\tan x + \dfrac{1}{\tan x} \equiv \dfrac{1}{\sin x\cos x}$. [2]

(ii) Solve the equation $\dfrac{2}{\sin x\cos x} = 1 + 3\tan x$, for $0° \leqslant x \leqslant 180°$. [4]

12 It is given that $a = \sin\theta - 3\cos\theta$ and $b = 3\sin\theta + \cos\theta$, where $0° \leqslant \theta \leqslant 360°$.

(i) Show that $a^2 + b^2$ has a constant value for all values of θ. [3]

(ii) Find the values of θ for which $2a = b$. [4]

13 The reflex angle θ is such that $\cos\theta = k$, where $0 < k < 1$.

(i) Find an expression, in terms of k, for

(a) $\sin\theta$, [1]

(b) $\tan\theta$. [2]

(ii) Explain why $\sin 2\theta$ is negative for $0 < k < 1$. [2]

14 (i) Prove the identity $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} \equiv \dfrac{\tan\theta - 1}{\tan\theta + 1}$. [1]

(ii) Hence solve the equation $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \dfrac{\tan\theta}{6}$, for $0° \leqslant \theta \leqslant 180°$. [4]

15 (i) Prove the identity $\dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} \equiv \dfrac{4}{\sin\theta\tan\theta}$. [4]

(ii) Hence solve, for $0° < \theta < 360°$, the equation

$$\sin\theta\left(\frac{1+\cos\theta}{1-\cos\theta} - \frac{1-\cos\theta}{1+\cos\theta}\right) = 3.$$ [3]

16 (i) Show that the equation $3\tan\theta = 2\cos\theta$ can be expressed as

$$2\sin^2\theta + 3\sin\theta - 2 = 0.$$ [3]

(ii) Hence solve the equation $3\tan\theta = 2\cos\theta$, for $0° \leqslant \theta \leqslant 360°$. [3]

17 (i) Show that the equation $4\sin^4\theta + 5 = 7\cos^2\theta$ may be written in the form $4x^2 + 7x - 2 = 0$, where $x = \sin^2\theta$. [1]

(ii) Hence solve the equation $4\sin^4\theta + 5 = 7\cos^2\theta$, for $0° \leqslant \theta \leqslant 360°$. [4]

18 (i) Sketch and label, on the same diagram, the graphs of $y = 2\sin x$ and $y = \cos 2x$, for the interval $0 \leqslant x \leqslant \pi$. [4]

(ii) Hence state the number of solutions of the equation $2\sin x = \cos 2x$ in the interval $0 \leqslant x \leqslant \pi$. [1]

19 Solve the equation $3\sin^2\theta - 2\cos\theta - 3 = 0$, for $0° \leqslant \theta \leqslant 180°$. [4]

20 Given that $x = \sin^{-1}\left(\frac{2}{5}\right)$, find the exact value of

(i) $\cos^2 x$, [2]

(ii) $\tan^2 x$. [2]

21 (i) Show that the equation $3\sin x\tan x = 8$ can be written as $3\cos^2 x + 8\cos x - 3 = 0$. [3]

(ii) Hence solve the equation $3\sin x\tan x = 8$ for $0° \leqslant x \leqslant 360°$. [3]

22 Prove the identity

$$\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} \equiv \frac{2}{\cos x}.$$ [4]

23 (i) Prove the identity $(\sin x + \cos x)(1 - \sin x \cos x) \equiv \sin^3 x + \cos^3 x$. [3]

(ii) Solve the equation $(\sin x + \cos x)(1 - \sin x \cos x) = 9 \sin^3 x$ for $0° \leqslant x \leqslant 360°$. [3]

24 Prove the identity

$$\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x.$$

[4]

25 (i) Sketch, on the same diagram, the graphs of $y = \sin x$ and $y = \cos 2x$ for $0° \leqslant x \leqslant 180°$. [3]

(ii) Verify that $x = 30°$ is a root of the equation $\sin x = \cos 2x$, and state the other root of this equation for which $0° \leqslant x \leqslant 180°$. [2]

(iii) Hence state the set of values of x, for $0° \leqslant x \leqslant 180°$, for which $\sin x < \cos 2x$. [2]

26 (i) Show that the equation $2 \cos x = 3 \tan x$ can be written as a quadratic equation in $\sin x$. [3]

(ii) Solve the equation $2 \cos 2y = 3 \tan 2y$, for $0° \leqslant y \leqslant 180°$. [4]

27 Given that $\cos x = p$, where x is an acute angle in degrees, find, in terms of p,

(i) $\sin x$, [1]

(ii) $\tan x$, [1]

(iii) $\tan(90° - x)$. [1]

28 (i) Show that the equation $1 + \sin x \tan x = 5 \cos x$ can be expressed as

$$6 \cos^2 x - \cos x - 1 = 0.$$

[3]

(ii) Hence solve the equation $1 + \sin x \tan x = 5 \cos x$ for $0° \leqslant x \leqslant 180°$. [3]

29 (i) Prove the identity $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 \equiv \frac{1 - \cos x}{1 + \cos x}$. [4]

(ii) Hence solve the equation $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 = \frac{2}{5}$ for $0 \leqslant x \leqslant 2\pi$. [3]

30 (i) Express the equation $\sin 2x + 3 \cos 2x = 3(\sin 2x - \cos 2x)$ in the form $\tan 2x = k$, where k is a constant. [2]

(ii) Hence solve the equation for $-90° \leqslant x \leqslant 90°$. [3]

Answers

1 $x = 38.9°, 98.9°, 158.9°$

2 (i) $\tan^2\theta + 3\tan\theta - 4 = 0$ (ii) $\theta = 45°, 104°$

3 (ii) $\theta = 71.6°, 251.6°$

4 $x = 54.2°, 144.2°$

6 $6\sqrt{6}$ cm

7 (ii) $\theta = 60°, 300°$

9 (ii) $143.1°, 323.1°$

10 (ii) $\theta = 19.5°, 160.5°$

11 (ii) $x = 45°, 116.6°$

12 (ii) $\theta = 98.1°, 278.1°$

13 (i) (a) $-\sqrt{1-k^2}$ (b) $\frac{-\sqrt{1-k^2}}{k}$ (ii) $540° \le 2\theta \le 720°$

14 (ii) $\theta = 63.4°, 71.6°$

15 (ii) $\theta = 53.11°, 233.1°$

16 (ii) $x = 30°, 150°$

17 (ii) $\theta = 30°, 150°, 210°, 330°$.

18 (ii) number of solution = 2

19 $\theta = 90°, 131.8°$

20 (i) $\cos^2 x = \frac{21}{25}$ (ii) $\tan^2 x = \frac{4}{21}$

21 (ii) $x = 70.5°, 289.5°$

23 (ii) $x = 26.6°, 206.6°$

25 (iii) $0° \le x < 30°$ and $150° < x \le 180°$

26 (ii) $y = 15°, 75°$

27 (i) $\sqrt{1-p^2}$ (ii) $-\frac{\sqrt{1-p^2}}{p}$ (iii) $\frac{p}{\sqrt{1-p^2}}$

28 (ii) $1 < x < 3$

29 (ii) $x = 1.13$ radians, 5.15 radians.

30 (i) $\tan 2x = 3$ (ii) $x = -54.2°, 35.8°$

DIFFERENTIATION

Syllabus:

- Understand the idea of the gradient of a curve, and use the notations f' (x), f''(x) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (the technique of differentiation from first principles is not required);
- Use the derivative of x^n (for any rational n), together with constant multiples, sums, differences of functions, and of composite functions using the chain rule;
- Apply differentiation to gradients, tangents and normal, increasing and decreasing functions and ratse of change (including connected rates of change);
- Locate stationary points, and use information about stationary points in sketching graphs (the ability to distinguish between maximum points and minimum points is required, but identification of points of inflexion is not included).

GRADIENT

Measurement of change of y with respect to x is called gradient. Gradient of curve at nay point is the gradient of tangent at that point. The gradient of straight line remain same at any point but gradient of curve varies with x-coordinate of point on curve.

Gradient function

The derivative of given equation is denoted as gradient function and the process of driving it is called differentiation. $\frac{d}{dx}$ is used as a symbol to denote 'derivative with respect to x' and dy/dx means derivative of y with respect to x. it also represents rate of change in y w.r.t x.

Formulae of differentiation

If a,b,c and 'n' are constants, then

1. $d/d_x \; (x^n) = n \; x^{n-1} \frac{d}{dx}(\text{base})$. If there is 'x' in base, then its derivative is 1, so d/dx (base) is not mention in formula. If there is composite base of any power, then derivative of base is always multiplied with the answer.

2. $\frac{d}{dx}(ax^n \pm b) = \frac{d}{dx}(ax^n) + \frac{d}{dx}(b)$

   $= anx^{n-1} \pm 0$

3. $\frac{d}{dx}$(constant) = 0:- Constant means the term which does not contain the variable by which the function is differentiated.

4. $\frac{d}{dt}$ is denoted as rate of change with respect to time.

Chain rule

The method which is used to change a derivative with respect to derivative in other variable i.e. to change derivative of y w.r.t x into derivative of y w.r.t time is dy/dx = dy/dt×dt/dx.

Stationary or turning point

To find co-ordinate of stationary or turning point put dy/dx = 0 and find value of x. then put that value into original equation and find y. write it into (x,y) form which is stationary or turning point.

Nature of turning point

Double derivative w.r.t x(d^2y/dx^2) is used to find nature of turning point. To find value of d^2y/dx^2, first put dy/dx = 0 and find x, then put that value of x into d^2y/dx^2 and simplify.

If value of d^2y/dx^2 is +ive, then turning point is minimum. If value of d^2y/dx^2 is –ive, then turning point is maximum f'(x) means dy/dx and f''(x) means d^2y/dx^2.

Range of x for increasing or decreasing function

When dy/dx is +ive, then function is increasing function. To find value of x on which given function is increasing function, put dy/dx >0 and write set of values of x. i.e. (domain in x).

When dy/dx is –ive, then given function is decreasing function. To find value of x on which given function is decreasing function, put dy/dx < 0 write set of values of x.

Gradient of function

Gradient (m) = dy/dx. To find gradient on any point, put value of x on that point in dy/dx. If θ is the angle of line with x-axis, then m = tan θ = dy / dx.

Intercepts of line or curve

Intercept means, the point on which line or curve cuts the axis. To find y-intercept or point on y-axis, put x=0 in given equation and to find x-intercept or point on x-axis, put y=0 in given equation.

Equation of tangent or normal (Perpendicular) to curve

If m = dy/dx is gradient and (x_1, y_1) is given point. Then equation of tangent to curve is

$y-y_1 = m(x-x_1)$

equation of normal is: $y-y_1 = \frac{-1}{m}(x-x_1)$

Example

Differentiate (a) $f(x)=2(x^2-3x-2)$, (b) $g(x)=(x+2)(2x-3)$.

$f(x)=2(x^2-3x-2)=2x^2-6x-4$, so $f'(x)=4x-6$.

Example

Find the equation of the tangent to the graph of $y=x^2-4x+2$ which is parallel to the x-axis.

a line parallel to the x-axis has gradient 0.

Let $f(x)=x^2-4x+2$. Then $f'(x)=2x-4$.

To find when the gradient is 0 you need to solve $2x-4=0$, giving $x=2$.

When $x=2$, $y=2^2-4\times2+2=-2$.

Example

Find the equation of the tangent to the graph of $y=2\sqrt{x}$ at the point where $x=9$.

Let $f(x)=2\sqrt{x}=2x^{\frac{1}{2}}$.

Then, using results in the boxes,

$f'(x)=2\times\frac{1}{2}x^{-\frac{1}{2}}=x^{-\frac{1}{2}}$.

When $x=9$, $f'(9)=9^{-\frac{1}{2}}=\frac{1}{\sqrt{9}}=\frac{1}{3}$.

The tangent passes through the point $(9,2\sqrt{9})=(9,6)$, so its equation is

$y-6=\frac{1}{3}(x-9)$, or $3y-x=9$.

Example

Differentiate each of the functions (a) $x(1+x^2)$, (b) $(1+\sqrt{x})^2$, (c) $\frac{x^2+x+1}{x}$.

(a) Let $f(x)=x(1+x^2)$.

Then $f(x)=x+x^3$, so $f'(x)=1+3x^2$.

(b) Let $f(x)=(1+\sqrt{x})^2$.

Then $f(x)=1+2\sqrt{x}+x=1+2x^{\frac{1}{2}}+x$, so $f'(x)=2\times\frac{1}{2}x^{-\frac{1}{2}}+1=x^{-\frac{1}{2}}+1=\frac{1}{\sqrt{x}}+1$.

(c) Let $f(x)=\frac{x^2+x+1}{x}$.

Then, by division, $f(x)=x+1+\frac{1}{x}=x+1+x^{-1}$, so $f'(x)=1+(-1)x^{-2}=1-\frac{1}{x^2}$.

Practice Questions

Exercise #1

1. Differentiate the following with respect to x (where a and b are constants).

(a) $3x^2 + 4x - 1$

(b) $x^4 - 7x^2 + 6x$

(c) $2x^3 + 5x^2 - 4x + 9$

(d) $4x + \frac{2}{x}$

(e) $9x^2 - \frac{3}{x^2}$

(f) $\frac{6}{x^3} - \frac{1}{x} + 3$

(g) $3a + bx^2$

(h) $5x^2 + \frac{4}{x} - 2$

(i) $3x + 2\sqrt{x} - 3$

(j) $8x^2 + 3x - \sqrt{x}$

(k) $2x^{\frac{5}{2}} - 4x^{\frac{3}{2}} - 6x + 8$

(l) $6x\sqrt{x} - 6\sqrt{x}$

(m) $4x^2\sqrt{x} - \frac{6}{\sqrt{x}}$

(n) $ax - \frac{b}{x}$

2. Differentiate the following with respect to x.

(a) $\frac{2x^2 + 4x}{x}$

(b) $\frac{x^2 - 6x + 4}{x}$

(c) $\frac{4x^3 - 5x - 3}{2x}$

(d) $\frac{x^2 + 4}{2x^2}$

(e) $\frac{3x^2 + x - 1}{\sqrt{x}}$

(f) $\frac{6x^2 - \sqrt{x} + 2}{2x}$

3. Find $\frac{dy}{dx}$ for the following functions of x.

(a) $(x + 1)(2x - 1)$

(b) $x(\sqrt{x} - 2)$

(c) $(1 + \sqrt{x})(1 - \sqrt{x})$

(d) $4x^2(3 - \sqrt{x})$

(e) $\frac{(2x + 1)(x - 2)}{x}$

(f) $\frac{(1 - x)(4x - 1)}{\sqrt{x}}$

4. Find the value of $f'(x)$ at the given value of x.

(a) $f(x) = 3x^2 - 2x - 4,\ x = 2$

(b) $f(x) = 6x - \frac{3}{x},\ x = -1$

(c) $f(x) = 3x - 4\sqrt{x},\ x = 4$

(d) $f(x) = (x - 4)(x + 3),\ x = 3$

5. Calculate the gradient of the tangent to the curve at the given point.

(a) $y = 4x^2 - 6x + 1,\ (2, 5)$

(b) $y = \frac{6 - 4x}{x},\ x = -2$

(c) $y = \sqrt{x}(2 - x),\ x = 9$

(d) $y = \frac{(x + 1)(2x + 3)}{x},\ x = -1$

6. Calculate the gradient(s) of the curve at the point(s) where y is given.

(a) $y = x^2 - 2x,\ y = -1$

(b) $y = 2x^2 + 3x,\ y = 2$

(c) $y = \frac{x - 9}{x},\ y = 4$

(d) $y = \frac{x^2 + 4}{x^2},\ y = 5$

7. Calculate the gradient(s) of the curve at the point(s) where it crosses the given line.

(a) $y = 2x^2 - 5x + 1$, y-axis

(b) $y = \frac{x - 4}{x}$, x-axis

(c) $y = 2x^2 - 8$, x-axis

(d) $y = \frac{x + 2}{x},\ y = x$

8. Find the coordinates of the point on the curve $y = x^3 - 3x^2 + 6x + 2$ at which the gradient is 3.

9. The curve $y = ax^2 + \frac{b}{x}$ has gradients 2 and –1 at $x = 1$ and $x = 4$ respectively. Find the value of a and of b.

10. The gradient of the tangent to the curve $y = ax^3 + bx$ at the point $(2, -4)$ is 6. Calculate the values of the constants a and b.

11. Given that the gradient of the curve $y = \frac{a}{x} + bx^2$ at the point $P(3, -15)$ is –13. Find the value of a and of b. Show that the tangent to the curve at the point where $x = 1$ has the same gradient as that at P.

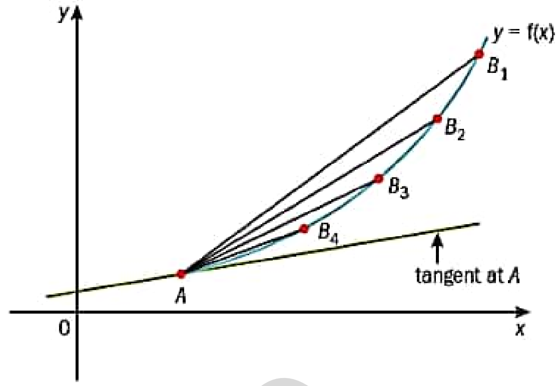
12. The tangent to the curve $y = \frac{a}{x} + bx$ at (1, 3) is parallel to the line $y = 2x + 1$. Calculate the value of a and of b.

Answers

1. (a) $6x + 4$ (b) $4x^3 - 14x + 6$ (c) $6x^2 + 10x - 4$ (d) $4 - \frac{2}{x^2}$
   (e) $18x + \frac{6}{x^3}$ (f) $-\frac{18}{x^4} + \frac{1}{x^2}$ (g) $2bx$ (h) $10x - \frac{4}{x^2}$
   (i) $3 + \frac{1}{\sqrt{x}}$ (j) $16x + 3 - \frac{1}{2\sqrt{x}}$ (k) $5x^{\frac{3}{2}} - 6x^{\frac{1}{2}} - 6$
   (l) $9x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$ (m) $10x^{\frac{3}{2}} + 3x^{-\frac{3}{2}}$ (n) $a + \frac{b}{x^2}$
2. (a) 2 (b) $1 - \frac{4}{x^2}$ (c) $4x + \frac{3}{2x^2}$ (d) $-\frac{4}{x^3}$
   (e) $\frac{9}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$ (f) $3 + \frac{1}{4}x^{-\frac{3}{2}} - x^{-2}$
3. (a) $4x + 1$ (b) $\frac{3}{2}\sqrt{x} - 2$ (c) –1 (d) $24x - 10x^{\frac{2}{3}}$
   (e) $2 + \frac{2}{x^2}$ (f) $-6x^{\frac{1}{2}} + \frac{5}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$
4. (a) 10 (b) 9 (c) 2 (d) 5
5. (a) 10 (b) $-1\frac{1}{2}$ (c) $-4\frac{1}{6}$ (d) 5
6. (a) 0 (b) –5, 5 (c) 1 (d) –8, 8
7. (a) –5 (b) $\frac{1}{4}$ (c) –8, 8 (d) $-2, -\frac{1}{2}$
8. (1, 6)
9. $a = -\frac{1}{7}$, $b = -\frac{16}{7}$
10. $a = 1$, $b = -6$
11. $a = 9$, $b = -2$
12. $a = \frac{1}{2}$, $b = \frac{5}{2}$

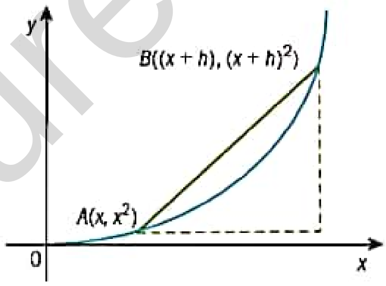
8.2 Gradient of a tangent as a limit

Consider the problem of finding the gradient of the tangent at point A on the curve $y = f(x)$.
If B is another point on the curve, fairly close to A, then the gradient of the chord AB gives an approximate value for the gradient of the tangent at A.
As we move B closer and closer to A (i.e. B_1, B_2, B_3, B_4), the approximate value for the gradient of the tangent at A gets more accurate.



So as $B \to A$, the gradient of chord $AB \to$ the gradient of the tangent at point A.

Consider the graph of $y = x^2$.
The point A has coordinates (x, x^2).
The point B has coordinates $((x + h), (x + h)^2)$.



$$\text{The gradient of } AB = \frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{x^2 + 2xh + h^2 - x^2}{x + h - x}$$

$$= \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h.$$

As h gets smaller and smaller, B gets closer to A, and the line AB gets closer to the tangent at A.
We say that as $h \to 0$, we can find the gradient of the tangent at A.

We read '$h \to 0$' as h tends to 0, i.e. h gets closer and closer to 0.

As $h \to 0$, the gradient of $AB \to 2x$.
Thus, the gradient of the tangent at any point on the curve $y = x^2$ is $2x$.

Note: The gradient of the tangent at any point is the rate of change of y with respect to x.

The definition of $f'(x)$ is given as $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

Example 2

Find an expression for $f'(x)$ when $f(x) = x^3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$ ← $f(x) = x^3$ so $f(x + h) = (x + h)^3$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$ ← $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$ ← Simplify and factorise.

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$ ← As $h \to 0$, we get $3x^2 + 0 + 0$.

When $f(x) = x^3$, $f'(x) = 3x^2$.

Exercise #2

1 Find the equation of the tangent(s) to each of the following graphs at the point(s) whose x- or y-coordinate is given.

(a) $y = x^2$ where $x = 2$
(b) $y = x^2 + 2$ where $x = -1$
(c) $y = x^2 - 2$ where $y = -1$
(d) $y = x^2 - 2$ where $y = -2$

2 Find the equation of the normal to each of the following graphs at the point whose x-coordinate is given.

(a) $y = x^2$ where $x = 1$
(b) $y = x^2 + 1$ where $x = -2$
(c) $y = x^2 + 1$ where $x = 0$
(d) $y = x^2 + c$ where $x = \sqrt{c}$

3 The tangent at P to the curve $y = x^2$ has gradient 3. Find the equation of the normal at P.

4 A normal to the curve $y = x^2 + 1$ has gradient -1. Find the equation of the tangent there.

5

Find the equation of the normal to the curve at the point with the given x-coordinate.

6 (a) $y = -x^2$ where $x = 1$
(b) $y = 3x^2 - 2x - 1$ where $x = 1$
(c) $y = 1 - 2x^2$ where $x = -2$
(d) $y = 1 - x^2$ where $x = 0$
(e) $y = 2(2 + x + x^2)$ where $x = -1$
(f) $y = (2x - 1)^2$ where $x = \frac{1}{2}$

7 Find the equation of the tangent to the curve $y = x^2$ which is parallel to the line $y = x$.

8 Find the equation of the tangent to the curve $y = x^2$ which is parallel to the x-axis.

9 Find the equation of the tangent to the curve $y = x^2 - 2x$ which is perpendicular to the line $2y = x - 1$.

Find the equation of the normal to the curve $y = 3x^2 - 2x - 1$ which is parallel to the line

10 $y = x - 3$.

11 Find the equation of the normal to the curve $y = (x - 1)^2$ which is parallel to the y-axis.

12 Find the equation of the normal to the curve $y = 2x^2 + 3x + 4$ which is perpendicular to the line $y = 7x - 5$.

Answers:

1 (a) $y=4x-4$ (b) $y=-2x+1$
(c) $y=2x-3$ and $y=-2x-3$
(d) $y=-2$

2 (a) $2y=-x+3$ (b) $4y=x+22$
(c) $x=0$
(d) $2y\sqrt{c}=-x+\sqrt{c}(4c+1)$

3 $12y=-4x+33$

4 $4y=4x+3$

5 $\left(-2\frac{1}{4},5\frac{1}{16}\right)$

6 (a) $2y=x-3$ (b) $4y=-x+1$
(c) $8y=-x-58$ (d) $x=0$
(e) $2y=x+9$ (f) $x=\frac{1}{2}$.

7 $4y=4x-1$

8 $y=0$

9 $y=-2x$

10 $12y=12x-17$

11 $x=1$

12 $7y=-x+64$

Exercise #3

1. Differentiate the following w.r.t. x.
   (a) $(x + 2)^5$ (b) $(2x - 1)^4$ (c) $(\frac{1}{4}x + 2)^5$
   (d) $(1 - 4x)^{10}$ (e) $(2 - 3x^2)^4$ (f) $(1 - x + x^2)^3$

2. Differentiate the following w.r.t. x.
   (a) $\frac{3}{(3 - 4x)^3}$ (b) $\frac{4}{(2x + 7)}$ (c) $\frac{6}{(2 - x)^2}$ (d) $\frac{2}{(6x^2 + 5)}$

3. Differentiate the following w.r.t. x.
   (a) $\sqrt{2x - 3}$ (b) $\sqrt{6 - 2x}$ (c) $\sqrt{x^2 - 2}$
   (d) $\sqrt{5 - 3x^2}$ (e) $\sqrt{x^2 - x + 1}$ (f) $\sqrt{x^2 + 2x + 2}$

4. Differentiate the following w.r.t. x
   (a) $(2 - \sqrt{x})^6$ (b) $\frac{1}{\left(1 - \frac{1}{x}\right)^3}$ (c) $\frac{1}{2(3x - 2)^2}$
   (d) $2(\sqrt{x} + 2)^{\frac{1}{2}}$ (e) $\left(x - \frac{1}{x}\right)^3$ (f) $(\sqrt{x} + 2x)^4$

5. Find $\frac{dy}{dx}$ and the gradient of the curve at the given value of x.
   (a) $y = (3x - 1)^4$, $x = 1$ (b) $y = \sqrt{5 - 2x}$, $x = \frac{1}{2}$
   (c) $y = \frac{1}{2x - 3}$, $y = 1$ (d) $y = (4x - 5)^3$, $y = 27$

6. Calculate the coordinates of the point on the curve $y = (1 - x)^4$ at which the gradient is -4.

7. Calculate the coordinates of the point on the curve $y = \sqrt{x^2 - 2x + 5}$ at which $\frac{dy}{dx} = 0$.

8. The curve $y = (a - x)^3$ has gradient $-\frac{1}{3}$ at $x = 2$. Find the possible values of a.

9. Find $\frac{dy}{dx}$ and calculate the gradient of the tangent to the curve
   (a) $y = (x^2 - 2x - 4)^3$ at the point where $x = -1$,
   (b) $y = \frac{1}{\sqrt{1 + x}}$ at the point where $x = 3$.

10. Find the equations of the tangent and the normal to the curve
    (a) $y = 2x^2 - 3x + 1$ at the point $(2, 3)$,
    (b) $y = x^3 + 3x^2$ at the point where $x = -1$
    (c) $y = x + \frac{2}{x}$ at the point where $x = 1$.

11. Find the equations of the tangents to the curve $y = 2x^2 - 3x$ at the point where $y = -1$. Find the coordinates of the point of intersection of the tangents.

12. Find the equations of the normals to the curve $y = 2x^2 - 7$ at the points where $x = 1$ and $x = -1$. Calculate the coordinates of the point of intersection of these normals.

13 Find the equation of the tangent to the curve $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$. Find the x-coordinate of the point at which the tangent is parallel to the tangent at $x = 1$. (C)

14. Find the equations of the normals to the curve $y = 2x + \frac{8}{x}$ at $x = 1$ and $x = 4$. Find the coordinates of the point where these normals intersect.

15. Find the equations of the tangent and normal to the curve $y = \sqrt{4x - x^2 + 1}$ at the point (1, 2). Show that the tangent is parallel to the line $6y - 3x = 1$.

16. Find the equations of the tangents to the curve $y = x^3 - 11x$ which are parallel to the line $y = x + 2$.

Answers

1. (a) $5(x + 2)^4$ (b) $8(2x - 1)^3$ (c) $\frac{5}{4}\left(\frac{1}{4}x + 2\right)^4$
(d) $-40(1 - 4x)^9$ (e) $-24x(2 - 3x^2)^3$ (f) $3(2x - 1)(1 - x + x^2)^2$

2. (a) $\frac{36}{(3 - 4x)^4}$ (b) $-\frac{8}{(2x + 7)^2}$ (c) $\frac{12}{(2 - x)^3}$ (d) $\frac{-24x}{(6x^2 + 5)^2}$

3. (a) $\frac{1}{\sqrt{2x - 3}}$ (b) $-\frac{1}{\sqrt{6 - 2x}}$ (c) $\frac{x}{\sqrt{x^2 - 2}}$
(d) $-\frac{3x}{\sqrt{5 - 3x^2}}$ (e) $\frac{2x - 1}{2\sqrt{x^2 - x + 1}}$ (f) $\frac{x + 1}{\sqrt{x^2 + 2x + 2}}$

4. (a) $-\frac{3(2 - \sqrt{x})^5}{\sqrt{x}}$ (b) $\frac{-3}{x^2\left(1 - \frac{1}{x}\right)^4}$ (c) $-\frac{3}{(3x - 2)^3}$
(d) $\frac{1}{2\sqrt{x(\sqrt{x} + 2)}}$ (e) $3\left(x - \frac{1}{x}\right)^2\left(1 + \frac{1}{x^2}\right)$ (f) $4(\sqrt{x} + 2x)^3\left(\frac{1}{2\sqrt{x}} + 2\right)$

5. (a) 96 (b) $-\frac{1}{2}$ (c) -2 (d) 108

6. (0, 1) 7. (1, 2) 8. $a = \frac{7}{3}$ or $\frac{5}{3}$ 9. (a) -12 (b) $-\frac{1}{16}$

10 (a) $y = 5x - 7$, $5y = 17 - x$ (b) $y = -3x - 1$, $3y = x + 7$ (c) $y = -x + 4$, $y = x + 2$

11 $2x + 2y = -1$, $y = x - 2$, $\left(\frac{3}{4}, -\frac{5}{4}\right)$ 12 $x + 4y = -19$, $x - 4y = 19$; $\left(0, -\frac{19}{4}\right)$

13 $y = 3x - 3$, $\frac{11}{3}$ 14 $6y = x + 59$, $3y + 2x = 38$, $\left(\frac{17}{5}, \frac{52}{5}\right)$

15 $2y = x + 3$, $y + 2x = 4$ 16 $y = x - 16$, $y = x + 16$

Application of Differentiation

Increasing and decreasing functions

x_1 and x_2 are two values of x in the interval $p \leqslant x \leqslant q$, and if $x_2 > x_1$, then $f(x_2) > f(x_1)$. A function with this property is said to be **increasing** over the interval $p \leqslant x \leqslant q$.

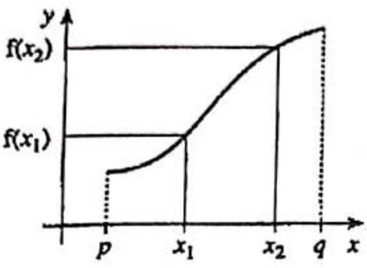


Fig. 7.6

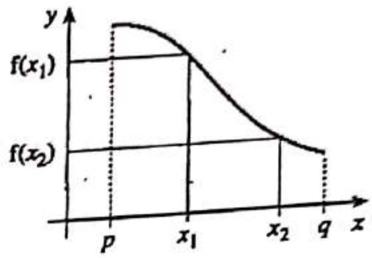


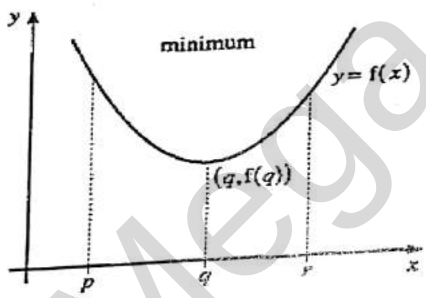
Fig. 7.7

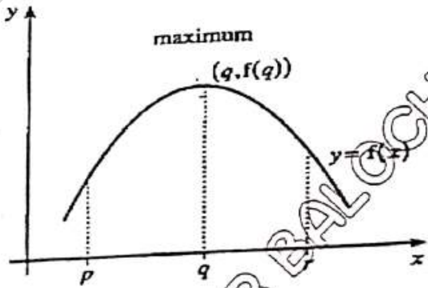
If $f'(x)$ is negative in the interval $p < x < q$, as in Fig. 7.7, the function has the opposite property; if $x_2 > x_1$, then $f(x_2) < f(x_1)$. A function with this property is **decreasing** over the interval $p \leqslant x \leqslant q$.

If $f'(x) > 0$ in an interval $p < x < q$, then $f(x)$ is increasing over the interval $p \leqslant x \leqslant q$.

If $f'(x) < 0$ in $p < x < q$, then $f(x)$ is decreasing over $p \leqslant x \leqslant q$.

Maximum and minimum points





If $f'(x) < 0$ in an interval $p < x < q$, and $f'(x) > 0$ in an interval $q < x < r$, then $(q, f(q))$ is a minimum point.

If $f'(x) > 0$ in $p < x < q$, and $f'(x) < 0$ in $q < x < r$, then $(q, f(q))$ is a maximum point.

If $f'(q) = 0$ and $f''(q) > 0$, then $f(x)$ has a minimum at $x = q$.

If $f'(q) = 0$ and $f''(q) < 0$, then $f(x)$ has a maximum at $x = q$.

To find the minimum and maximum points on the graph of $y = f(x)$:

Step 1 Decide the domain in which you are interested.

Step 2 Find an expression for $f'(x)$.

Step 3 List the values of x in the domain for which $f'(x)$ is either 0 or undefined.

Step 4 Taking each of these values of x in turn, find the sign of $f'(x)$ in intervals immediately to the left and to the right of that value.

Step 5 If these signs are – and + respectively, the graph has a minimum point. If they are + and – it has a maximum point. If the signs are the same, it has neither.

Step 6 For each value of x which gives a minimum or maximum, calculate $f(x)$.

Example

Find the minimum point on the graph with equation $y = \sqrt{x} + \frac{4}{x}$.

Let $f(x) = \sqrt{x} + \frac{4}{x}$.

Step 1 As $\sqrt{x}$ is defined for $x \geqslant 0$ but $\frac{1}{x}$ is not defined for $x = 0$, the largest possible domain for $f(x)$ is the positive real numbers.

Step 2 The derivative $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 4x^{-2}$ can be written as $f'(x) = \frac{x^{\frac{3}{2}} - 8}{2x^2}$.

Step 3 The derivative is defined for all positive real numbers, and is 0 when $x^{\frac{3}{2}} = 8$. Raising both sides to the power $\frac{2}{3}$ and using the power-on-power rule,

$$x = \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 8^{\frac{2}{3}} = 4.$$

Step 4 If $0 < x < 4$, the bottom line, $2x^2$, is positive, and

$$x^{\frac{3}{2}} - 8 < 4^{\frac{3}{2}} - 8 = 8 - 8 = 0, \text{ so that } f'(x) < 0.$$

If $x > 4$, $2x^2$ is still positive, but $x^{\frac{3}{2}} - 8 > 4^{\frac{3}{2}} - 8 = 0$, so that $f'(x) > 0$.

Step 5 The sign of $f'(x)$ is – on the left of 4 and + on the right, so the function has a minimum at $x = 4$.

Step 6 Calculate $f(4) = \sqrt{4} + \frac{4}{4} = 2 + 1 = 3$. The minimum point is $(4,3)$.

Exercise #4

1 For each of the following functions $f(x)$, find $f'(x)$ and the interval in which $f(x)$ is increasing.

(a) x^2-5x+6 (b) x^2+6x-4 (c) $7-3x-x^2$
(d) $3x^2-5x+7$ (e) $5x^2+3x-2$ (f) $7-4x-3x^2$

2 For each of the following functions $f(x)$, find $f'(x)$ and the interval in which $f(x)$ is decreasing.

(a) x^2+4x-9 (b) x^2-3x-5 (c) $5-3x+x^2$
(d) $2x^2-8x+7$ (e) $4+7x-2x^2$ (f) $3-5x-7x^2$

3 For the graphs of each of the following functions:
(i) find the coordinates of the stationary point;
(ii) say, with reasoning, whether this is a maximum or a minimum point;
(iii) check your answer by using the method of 'completing the square' to find the vertex;
(iv) state the range of values which the function can take.

(a) x^2-8x+4 (b) $3x^2+12x+5$ (c) $5x^2+6x+2$
(d) $4-6x-x^2$ (e) x^2+6x+9 (f) $1-4x-4x^2$

4 Find the coordinates of the stationary points on the graphs of the following functions, and find whether these points are maxima or minima.

(a) $2x^3+3x^2-72x+5$ (b) $x^3-3x^2-45x+7$ (c) $3x^4-8x^3+6x^2$
(d) $3x^5-20x^3+1$ (e) $2x+x^2-4x^3$ (f) x^3+3x^2+3x+1
(g) $x+\frac{1}{x}$ (h) $x^2+\frac{54}{x}$ (i) $x-\frac{1}{x}$
(j) $x-\sqrt{x}$, for $x>0$ (k) $\frac{1}{x}-\frac{3}{x^2}$ (l) $x^2-\frac{16}{x}+5$
(m) $x^{\frac{1}{3}}(4-x)$ (n) $x^{\frac{1}{3}}(x+6)$ (o) $x^{\frac{1}{4}}(5-x)$

5 A metal bar is heated to a certain temperature and then the heat source is removed. At time t minutes after the heat source is removed, the temperature, θ degrees Celsius, of the metal bar is given by $\theta=\frac{280}{1+0.02t}$. At what rate is the temperature decreasing 100 minutes after the removal of the heat source?

6 The length of the side of a square is increasing at a constant rate of 1.2 cm s^{-1}. At the moment when the length of the side is 10 cm, find
(a) the rate of increase of the perimeter,
(b) the rate of increase of the area.

Answers:

1 (a) $2x-5, x \geqslant \frac{5}{2}$ (b) $2x+6, x \geqslant -3$
(c) $-3-2x, x \leqslant -\frac{3}{2}$ (d) $6x-5, x \geqslant \frac{5}{6}$
(e) $10x+3, x \geqslant -\frac{3}{10}$ (f) $-4-6x, x \leqslant -\frac{2}{3}$

2 (a) $2x+4, x \leqslant -2$ (b) $2x-3, x \leqslant \frac{3}{2}$
(c) $-3+2x, x \leqslant \frac{3}{2}$ (d) $4x-8, x \leqslant 2$
(e) $7-4x, x \geqslant \frac{7}{4}$ (f) $-5-14x, x \geqslant -\frac{5}{14}$

3 (a) (i) $(4,-12)$ (ii) minimum (iv) $f(x) \geqslant -12$
(b) (i) $(-2,-7)$ (ii) minimum (iv) $f(x) \geqslant -7$
(c) (i) $\left(-\frac{3}{5},\frac{1}{5}\right)$ (ii) minimum (iv) $f(x) \geqslant \frac{1}{5}$
(d) (i) $(-3,13)$ (ii) maximum (iv) $f(x) \leqslant 13$
(e) (i) $(-3,0)$ (ii) minimum (iv) $f(x) \geqslant 0$
(f) (i) $\left(-\frac{1}{2},2\right)$ (ii) maximum (iv) $f(x) \leqslant 2$

4 (a) $(-4,213)$, maximum; $(3,-130)$, minimum
(b) $(-3,88)$, maximum; $(5,-168)$, minimum
(c) $(0,0)$, minimum; $(1,1)$, neither
(d) $(-2,65)$, maximum; $(0,1)$, neither; $(2,-63)$, minimum
(e) $\left(-\frac{1}{3},-\frac{11}{27}\right)$, minimum; $\left(\frac{1}{2},\frac{3}{4}\right)$, maximum
(f) $(-1,0)$, neither
(g) $(-1,-2)$, maximum; $(1,2)$, minimum
(h) $(3,27)$, minimum
(i) none
(j) $\left(\frac{1}{4},-\frac{1}{4}\right)$, minimum
(k) $\left(6,\frac{1}{12}\right)$, maximum
(l) $(-2,17)$, minimum
(m) $(1,3)$, maximum
(n) $(-1,-5)$, minimum
(o) $(0,0)$, minimum; $\left(\frac{4}{5},\frac{256}{3125}\right)$, maximum

5 (a) 4.8 cm s^{-1} (b) $24 \text{ cm}^2 \text{ s}^{-1}$

6 (a) $240 \text{ mm}^2 \text{ s}^{-1}$ (b) $2400 \text{ mm}^3 \text{ s}^{-1}$

Example

Suppose that a spherical balloon is being inflated at a constant rate of $5\text{ m}^3\text{ s}^{-1}$. At a particular moment, the radius of the balloon is 4 metres. Find how fast the radius of the balloon is increasing at that instant.

First translate the information into a mathematical form.

Let V m^3 be the volume of the balloon, and let r metres be its radius. Let t seconds be the time for which the balloon has been inflating. Then you are given that $\frac{dV}{dt} = 5$ and $r = 4$, and you are asked to find $\frac{dr}{dt}$ at that moment.

Your other piece of information is that the balloon is spherical, so that $V = \frac{4}{3}\pi r^3$.

The key to solving the problem is to use the chain rule in the form

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}.$$

You can now use $\frac{dV}{dr} = 4\pi r^2$. Substituting the various values into the chain rule formula gives

$$5 = \left(4\pi \times 4^2\right) \times \frac{dr}{dt}.$$

Therefore, rearranging this equation, you find that $\frac{dr}{dt} = \frac{5}{64\pi}$, so the radius is increasing at $\frac{5}{64\pi}$ m s^{-1}.

Example

The surface area of a cube is increasing at a constant rate of $24\text{ cm}^2\text{ s}^{-1}$. Find the rate at which its volume is increasing at the moment when the volume is 216 cm^3.

Let the side of the cube be x cm at time t seconds, let the surface area be S cm^2 and let the volume be V cm^3.

Then $S = 6x^2$, $V = x^3$ and $\frac{dS}{dt} = 24$, and you need to find $\frac{dV}{dt}$ when $V = 216$, which is when $x^3 = 216$, or $x = 6$.

If you know S and want to find V you need to find x first. Similarly, when you know $\frac{dS}{dt}$ and want to find $\frac{dV}{dt}$ you should expect to find $\frac{dx}{dt}$ first.

From the chain rule, $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 3x^2\frac{dx}{dt}$, so, when $x = 6$, $\frac{dV}{dt} = 108\frac{dx}{dt}$.

But $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt} = 12x\frac{dx}{dt}$, so $24 = (12 \times 6) \times \frac{dx}{dt}$, giving $\frac{dx}{dt} = \frac{1}{3}$. Substituting this in the equation $\frac{dV}{dt} = 108\frac{dx}{dt}$ gives $\frac{dV}{dt} = 108 \times \frac{1}{3} = 36$.

Therefore the volume is increasing at a rate of $36\text{ cm}^3\text{ s}^{-1}$.

Example

Find the minimum and maximum points on the graph of $y = \frac{(x+1)^2}{x}$.

The function is defined for all real numbers except 0.

To differentiate, write $\frac{(x+1)^2}{x}$ as $\frac{x^2+2x+1}{x} = x+2+x^{-1}$.

Then $\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$, so $\frac{dy}{dx} = 0$ gives $x^2 - 1 = 0$, or $x = \pm 1$.

The second derivative is $\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$. This has values -2 when $x = -1$, and 2 when $x = 1$. So $(-1,0)$ is a maximum point and $(1,4)$ is a minimum point.

Exercise #5

1. For each of the following equations connecting x and y, if the rate of change of x is 2 units s^{-1}, find the rate of change of y at the given instant.
   (a) $y = 3x^2 - 1$; $x = 2$
   (b) $y = 2x^2 + \frac{1}{x}$; $x = 1$
   (c) $y = \frac{3}{(2x-3)^3}$; $x = 2$
   (d) $y = (3x - 5)^5$; $x = \frac{4}{3}$
   (e) $y = x^3 + 2$; $y = 10$
   (f) $y = \frac{x}{x+1}$; $y = 2$

2. For each of the following equations connecting x and y, if the rate of change of y is 4 units s^{-1}, find the rate of change of x at the given instant.
   (a) $y = x^3 - 2x^2$; $x = 3$
   (b) $y = \frac{3x^2}{1+x}$; $x = 2$
   (c) $y = \sqrt{2x+7}$; $y = 3$
   (d) $y = x(x - 4)$, $x > 0$; $y = 5$

3. The radius of a circle increases at a rate of 2 cm s^{-1}. Find the rate of increase of its area when
   (a) the radius is 4 cm,
   (b) the area is 9π cm^2.

4. The area of a circle increases at a rate of 2π cm^2 s^{-1}. Calculate the rate of increase of the radius when the radius is 6 cm.

5. The radius of a circular disc increases at a constant rate of 0.02 cm s^{-1}. Find the rate at which the area is increasing when the radius is 10 cm.

6. A circular ripple spreads across a lake. If the area of the ripple increases at a rate of 10π m^2 s^{-1}, find the rate at which the radius is increasing when the radius is 2 m.

7. The radius of a sphere increases at a rate of 2 cm s^{-1}. Find the rate of increase of its volume when the radius is 3 cm.

8. Air is let out of a spherical balloon at a rate of 300 cm^3 s^{-1}. Find the rate at which the radius is decreasing when
   (a) the radius is 2 cm,
   (b) the volume is 36π cm^3.

9. The area of a square increases at a rate of 10 cm^2 s^{-1}. Find the rate of change in the length of its side when the area is 4 cm^2.

10. A metal cube is being expanded by heat. At the instant when the length of an edge is 2 cm, the volume of the cube is increasing at the rate of 0.012 cm^3 per second. At what rate is the length of the edge increasing at this instant? *(C)*

11. The surface area of a cube is increasing at 0.2 cm^2 s^{-1}. Find the rate of increase of the volume when the length of a side is 1 cm.

12. (a) Variables x and y are connected by the equation $y = x^2 + \frac{1}{x}$. Given that x is increasing at the rate of $\frac{1}{5}$ unit s^{-1}, find the rate of increase of y when $x = 2$.
    (b) A rectangle has sides of length $2x$ cm and $3x$ cm. Given that the area is increasing at a rate of 36 cm^2 s^{-1}, find the rate of increase of the perimeter when $x = 3$.

Answers

1. (a) 24 units s^{-1} (b) 6 units s^{-1} (c) -36 units s^{-1}
(d) 30 units s^{-1} (e) 24 units s^{-1} (f) 2 units s^{-1}

2. (a) $\frac{4}{15}$ units s^{-1} (b) $\frac{3}{2}$ units s^{-1} (c) 12 units s^{-1} (d) $\frac{2}{3}$ units s^{-1}

3. (a) 16π cm^2 s^{-1} (b) 12π cm^2 s^{-1}

4. $\frac{1}{6}$ cm s^{-1}

5. 0.4π cm^2 s^{-1}

6. $\frac{5}{2}$ m s^{-1}

7. 72π cm^3 s^{-1}

8. (a) $\frac{75}{4\pi}$ cm s^{-1} (b) $\frac{25}{3\pi}$ cm s^{-1}

9. $\frac{5}{2}$ cm s^{-1}

10. 0.001 cm s^{-1}

11. 0.05 cm^3 s^{-1}

12. (a) $\frac{3}{4}$ units s^{-1} (b) 10 cm s^{-1}

Application of Differentiation

Exercise #6

1. For each of the following expressions, calculate the value of x which gives y a stationary value. Determining whether this value of y is a minimum or a maximum. Give your answers correct to 3 significant figures where necessary.
   (a) $y = 2x^2 - 8x + 3$ (b) $y = x^2 + (2x - 1)^2$
   (c) $y = \sqrt{3}x - 6x^2 + 1$ (d) $y = 2\pi x - (x - 2)^2$

2. Find the stationary values of the following functions.
   (a) $y = (x - 2)^4 + 3$ (b) $y = (x - 3)(x + 2)$
   (c) $y = x\sqrt{1 - 2x}$ (d) $y = \frac{x}{x^2 + 1}$

3. If $2x + y = 10$ and $A = xy$, find the maximum value of A.

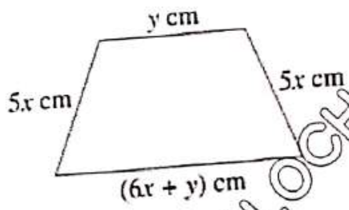
4. Two variables x and y vary in such a way that $x + y = 2$. Another variable z is defined by $z = x^2 + y^2$. Find the values of x and y that make z a minimum.

5. The positive variables x and y are such that $x^4y = 32$. A third variable z is defined by $z = x^2 + y$. Find the values of x and y that give z a stationary value and show that this value of z is a minimum.

6. A rectangle has sides x cm and y cm. If the area of the rectangle is 16 cm², show that its perimeter, P cm, is given by $P = 2x + \frac{32}{x}$. Hence, calculate the value of x which gives P a stationary value and show that this value of P is a minimum.

7. If the perimeter of a rectangle is to be 80 m, calculate the maximum area.

8. A piece of wire of length 104 cm, is bent to form a trapezium as shown in the diagram. Express y in terms of x and show that the area, A cm², enclosed by the wire is given by $A = 208x - 20x^2$. Find the value of x and of y for which A is a maximum.



9. A rectangle block has a total surface area of 1.08 m². The dimensions of the block are x m, $2x$ m and h m. Show that $h = \frac{1.08 - 4x^2}{6x}$ and hence express the volume of the block in terms of x. Find the value of x that makes this volume a maximum. (Proof that it is a maximum is not required.) *(C)*

10. In an upright triangular prism, the triangular base ABC is right-angled at B, $AB = 5x$ cm and $BC = 12x$ cm. The sum of the lengths of all its edges is 180 cm.
   (a) Show that the volume, V cm³, is given by $V = 1800x^2 - 600x^3$.
   (b) Find the value of x for which V has a maximum value.

Answers

1. (a) 2, min. (b) $\frac{2}{5}$, min. (c) 0.144, max. (d) 5.14, max.

2. (a) 3 (b) $-6\frac{1}{4}$ (c) $\frac{1}{3\sqrt{3}}$ (d) $-\frac{1}{2}$, $\frac{1}{2}$

3. 12.5
4. $x = 1$, $y = 1$
5. $x = 2$, $y = 2$
6. 4
7. 400 m^2
8. $y = 52 - 8x$, $x = 5.2$, $y = 10.4$
9. $\frac{1}{3}x(1.08 - 4x^2)$, $x = 0.3$
10. (b) 2

Past Paper Questions

1 The equation of a curve is $y = \sqrt{(5x+4)}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates (x, y) moves along the curve in such a way that the rate of increase of x has the constant value 0.03 units per second. Find the rate of increase of y at the instant when $x = 1$. [2]

2 A curve has equation $y = \frac{k}{x}$. Given that the gradient of the curve is -3 when $x = 2$, find the value of the constant k. [3]

3 The equation of a curve is $y = 2x + \frac{8}{x^2}$.

(i) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary point on the curve and determine the nature of the stationary point. [3]

(iii) Show that the normal to the curve at the point $(-2, -2)$ intersects the x-axis at the point $(-10, 0)$. [3]

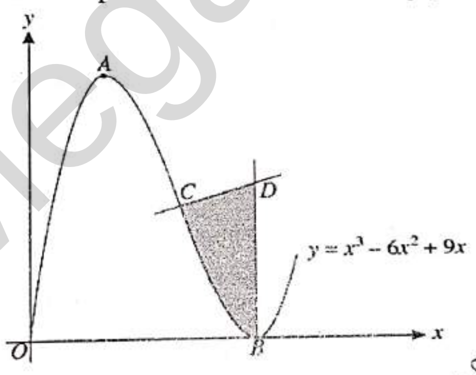
(iv) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$. [3]

4 The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$.

(i) Find the x-coordinates of the points of intersection of L and C. [4]

(ii) Show that one of these points is also the stationary point of C. [3]

5



The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geq 0$. The curve has a maximum point at A and a minimum point on the x-axis at B. The normal to the curve at C (2, 2) meets the normal to the curve at B at the point D.

(i) Find the coordinates of A and B. [3]

(ii) Find the equation of the normal to the curve at C. [3]

(iii) Find the area of the shaded region. [5]

6 A solid rectangular block has a square base of side x cm. The height of the block is h cm and the total surface area of the block is 96 cm^2.

(i) Express h in terms of x and show that the volume, V cm^3, of the block is given by

$$V = 24x - \tfrac{1}{2}x^3.$$ [3]

Given that x can vary,

(ii) find the stationary value of V, [3]

(iii) determine whether this stationary value is a maximum or a minimum. [2]

7 A curve has equation $y = \dfrac{4}{3x-4}$ and $P\,(2, 2)$ is a point on the curve.

(i) Find the equation of the tangent to the curve at P. [4]

(ii) Find the angle that this tangent makes with the x-axis. [2]

8 The equation of a curve is $y = 4\sqrt{x} + \dfrac{2}{\sqrt{x}}$.

(i) Obtain an expression for $\dfrac{dy}{dx}$. [3]

(ii) A point is moving along the curve in such a way that the x-coordinate is increasing at a constant rate of 0.12 units per second. Find the rate of change of the y-coordinate when $x = 4$. [2]

9 The volume of a solid circular cylinder of radius r cm is 250π cm^3.

(i) Show that the total surface area, S cm^2, of the cylinder is given by

$$S = 2\pi r^2 + \frac{500\pi}{r}.$$ [2]

(ii) Given that r can vary, find the stationary value of S. [4]

(iii) Determine the nature of this stationary value. [2]

10 A curve has equation $y = x^3 + 3x^2 - 9x + k$, where k is a constant.

(i) Write down an expression for $\dfrac{dy}{dx}$. [2]

(ii) Find the x-coordinates of the two stationary points on the curve. [2]

(iii) Hence find the two values of k for which the curve has a stationary point on the x-axis. [3]

11 A solid rectangular block has a base which measures $2x$ cm by x cm. The height of the block is y cm and the volume of the block is 72 cm^3.

(i) Express y in terms of x and show that the total surface area, A cm^2, of the block is given by

$$A = 4x^2 + \frac{216}{x}.$$ [3]

Given that x can vary,

(ii) find the value of x for which A has a stationary value, [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

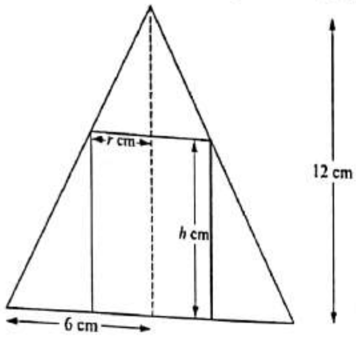
12 A curve has equation $y = x^2 + \frac{2}{x}$.

(i) Write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary point on the curve and determine its nature. [4]

(iii) Find the volume of the solid formed when the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is rotated completely about the x-axis. [6]

13



The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.

(i) Express h in terms of r and hence show that the volume, $V\text{ cm}^3$, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3.$$ [3]

(ii) Given that r varies, find the stationary value of V. [4]

14 The equation of a curve is $y = \frac{6}{5 - 2x}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates (x, y) moves along the curve in such a way that the rate of increase of y has a constant value of 0.02 units per second. Find the rate of increase of x when $x = 1$. [2]

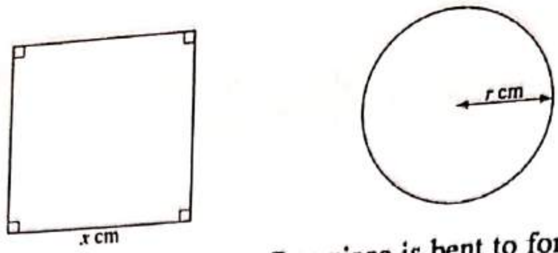
(iii) The region between the curve, the x-axis and the lines $x = 0$ and $x = 1$ is rotated through 360° about the x-axis. Show that the volume obtained is $\frac{12}{5}\pi$. [5]

15 The equation of a curve is $y = (2x - 3)^3 - 6x$.

(i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x. [3]

(ii) Find the x-coordinates of the two stationary points and determine the nature of each stationary point. [5]

16

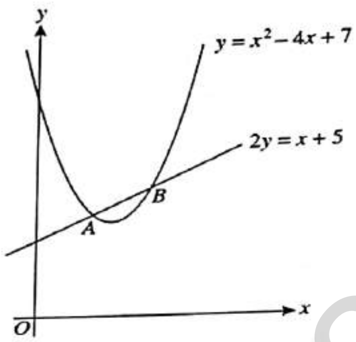


A wire, 80 cm long, is cut into two pieces. One piece is bent to form a square of side x cm and the other piece is bent to form a circle of radius r cm (see diagram). The total area of the square and the circle is A cm^2.

(i) Show that $A = \dfrac{(\pi + 4)x^2 - 160x + 1600}{\pi}$. [4]

(ii) Given that x and r can vary, find the value of x for which A has a stationary value. [4]

17



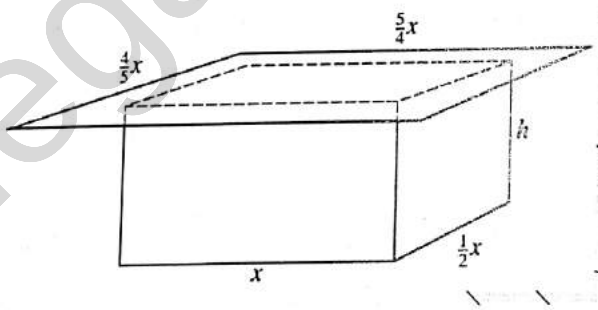
The diagram shows the line $2y = x + 5$ and the curve $y = x^2 - 4x + 7$, which intersect at the points A and B. Find

(a) the x-coordinates of A and B, [3]

(b) the equation of the tangent to the curve at B, [3]

(c) the acute angle, in degrees correct to 1 decimal place, between this tangent and the line $2y = x + 5$. [3]

18



The diagram shows an open rectangular tank of height h metres covered with a lid. The base of the tank has sides of length x metres and $\frac{1}{2}x$ metres and the lid is a rectangle with sides of length $\frac{5}{4}x$ metres and $\frac{4}{5}x$ metres. When full the tank holds 4 m^3 of water. The material from which the tank is made is of negligible thickness. The external surface area of the tank together with the area of the top of the lid is A m^2.

(i) Express h in terms of x and hence show that $A = \frac{3}{2}x^2 + \dfrac{24}{x}$. [5]

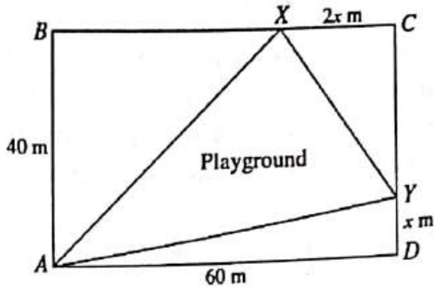
(ii) Given that x can vary, find the value of x for which A is a minimum, showing clearly that A is a minimum and not a maximum. [5]

19 The equation of a curve is $y = \sqrt{(8x - x^2)}$. Find

(i) an expression for $\frac{dy}{dx}$, and the coordinates of the stationary point on the curve, [4]

(ii) the volume obtained when the region bounded by the curve and the x-axis is rotated through 360° about the x-axis. [4]

20



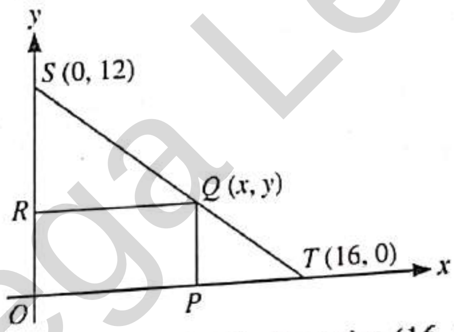
The diagram shows a plan for a rectangular park $ABCD$, in which $AB = 40$ m and $AD = 60$ m. Points X and Y lie on BC and CD respectively and AX, XY and YA are paths that surround a triangular playground. The length of DY is x m and the length of XC is $2x$ m.

(i) Show that the area, A m², of the playground is given by [2]

$$A = x^2 - 30x + 1200.$$

(ii) Given that x can vary, find the minimum area of the playground. [3]

21



In the diagram, S is the point (0, 12) and T is the point (16, 0). The point Q lies on ST, between S and T, and has coordinates (x, y). The points P and R lie on the x-axis and y-axis respectively and $OPQR$ is a rectangle.

(i) Show that the area, A, of the rectangle $OPQR$ is given by $A = 12x - \frac{3}{4}x^2$. [3]

(ii) Given that x can vary, find the stationary value of A and determine its nature. [4]

22 A curve has equation $y = \frac{12}{3 - 2x}$.

(i) Find $\frac{dy}{dx}$. [2]

A point moves along this curve. As the point passes through A, the x-coordinate is increasing at a rate of 0.15 units per second and the y-coordinate is increasing at a rate of 0.4 units per second.

(ii) Find the possible x-coordinates of A. [4]

Answers

1 (i) $\frac{5}{6}$ (ii) 0.025 units/sec

2 $k=12$

3 (i) $2-\frac{16}{x^3}$; $\frac{48}{x^4}$ (ii) (2, 6) minimum

(iii) (−10, 0) (iv) 7 sq. units

4 (i) $x=2$, $1\frac{1}{2}$ (ii) (2, 1)

5 (i) $A(1, 4)$ $B(3, 0)$ (ii) $3y-x=4$ (iii) $2\frac{1}{6}$ units^2

(iv) $1\frac{5}{12}$ unit^2

6 (ii) 64 (iii) the stationary value is a maximum.

7 (i) $3x+y=8$ (ii) 108.4°

8 (i) $\frac{2x-1}{x\sqrt{x}}$ (ii) 0.105 units/sec

9 (ii) 471 cm^2 (iii) minimum value.

10 (i) $3x^2+6x-9$ (ii) $x=1$, or $x=-3$

(iii) $k=5$, -27

11 (ii) $x=3$

12 (i) $2+\frac{4}{x^3}$ (ii) (1, 3) (iii) minimum (iv) $14\frac{1}{5}\pi$

13 (ii) 64π

14 (i) $\frac{4}{3}$ (ii) 0.015 units/sec

15 (i) $24(2x-3)$ (ii) $x=1$ and $x=2$

at $x=1$, the stationary point is a maximum point.

at $x=2$, the stationary point is a minimum

16 (ii) 11.2 cm

17 (a) x-coordinate of A is: $x=\frac{3}{2}$

x-coordinate of B is: $x=3$

(b) $y-2x+2=0$ (c) 36.8°

18 (ii) A is a minimum at $x=2$

19 (i) $\frac{4-x}{\sqrt{8x-x^2}}$ (4, 4) (ii) $85\frac{1}{3}\pi$ units^3

20 (ii) 975 m^2

21 (ii) 48 stationary value of A is a maximum

22 (i) $\frac{24}{(3-2x)^2}$ (ii) 0 or 3.

23 (ii) $\frac{1}{2\sqrt{3}}$ cm s^{-1}

24 (i) $-\frac{6}{(2x-1)^2}$ (ii) $-\frac{6}{(2x-1)^2}<0$ (iv) 0.04 units/sec

INTEGRATION

Syllabus:

- Understand integration as the reverse process of differentiation, and integrate $(ax+b)^n$ (for any rational n except -1) Together with constant multiples, sums and differences:
- Solve problems involving the evaluation of a constant of integration. e.g to find the equation of the curve through (1-2) for which $\frac{dy}{dx} = 2x+1$
- Evaluate definite integrals (including simple cases of 'improper' integrals, such as $\int_0^1 x^{-\frac{1}{2}} dx$ and $\int_1^x x^{-2} dx$);
- Use definite integration to find.
- The area of a region bounded by a curve and lines parallel to the axes, or between two curves.
- A volume of revolution about one of the axes.

INTEGRATION

In the process of differentiation, if y = $\frac{1}{2}x^2 + k$, where k is any constant, then dy/dx = x. This means that for any curve define by y = $\frac{1}{2}x^2 + k$ have the same gradient function dy/dx = x.

Conversely, for gradient function dy/dx = x. The equation of the curve is of the form y = $\frac{1}{2}x^2$+K. this process is of the reverse of differentiation and is called integration, where K is any arbitrary constant and known as the constant of integration.

Indefinite integral

There is an arbitrary constant 'k' in the expression $\frac{1}{2}x^2$ +K, which shows that it is not a unique function. That is why the expression is called indefinite integral.

i.e. $\frac{d}{dx}\left(\frac{1}{2}x^2 + K\right) = x \quad \Rightarrow \quad \int xdx = \frac{1}{2}x^2 + K$

Power rule of integration

In general if n ≠ -1, a and n are constants, then $\int ax^n\,dx = \frac{ax^{n+1}}{(n+1)\frac{d}{dx}(base)} + K$, where $\int$ dx means, the integral of (...) w.r.t x.

Reverese process of differentiation

An integration reverses the process of differentiation. For a function f (x), we have $\int \frac{d}{dx} f(x)\,dx = f(x) + K$

In integrating a constant 'c' $\int d/dx((x) = \int c$ thus

$$\int c\,d\,x == cx + k$$

Integration of sum and difference function

Integration is distributive in case of sum or difference of functions:

i.e. $\int (ax+b)\,dx = \int a\,x\,dx \pm \int b dx$

Definite integral

In general if d/dx F(x) = f(x), then the change in F(x) when x changes form α to b is

$$F(b) - F(a) = [F(x)]^b_a = \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)dx = F(b) - F(a)$$

Area between the curve and axis (Shaded area):

If 'a' and 'b' are the limits of region under the curve on x-axis, then area from x=a to x = b is. Area under the curve = $\int_a^b f(x)\,dx$ where f(x) >, 0 for a < x < b i.e. The curve lie upward to the x-axis. F (x) < 0 means that the curve lie downward to the x-axis. In this case the area enclosed by the curve and x-axis form x=a to x=b is Area = $\int_a^b -f(x)dx$ where f(x) < 0 for α < x < b.

Area between a curve and y axis

Area bounded by the curve x = h(y), the y-axis and the lines y = a to y = b is given by

Area = $\int_a^b x dy$ where x>, 0 for a < y < b

X≤0 means that the curve lies in 2nd or 3rd quadrent then area = $\int_a^b -x dy$ where x < 0 for a < y < b

Area between two curves

If y = f (x) is equation of upper curve and y = g (x) is the equation of below curve of the region and 'a' and 'b' are the limits of area on x-axis then

Area bounded by the curves = $\int_a^b (f(x)-g(x))dx$ where f(x), ≥ g (x) for the interval a< x<b.

If limits of bounded region on x-axis are not given, then limits can be found by finding the points of intersection of line and curve of two curves.

Volume of revolution about -axis

If y = f(x) is the final equation of bounded area and 'a' and 'b' are the limits of required region on x-axis. Then

volume of rotation about x-axis = $\pi \int_a^b y^2 dx.$ 106

Where y = f(x) and a< x < b

When x = h (y) and y = a and y = b are the limits on y-axis

Then volume of rotation about y-axis = $\pi \int_a^b x^2 dy.$

Equation of curve

To find equation of curve always integrate its derivative. First separate and then apply integration on both sides. To find value of integration constant, put coordinate of the point lie on the curve in the equation and then put that value of integration constant in final

$$\int g(ax+b)\,dx = \frac{1}{a}f(ax+b)+k$$ where $f(x)$ is the simplest integral of $g(x)$.

Example:

Find the integrals of (a) $\sqrt{5-2x}$, (b) $\frac{1}{(3-x)^2}$.

$$\int (5-2x)^{\frac{1}{2}}\,dx = -\tfrac{1}{3}(5-2x)^{\frac{3}{2}}+k.$$

$$\int \frac{1}{(3-x)^2}\,dx = \int (3-x)^{-2}\,dx = \frac{1}{-1}\times\frac{1}{-1}(3-x)^{-1}+k = \frac{1}{3-x}+k.$$

Example:

$$\int_{-\frac{3}{2}}^{\frac{1}{2}} \left(16-(2x+1)^4\right)dx$$

$$= \left[16x - \tfrac{1}{10}(2x+1)^5\right]_{-\frac{3}{2}}^{\frac{1}{2}}$$

$$= \left(16\times\tfrac{1}{2} - \tfrac{1}{10}\left(2\times\tfrac{1}{2}+1\right)^5\right) - \left(16\times\left(-\tfrac{3}{2}\right) - \tfrac{1}{10}\left(2\times\left(-\tfrac{3}{2}\right)+1\right)^5\right)$$

$$= \left(8-\tfrac{1}{10}\times 2^5\right) - \left(-24-\tfrac{1}{10}\times(-2)^5\right)$$

$$= 4.8-(-20.8) = 25.6.$$

Example

The graph of $y = f(x)$ passes through $(2,3)$, and $f'(x) = 6x^2 - 5x$. Find its equation.

The indefinite integral is $6\left(\tfrac{1}{3}x^3\right) - 5\left(\tfrac{1}{2}x^2\right)+k$, so the graph has equation

$$y = 2x^3 - \tfrac{5}{2}x^2 + k$$

for some constant k. The coordinates $x=2$, $y=3$ have to satisfy this equation, so

$$3 = 2\times 8 - \tfrac{5}{2}\times 4 + k, \text{ giving } k = 3-16+10 = -3.$$

The equation of the graph is therefore $y = 2x^3 - \tfrac{5}{2}x^2 - 3$.

Practice Questions

Exercise #1

1. Find an expression for y if $\frac{dy}{dx}$ is each of the following:

(a) $2x^3$ (b) -5 (c) $\sqrt{x}$ (d) $-\frac{1}{x^2}$ (e) $\frac{2}{\sqrt{x}}$ (f) $\frac{1}{2x^3}$

2. Find an expression for y if $\frac{dy}{dx}$ is each of the following:

(a) $6x + 3$ (b) 4 (c) $3x(x + 2)$

(d) $(x - 1)(x + 2)$ (e) $x(2 + \frac{1}{x})$ (f) $\frac{2x^2 + 3}{x^2}$

3. Integrate with respect to x.

(a) $x^2 + \frac{1}{x^2}$ (b) $\frac{x^2 + 1}{2x^2}$

(c) $3 - \sqrt{x}$ (d) $\sqrt{x}(\sqrt{x} + 3)$

4. Given that a and b are constants, integrate with respect to x.

(a) $ax + b$ (b) $a - bx^2$

5. Find

(a) $\int (2 + 4x - 3x^2)\, dx$, (b) $\int \left(x^4 - \frac{1}{x^2}\right) dx$,

(c) $\int (2x - \sqrt{x})^2\, dx$, (d) $\int \frac{x + 1}{\sqrt{x}}\, dx$.

6. Find the equation of the curve which passes through the point (2, 4) and for which $\frac{dy}{dx} = x(3x - 1)$.

7. Find the equation of the curve which passes through the points (2, –2) and (4, 2) and for which $\frac{dy}{dx} = x^2(x - k)$ where k is a constant.

8. Given that the gradient of a curve is $2x + \frac{3}{x^2}$ and that the curve passes through the point (–1, 5), determine the equation of the curve.

9 Integrate with respect to x.

(a) $(3x + 1)^4$ (b) $(1 - x)^3$ (c) $(2x + 5)^{-3}$

(d) $\sqrt{6x - 1}$ (e) $\frac{2}{(2x - 7)^2}$ (f) $\frac{1}{\sqrt{3 - 2x}}$

(g) $\frac{3}{5(3x - 1)^6}$ (h) $\frac{4}{3\sqrt{6x - 1}}$ (i) $\left(\frac{4}{1 - 2x}\right)^2$

10 Find

(a) $\int (1 - x)^6\, dx$, (b) $\int 3(2x - 5)^2\, dx$, (c) $\int \frac{2}{(1 - x)^2}\, dx$, (d) $\int \sqrt{4t - 1}\, dt$.

11 Given that $\frac{dy}{dx} = (3x - 2)^2$ and that $y = 0$ when $x = 1$, calculate the value of y when $x = 1.5$.

12. The gradient of a curve is $6(4x - 1)^2$ and the curve passes through the origin. Find the equation of the curve.

Answers

1. (a) $y = \frac{1}{2}x^4 + c$ (b) $y = -5x + c$ (c) $y = \frac{2}{3}x^{\frac{3}{2}} + c$

(d) $y = \frac{1}{x} + c$ (e) $y = 4\sqrt{x} + c$ (f) $y = -\frac{1}{4x^2} + c$

2. (a) $y = 3x^2 + 3x + c$ (b) $y = 4x + c$ (c) $y = x^3 + 3x^2 + c$

(d) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + c$ (e) $y = x^2 + x + c$ (f) $y = 2x - \frac{3}{x} + c$

3. (a) $\frac{1}{3}x^3 - \frac{1}{x} + c$ (b) $\frac{1}{2}x - \frac{1}{2x} + c$ (c) $3x - \frac{2}{3}x^{\frac{3}{2}} + c$ (d) $\frac{1}{2}x^2 + 2x^{\frac{3}{2}} + c$

4. (a) $\frac{1}{2}ax^2 + bx + c$ (b) $ax - \frac{1}{3}bx^3 + c$

5. (a) $2x + 2x^2 - x^3 + c$ (b) $\frac{1}{5}x^5 + \frac{1}{x} + c$ (c) $\frac{4}{3}x^3 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$ (d) $\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + c$

6. $y = x^3 - \frac{1}{2}x^2 - 2$ **7.** $y = \frac{1}{4}x^4 - x^3 + 2$ **8.** $y = x^2 - \frac{3}{x} + 1$

9 (a) $\frac{1}{15}(3x+1)^5 + c$ (b) $-\frac{1}{4}(1-x)^4 + c$ (c) $-\frac{1}{4}(2x+5)^{-2} + c$

(d) $\frac{1}{9}(6x-1)^{\frac{3}{2}} + c$ (e) $-\frac{1}{2x-7} + c$ (f) $-\sqrt{3-2x} + c$

(g) $-\frac{1}{25(3x-1)^5} + c$ (h) $\frac{4}{9}\sqrt{6x-1} + c$ (i) $\frac{8}{1-2x} + c$

10 (a) $-\frac{1}{7}(1-x)^7 + c$ (b) $\frac{1}{2}(2x-5)^3 + c$ (c) $\frac{2}{1-x} + c$ (d) $\frac{1}{6}(4t-1)^{\frac{3}{2}} + c$

11 1.625 12 $y = \frac{1}{2}(4x-1)^3 + \frac{1}{2}$

Exercise #3

1 Integrate the following with respect to x.

(a) $(2x+1)^6$ (b) $(3x-5)^4$ (c) $(1-7x)^3$ (d) $\left(\frac{1}{2}x+1\right)^{10}$

(e) $(5x+2)^{-3}$ (f) $2(1-3x)^{-2}$ (g) $\dfrac{1}{(x+1)^5}$ (h) $\dfrac{3}{2(4x+1)^4}$

(i) $\sqrt{10x+1}$ (j) $\dfrac{1}{\sqrt{2x-1}}$ (k) $\left(\frac{1}{2}x+2\right)^{\frac{3}{2}}$ (l) $\dfrac{8}{\sqrt[4]{2+6x}}$

2 Evaluate the following integrals.

(a) $\displaystyle\int_1^5 (2x-1)^3\,dx$ (b) $\displaystyle\int_1^5 \sqrt{2x-1}\,dx$ (c) $\displaystyle\int_1^3 \frac{1}{(x+2)^2}\,dx$ (d) $\displaystyle\int_1^3 \frac{2}{(x+2)^3}\,dx$

3 Given that $\displaystyle\int_{1.25}^{p} (4x-5)^4\,dx = 51.2$, find the value of p.

4 Find the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{x^3}\,dx$ (b) $\displaystyle\int \left(x^2-\frac{1}{x^2}\right)dx$ (c) $\displaystyle\int \sqrt{x}\,dx$

(d) $\displaystyle\int 6x^{\frac{2}{3}}\,dx$ (e) $\displaystyle\int \frac{6x^4+5}{x^2}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$

5 Evaluate the following definite integrals.

(a) $\displaystyle\int_0^8 12\sqrt[3]{x}\,dx$ (b) $\displaystyle\int_1^2 \frac{3}{x^2}\,dx$ (c) $\displaystyle\int_1^4 \frac{10}{\sqrt{x}}\,dx$

(d) $\displaystyle\int_1^2 \left(\frac{8}{x^3}+x^3\right)dx$ (e) $\displaystyle\int_4^9 \frac{2\sqrt{x}+3}{\sqrt{x}}\,dx$ (f) $\displaystyle\int_1^8 \frac{1}{\sqrt[3]{x^2}}\,dx$

6 Find a general expression for the function $f(x)$ in each of the following cases.

(a) $f'(x) = x^{-2}$ (b) $f'(x) = 3x^{-4}$ (c) $f'(x) = \dfrac{6}{x^3}$

(d) $f'(x) = 4x - \dfrac{3}{x^2}$ (e) $f'(x) = \dfrac{1}{x^3} - \dfrac{1}{x^4}$ (f) $f'(x) = \dfrac{2}{x^2} - 2x^2$

7 The graph of $y = f(x)$ passes through $\left(\frac{1}{2}, 5\right)$ and $f'(x) = \dfrac{4}{x^2}$. Find its equation.

8 A curve passes through the point $(25, 3)$ and is such that $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$. Find the equation of the curve.

9 A curve passes through the point $(1, 5)$ and is such that $\dfrac{dy}{dx} = \sqrt[3]{x} - \dfrac{6}{x^3}$. Find the equation of the curve.

Answers:

1 (a) $\frac{1}{14}(2x+1)^7+k$ (b) $\frac{1}{15}(3x-5)^5+k$

(c) $-\frac{1}{28}(1-7x)^4+k$ (d) $\frac{2}{11}\left(\frac{1}{2}x+1\right)^{11}+k$

(e) $-\frac{1}{10}(5x+2)^{-2}+k$ (f) $\frac{2}{3}(1-3x)^{-1}+k$

(g) $-\frac{1}{4}(x+1)^{-4}+k$ (h) $-\frac{1}{8}(4x+1)^{-3}+k$

(i) $\frac{1}{15}(10x+1)^{\frac{3}{2}}+k$ (j) $\sqrt{2x-1}+k$

(k) $\frac{6}{5}\left(\frac{1}{2}x+2\right)^{\frac{5}{3}}+k$ (l) $\frac{16}{9}(2+6x)^{\frac{3}{4}}+k$

2 (a) 820 (b) $\frac{26}{3}$ (c) $\frac{2}{15}$ (d) $\frac{16}{225}$

3 2.25

4 (a) $-\frac{1}{2x^2}+k$ (b) $\frac{1}{3}x^3+\frac{1}{x}+k$

(c) $\frac{2}{3}x\sqrt{x}+k$ (d) $\frac{18}{5}x^{5/3}+k$

(e) $2x^3-\frac{5}{x}+k$ (f) $2\sqrt{x}+k$

5 (a) 144 (b) $1\frac{1}{2}$ (c) 20
(d) $6\frac{3}{4}$ (e) 16 (f) 3

6 (a) $-\frac{1}{x}+k$ (b) $-\frac{1}{x^3}+k$

(c) $-\frac{3}{x^2}+k$ (d) $2x^2+\frac{3}{x}+k$

(e) $-\frac{1}{2x^2}+\frac{1}{3x^3}+k$ (f) $-\frac{2}{x}-\frac{2}{3}x^3+k$

7 $y=-4x^{-1}+13$

8 $y=\sqrt{x}-2$

9 $y=\frac{3}{4}x^{4/3}+3x^{-2}+\frac{5}{4}$

Exercise #4

1. Evaluate the following definite integrals.

(a) $\int_2^5 3x \, dx$ (b) $\int_1^9 x^{\frac{1}{2}} \, dx$ (c) $\int_1^8 \frac{1}{2} x^{-\frac{1}{3}} \, dx$

(d) $\int_2^3 \frac{1}{3x^2} \, dx$ (e) $\int_4^9 \frac{1}{\sqrt{x}} \, dx$ (f) $\int_1^4 x\sqrt{x} \, dx$

2. Evaluate the following definite integrals.

(a) $\int_{-1}^1 (8x - 4) \, dx$ (b) $\int_{-1}^0 (3x^2 - 2x + 5) \, dx$

(c) $\int_1^4 (6x - 3\sqrt{x}) \, dx$ (d) $\int_1^4 \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right) dx$

(e) $\int_1^2 \left(x^2 - \frac{4}{x^2}\right) dx$ (f) $\int_1^2 \left(8x^3 - 2 + \frac{1}{2x^2}\right) dx$

3. Evaluate the following definite integrals.

(a) $\int_0^2 x(x^2 - 2) \, dx$ (b) $\int_1^2 (x + 1)(x - 2) \, dx$

(c) $\int_{-1}^0 x(x - 2)(x + 2) \, dx$ (d) $\int_0^4 \sqrt{x}(1 - \sqrt{x}) \, dx$

(e) $\int_1^3 \frac{1}{x^2}(4x^2 - 9) \, dx$ (f) $\int_0^1 x^2(2 - 3\sqrt{x}) \, dx$

4. Evaluate the following definite integrals.

(a) $\int_1^4 \frac{x^2 + 1}{x^2} \, dx$ (b) $\int_1^2 \frac{1 - 2x^3}{x^2} \, dx$

(c) $\int_1^4 \frac{2x - 1}{\sqrt{x}} \, dx$ (d) $\int_1^9 \frac{3 - 2\sqrt{x}}{x^2} \, dx$

(e) $\int_1^3 \frac{1 - 4x + x^3}{2x^3} \, dx$ (f) $\int_1^2 \frac{(x + 3)(x - 3)}{x^2} \, dx$

Answers

1. (a) $31\frac{1}{2}$ (b) $17\frac{1}{3}$ (c) $2\frac{1}{4}$ (d) $\frac{1}{18}$ (e) 2 (f) $12\frac{2}{5}$

2. (a) -8 (b) 7 (c) 31 (d) $\frac{2}{3}$ (e) $\frac{1}{3}$ (f) $28\frac{1}{4}$

3. (a) 0 (b) $-1\frac{1}{6}$ (c) $1\frac{3}{4}$ (d) $-2\frac{2}{3}$ (e) 2 (f) $-\frac{4}{21}$

4. (a) $3\frac{3}{4}$ (b) $-2\frac{1}{2}$ (c) $7\frac{1}{3}$ (d) 0 (e) $-\frac{1}{9}$ (f) $-3\frac{1}{2}$

Past Paper Questions

Differentiation&Integration

1 A curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 6$ and the point (9, 2) lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the x-coordinate of the stationary point on the curve and determine the nature of the stationary point. [3]

2 The equation of a curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(3x-2)}}$. Given that the curve passes through the point $P(2, 11)$, find

(i) the equation of the normal to the curve at P, [3]

(ii) the equation of the curve. [4]

3 A curve is such that $\frac{dy}{dx} = \frac{3}{(1+2x)^2}$ and the point $(1, \frac{1}{2})$ lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the set of values of x for which the gradient of the curve is less than $\frac{1}{3}$. [3]

4 A curve is such that $\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 1$ and $P(9, 5)$ is a point on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the coordinates of the stationary point on the curve. [3]

(iii) Find an expression for $\frac{d^2y}{dx^2}$ and determine the nature of the stationary point. [2]

(iv) The normal to the curve at P makes an angle of $\tan^{-1} k$ with the positive x-axis. Find the value of k. [2]

5 A curve is such that $\frac{d^2y}{dx^2} = -4x$. The curve has a maximum point at (2, 12).

(i) Find the equation of the curve. [6]

A point P moves along the curve in such a way that the x-coordinate is increasing at 0.05 units per second.

(ii) Find the rate at which the y-coordinate is changing when $x = 3$, stating whether the y-coordinate is increasing or decreasing. [2]

6 A curve has equation $y = f(x)$ and is such that $f'(x) = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} - 10$.

(i) By using the substitution $u = x^{\frac{1}{2}}$, or otherwise, find the values of x for which the curve $y = f(x)$ has stationary points. [4]

(ii) Find $f''(x)$ and hence, or otherwise, determine the nature of each stationary point. [3]

(iii) It is given that the curve $y = f(x)$ passes through the point (4, −7). Find $f(x)$. [4]

7 The equation of a curve is $y = \frac{9}{2-x}$.

(i) Find an expression for $\frac{dy}{dx}$ and determine, with a reason, whether the curve has any stationary points. [3]

(ii) Find the volume obtained when the region bounded by the curve, the coordinate axes and the line $x = 1$ is rotated through 360° about the x-axis. [4]

(iii) Find the set of values of k for which the line $y = x + k$ intersects the curve at two distinct points. [4]

8 A curve has equation $y = f(x)$. It is given that $f'(x) = 3x^2 + 2x - 5$.

(i) Find the set of values of x for which f is an increasing function.

(ii) Given that the curve passes through (1, 3), find $f(x)$.

9 A function f is defined for $x \in \mathbb{R}$ and is such that $f'(x) = 2x - 6$. The range of the function is given by $f(x) \geqslant -4$.

(i) State the value of x for which $f(x)$ has a stationary value. [1]

(ii) Find an expression for $f(x)$ in terms of x. [4]

10 A curve $y = f(x)$ has a stationary point at $P(3, -10)$. It is given that $f'(x) = 2x^2 + kx - 12$, where k is a constant.

(i) Show that $k = -2$ and hence find the x-coordinate of the other stationary point, Q. [4]

(ii) Find $f''(x)$ and determine the nature of each of the stationary points P and Q. [2]

(iii) Find $f(x)$. [4]

11 A curve is such that

$$\frac{dy}{dx} = 2(3x+4)^{\frac{3}{2}} - 6x - 8.$$

(i) Find $\frac{d^2y}{dx^2}$. [2]

(ii) Verify that the curve has a stationary point when $x = -1$ and determine its nature. [2]

(iii) It is now given that the stationary point on the curve has coordinates $(-1, 5)$. Find the equation of the curve. [5]

Answers

(i) $y=\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}-6x\ (+c)$

1 (9, 2) $2=54-54+c$
$\rightarrow c=2.$

$\rightarrow$ +ve (or ¾) Minimum

2 $y-11=-\frac{1}{3}(x-2)$

$y=4\sqrt{3x-2}+3$

3 $y=\frac{3(1+2x)^{-1}}{-2}+(c)$
Sub (1, (1/2))
$\frac{1}{2}=\frac{3}{-6}+c\Rightarrow c=1$

(ii) $(1+2x)^2(>)9$ or $4x^2$.
1, −2
$x>1, x<-2$ ISW

4 $y=4\sqrt{x}-x\,(+c)$
Uses (9, 5) in an ir
$\rightarrow c=2$

$x=4, y=6$

(ii) $\frac{d^2y}{d^2x}=-x^{\frac{-3}{2}}\rightarrow -ve\rightarrow$ Max

$k=3$

5 $y=-\frac{2x^3}{3}+8x\ \ (+C)$
Subs (2, 12) $\rightarrow C=\frac{4}{3}$

$\rightarrow$ decreasing at 0.5 units per second

6 $\sqrt{x}=\frac{1}{3}$ or 3

$\sqrt{x}=\frac{1}{9}$ or 9

$=\frac{4}{9}) > 0\rightarrow$ Min

(iii) $f(x)=2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-10x\ (+c)$
$-7=16+12-40+c$
$c=5$

7 $\frac{9}{(2-x)^2}\neq 0.$ No turning points.

$\rightarrow \frac{81\pi}{2}$ (or 127)

$k<-8,\ k>4.$

8 $x<-5/3, x>1$
$f(x)=x^3+x^2-5x+6$

9 (i) 3

$f(x)=x^2-6x+5$

10 $x=-2$, (Allow also = 3)

(ii) $f'(x)=4x-2$
$f'(3)>0$ hence min at P
$f'(-2)<0$ hence max at Q

(iii) $f(x)=\frac{2}{3}x^3-x^2-12x\ (+c)$
$c=17$

11 (i) $f''(x)=[9]\times\left[(3x+4)^{\frac{1}{2}}\right]-[6]$

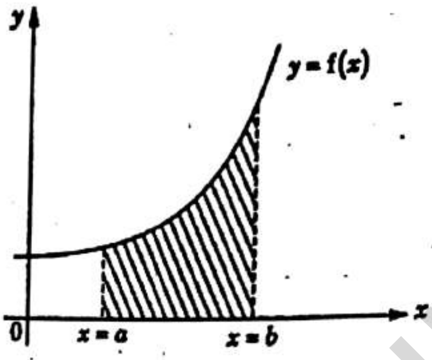
$=3$ >0 hence minimum

$c=-\frac{4}{15}$

Area under A Curve

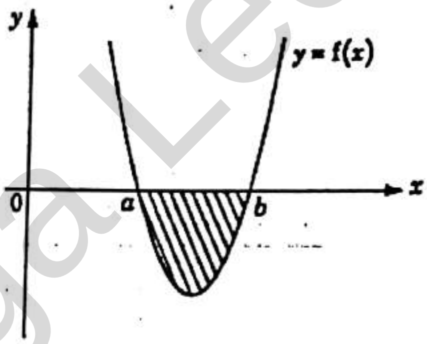
An important application of integration is in finding the area under a given curve $y=f(x)$.

$A=\int_a^b f(x)\,dx$	area bounded by $y=f(x)$, $x=a$, $x=b$ and x–axis



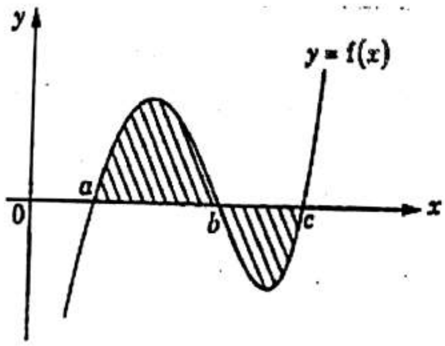
If area is below the x-axis:

$A=\int_b^a f(x)\,dx$ or $\left\vert\int_a^b f(x)\,dx\right\vert$	interchange limits or use | | to indicate



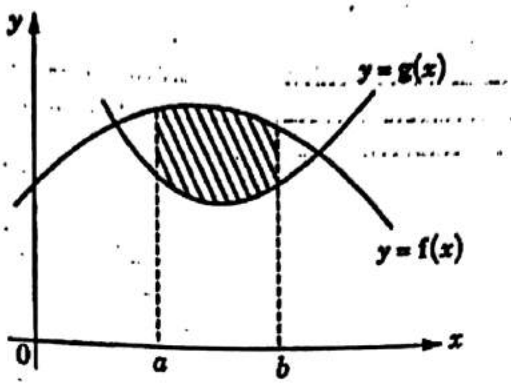
If area is both above and below the x-axis:

$A=\int_a^b f(x)\,dx+\left\vert\int_b^c f(x)\,dx\right\vert$	just add area above and area below



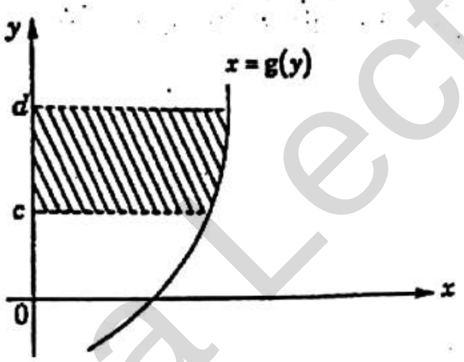
If **area** is between two curves $y = f(x)$ and $y = g(x)$:

$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ $= \int_a^b [f(x) - g(x)]\,dx$	check the limits so that the required area is always area of higher curve **less** area of lower curve



If **area** is between a curve and the y-axis:

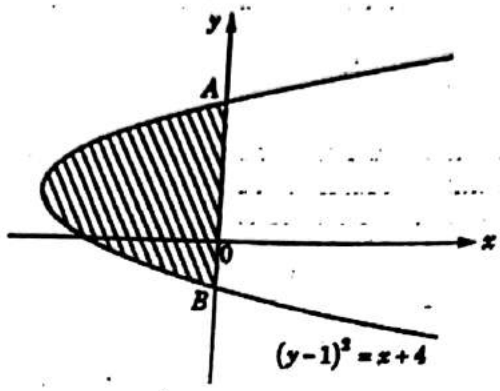
$A = \int_c^d x\,dy$ or $\int_c^d g(y)\,dy$	c and d are limits on y-axis and the curve must be expressed in the form $x = g(y)$



If the region is on the left side of y-axis, you need to interchange the limits.

Example

The curve $(y-1)^2 = x+4$ meets the y-axis at A and B. Calculate the area of the region bounded by the curve and the y-axis.



$$(y-1)^2 = x+4$$
$$x = (y-1)^2 - 4$$

At $x = 0$, $(y-1)^2 = x+4$
$$(y-1)^2 = 4$$
$$y-1 = 2 \text{ or } -2$$
$$y = 3 \text{ or } -1$$

$\therefore$ A is $(0,3)$, B is $(0,-1)$

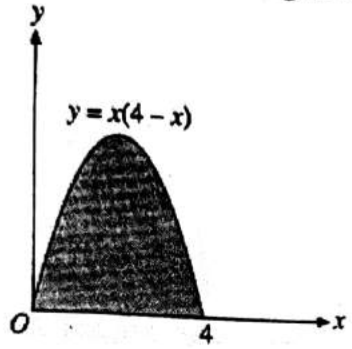
$$\text{Area} = \left|\int_{-1}^{3} x\,dy\right|$$
$$= \left|\int_{-1}^{3} (y-1)^2 - 4\,dy\right| \quad \leftarrow \boxed{\text{or } \int_{3}^{-1} (y-1)^2 - 4\,dy}$$
$$= \left|\int_{-1}^{3} y^2 - 2y - 3\,dy\right|$$
$$= \left|\left[\frac{y^3}{3} - y^2 - 3y\right]_{-1}^{3}\right|$$
$$= \left|(9-9-9) - \left(-\frac{1}{3} - 1 + 3\right)\right|$$
$$= \left|-\frac{32}{3}\right|$$
$$= 10\frac{2}{3} \text{ units}^2 \quad \text{(Ans)}$$

Application of Integration

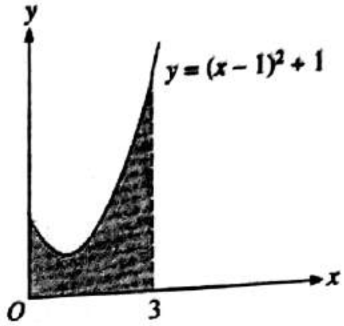
Exercise #1

1. For each of the following, find the shaded area.

(a)

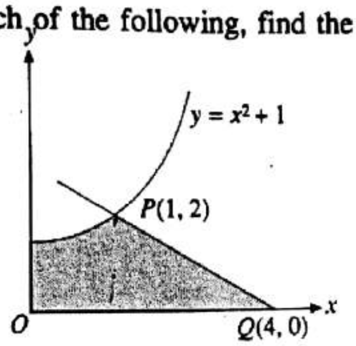


(b)

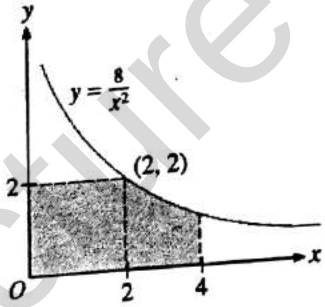


2. For each of the following, find the shaded area.

(a)



(b)



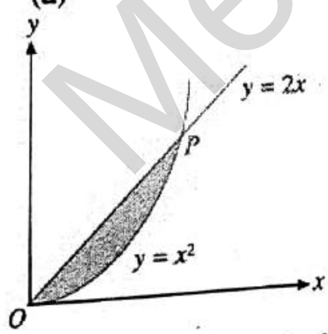
3. Find the area bounded by the following:

(a) $y = x^3$; $x = 2$, $x = 3$, x-axis

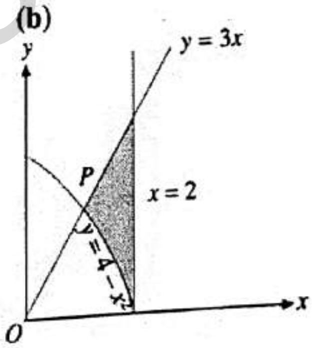
(b) $y = \frac{4}{x^2}$; $x = 1$, $x = 3$, x-axis.

4 For each of the following, find the x-coordinate of the point P and calculate the area of the shaded region.

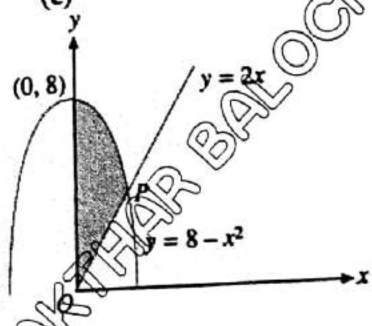
(a)



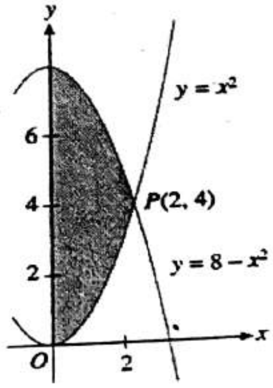
(b)



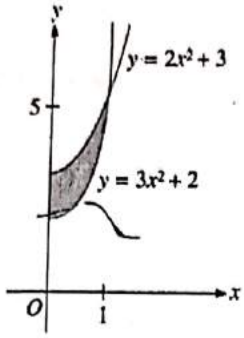
(c)



5 The diagram shows a shaded region bounded by the curves $y = x^2$, $y = 8 - x^2$ and the y-axis. Find the area of the shaded region.



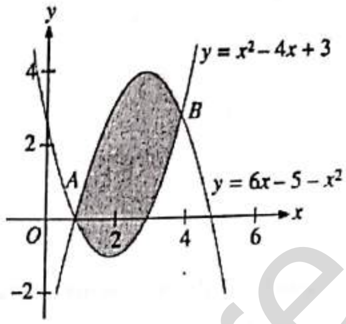
6 The diagram shows part of the curves $y = 2x^2 + 3$ and $y = 3x^2 + 2$, intersecting at (1, 5). Find the area of the shaded region.



7 The diagram shows part of the curves $y = x^2 - 4x + 3$ and $y = 6x - 5 - x^2$, intersecting at the points A and B. Calculate

(a) the coordinates of A and B,

(b) the area of the shaded region.



Answers

1. (a) $10\frac{2}{3}$ sq. units (b) 6 sq. units

2. (a) $4\frac{1}{3}$ sq. units (b) 6 sq. units

3. (a) $16\frac{1}{4}$ sq. units (b) $2\frac{2}{3}$ sq. units

4 (a) 2, $\frac{4}{3}$ sq. units (b) 1, $\frac{17}{6}$ sq. units (c) 2, $9\frac{1}{3}$ sq. units

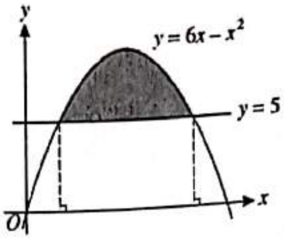
5 $10\frac{2}{3}$ sq. units

6 $\frac{2}{3}$ sq. units

7 (a) $A(1, 0)$, $B(4, 3)$ (b) 9 sq. units

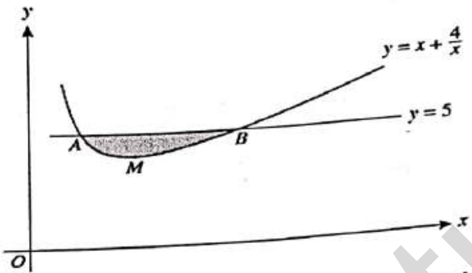
Exercise #2

1



The diagram shows the curve $y = 6x - x^2$ and the line $y = 5$. Find the area of the shaded region. [6]

2



The diagram shows part of the curve $y = x + \dfrac{4}{x}$ which has a minimum point at M. The line $y = 5$ intersects the curve at the points A and B.

(i) Find the coordinates of A, B and M. [5]

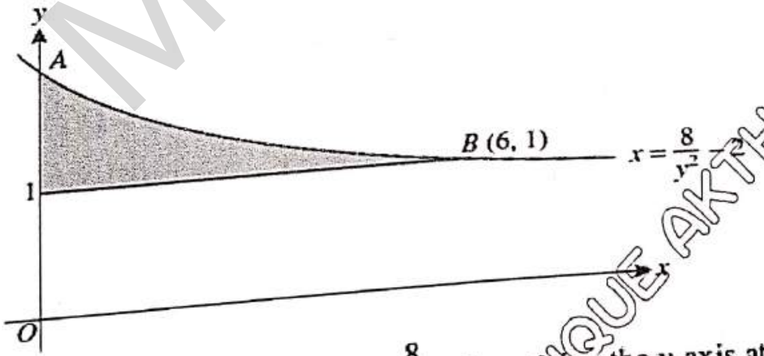
(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [6]

3 (i) Sketch the curve $y = (x - 2)^2$. [1]

(ii) The region enclosed by the curve, the x-axis and the y-axis is rotated through 360° about the x-axis. Find the volume obtained, giving your answer in terms of π. [4]

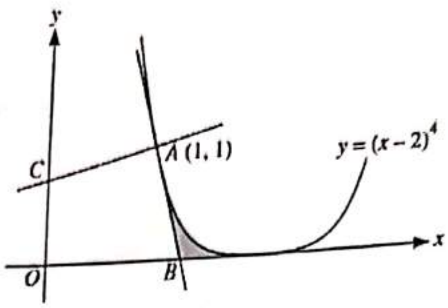
4 Find $\int (3x - 2)^5 \, dx$ and hence find the value of $\int_0^1 (3x - 2)^5 \, dx$. [4]

5



The diagram shows part of the curve $x = \dfrac{8}{y^2} - 2$, crossing the y-axis at the point A. The point B (6, 1) lies on the curve. The shaded region is bounded by the curve, the y-axis and the line $y = 1$. Find the exact volume obtained when this shaded region is rotated through 360° about the **y-axis**. [6]

6



The diagram shows part of the curve $y = (x-2)^4$ and the point $A\,(1,\ 1)$ on the curve. The tangent at A cuts the x-axis at B and the normal at A cuts the y-axis at C.

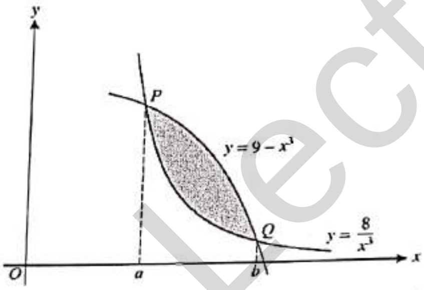
(i) Find the coordinates of B and C. [6]

(ii) Find the distance AC, giving your answer in the form $\frac{\sqrt{a}}{b}$, where a and b are integers. [2]

(iii) Find the area of the shaded region. [4]

7 Find $\int \left(x + \frac{1}{x}\right)^2 \mathrm{d}x$. [3]

8



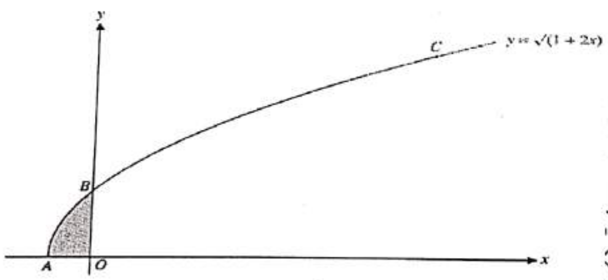
The diagram shows parts of the curves $y = 9 - x^3$ and $y = \frac{8}{x^3}$ and their points of intersection P and Q. The x-coordinates of P and Q are a and b respectively.

(i) Show that $x = a$ and $x = b$ are roots of the equation $x^6 - 9x^3 + 8 = 0$. Solve this equation and hence state the value of a and the value of b. [4]

(ii) Find the area of the shaded region between the two curves. [5]

(iii) The tangents to the two curves at $x = c$ (where $a < c < b$) are parallel to each other. Find the value of c. [4]

9



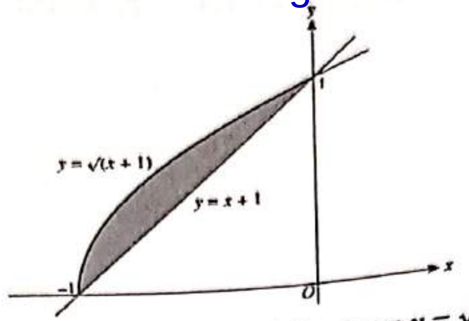
The diagram shows the curve $y = \sqrt{(1 + 2x)}$ meeting the x-axis at A and the y-axis at B. The y-coordinate of the point C on the curve is 3.

(i) Find the coordinates of B and C. [2]

(ii) Find the equation of the normal to the curve at C. [4]

(iii) Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [5]

10



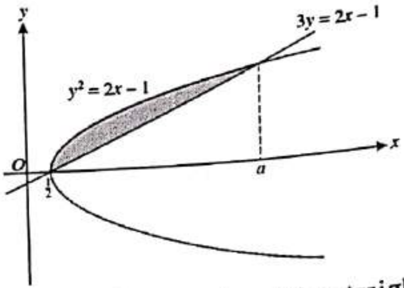
The diagram shows the line $y = x + 1$ and the curve $y = \sqrt{(x + 1)}$, meeting at $(-1, 0)$ and $(0, 1)$.

(i) Find the area of the shaded region. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [7]

11 A curve is such that $\frac{dy}{dx} = -\frac{8}{x^3} - 1$ and the point $(2, 4)$ lies on the curve. Find the equation of the curve. [4]

12

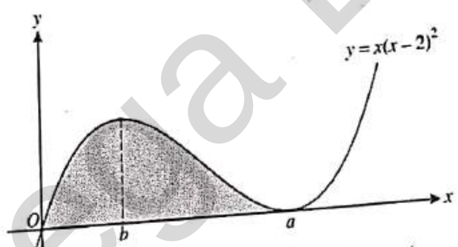


The diagram shows the curve $y^2 = 2x - 1$ and the straight line $3y = 2x - 1$. The curve and straight line intersect at $x = \frac{1}{2}$ and $x = a$, where a is a constant.

(i) Show that $a = 5$. [2]

(ii) Find, showing all necessary working, the area of the shaded region. [6]

13



The diagram shows the curve with equation $y = x(x - 2)^2$. The minimum point on the curve has coordinates $(a, 0)$ and the x-coordinate of the maximum point is b, where a and b are constants.

(i) State the value of a. [1]

(ii) Find the value of b. [4]

(iii) Find the area of the shaded region. [4]

(iv) The gradient, $\frac{dy}{dx}$, of the curve has a minimum value m. Find the value of m. [4]

14 A curve has equation $y = f(x)$. It is given that $f'(x) = \frac{1}{\sqrt{(x + 6)}} + \frac{6}{x^2}$ and that $f(3) = 1$. Find $f(x)$. [5]

15 A curve has equation $y = f(x)$. It is given that $f'(x) = x^{-\frac{3}{2}} + 1$ and that $f(4) = 5$. Find $f(x)$. [4]

Answers

1 Shaded area = 10⅔

2 A (1, 5), B(4, 5)

when $x = 2$, $M(2, 4)$.

$= 18\pi$

3 $\frac{32\pi}{5}$ or 6.4π

4 $= \frac{(3x-2)^6}{6} \div 3\ (+c)$

$-3\frac{1}{2}$

5 $x^2 = \frac{64}{y^4} - \frac{32}{y^2} + 4$

$6\frac{2}{3}\pi$

6 $\rightarrow B\left(\frac{5}{4}, 0\right) \rightarrow C\left(0, \frac{3}{4}\right)$

$\frac{\sqrt{17}}{4}$

$\frac{3}{40}$ or 0.075

7 $\frac{x^3}{3} - \frac{1}{x} + 2x + (c)$

8 $a = 1, b = 2$

$2\frac{1}{4}$

$c = \sqrt{2}$

9 (i) $B = (0,1)$ $C = (4,3)$

$y = -3x + 15$

$\frac{2}{15}\pi$

10 $\frac{1}{6}$

$V_1 = \frac{8}{15(\pi)}$ or $0.533(\pi)$

11 $y = \frac{4}{x^2} - x\ \ (+c)$

Sub $(2, 4) \rightarrow c = 5$

12 $\frac{9}{4}$

13 (i) $a = 2$

$b = \frac{2}{3}$

$\frac{4}{3}$

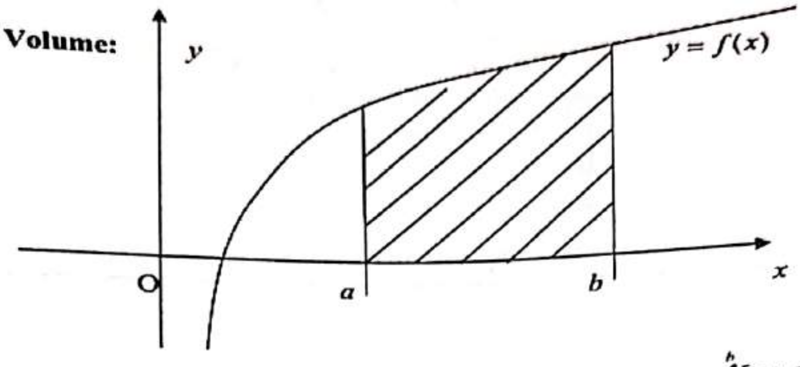
$= -\frac{4}{3}$

14 $f(x) = 2(x+6)^{\frac{1}{2}} - \frac{6}{x}\ (-3$

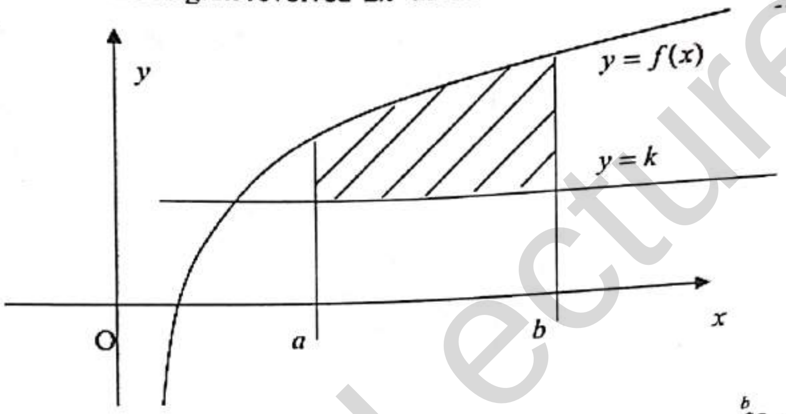
15 $f(x) = 2x^{-\frac{1}{2}} + x\ (+ 2$

Volume of Revolution

2. Volume:



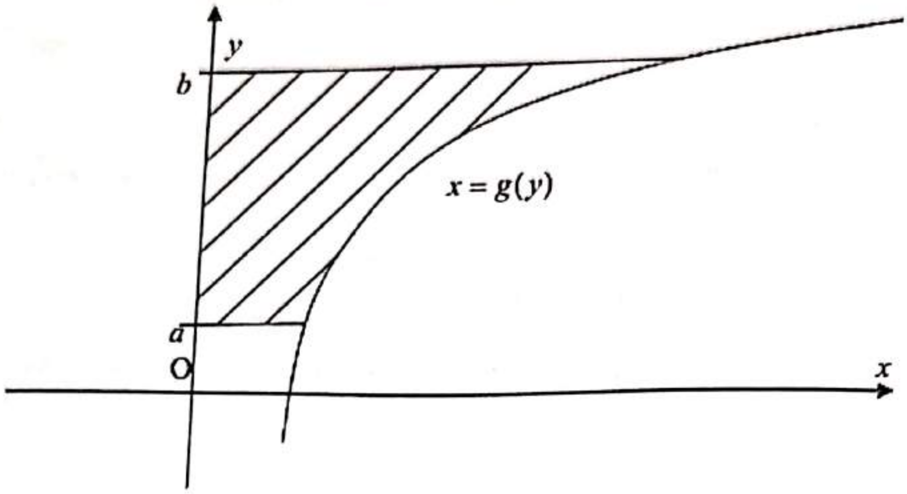
Volume of shaded region revolved 2π about the $x-$axis $=\pi\int_a^b [f(x)]^2\,dx$



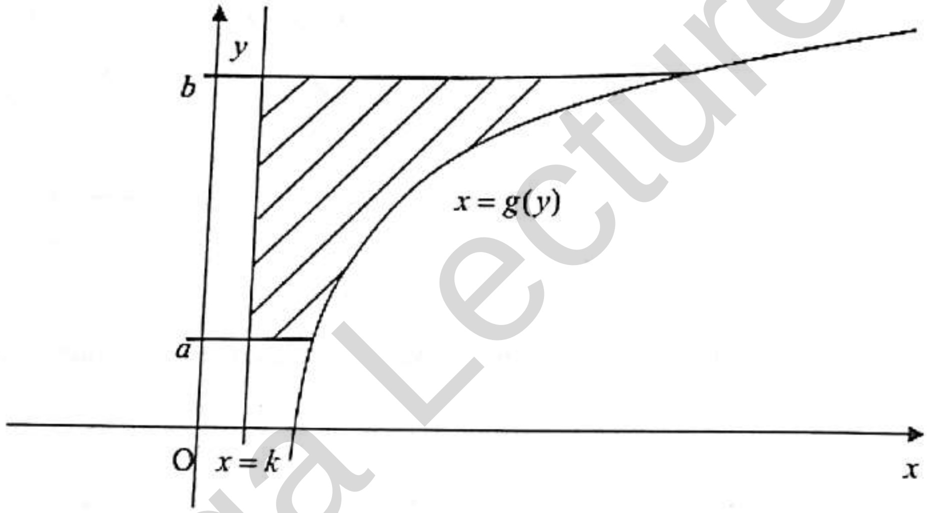
Volume of shaded region revolved 2π about the $y = k$ line $=\pi\int_a^b [f(x)-k]^2\,dx$

Volume of shaded region revolved 2π about the $x-$axis

$$=\pi\int_a^b [f(x)]^2\,dx-\pi(k^2)(b-a)$$



Volume of shaded region revolved 2π about the $y-$axis $=\pi\int_a^b [g(y)]^2 \, dy$



Volume of shaded region revolved 2π about the $x = k$ line $=\pi\int_a^b [g(y)-k]^2 \, dy$

Volume of shaded region revolved 2π about the $y-$axis

$=\pi\int_a^b [g(y)]^2 \, dy - \pi(k^2)(b-a)$

When the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ (where $a < b$) is rotated about the x-axis, the volume of the solid of revolution formed is

$$\int_a^b \pi(f(x))^2\, dx, \quad \text{or} \quad \int_a^b \pi y^2\, dx.$$

When the region bounded by the graph of $y = f(x)$, the lines $y = c$ and $y = d$ and the y-axis is rotated about the y-axis, the volume of the solid of revolution formed is

$$\int_c^d \pi x^2\, dy.$$

Remember that the limits in the integral are limits for y, not for x.

Example

Find the volume generated when the region under the graph of $y = 1 + x^2$ between $x = -1$ and $x = 1$ is rotated through four right angles about the x-axis.

The phrase 'four right angles' is sometimes used in place of 360° for describing a full rotation about the x-axis.

The required volume is V, where

$$V = \int_{-1}^{1} \pi y^2\, dx = \int_{-1}^{1} \pi(1 + x^2)^2\, dx = \int_{-1}^{1} \pi(1 + 2x^2 + x^4)\, dx$$
$$= \left[\pi\left(x + \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right)\right]_{-1}^{1}$$
$$= \pi\left\{\left(1 + \tfrac{2}{3} + \tfrac{1}{5}\right) - \left((-1) + \tfrac{2}{3}(-1)^3 + \tfrac{1}{5}(-1)^5\right)\right\} = \tfrac{56}{15}\pi.$$

The volume of the solid is $\frac{56}{15}\pi$.

It is usual to give the result as an exact multiple of π, unless you are asked for an answer correct to a given number of significant figures or decimal places.

Example

Find the volume generated when the region bounded by $y = x^3$ and the y-axis between $y = 1$ and $y = 8$ is rotated through 360° about the y-axis.

Since the volume is given by $\int_1^8 \pi x^2\, dy$, you need to express x^2 in terms of y.

The equation $y = x^3$ can be inverted to give $x = y^{\frac{1}{3}}$, so that $x^2 = y^{\frac{2}{3}}$. Then

$$V = \int_1^8 \pi y^{\frac{2}{3}}\, dy = \pi\left[\tfrac{3}{5}y^{\frac{5}{3}}\right]_1^8 = \pi\left(\tfrac{3}{5} \times 8^{\frac{5}{3}}\right) - \pi\left(\tfrac{3}{5} \times 1^{\frac{5}{3}}\right)$$
$$= \pi\left(\tfrac{3}{5} \times 32\right) - \pi\left(\tfrac{3}{5} \times 1\right) = \tfrac{93}{5}\pi.$$

The required volume is $\frac{93}{5}\pi$.

Practice Questions

Exercise #3

1 In each case the region enclosed between the following curves and the x-axis is rotated through $360°$ about the x-axis. Find the volume of the solid generated.

(a) $y=(x+1)(x-3)$ (b) $y=1-x^2$

(c) $y=x^2-5x+6$ (d) $y=x^2-3x$

2 The region enclosed between the graphs of $y=x$ and $y=x^2$ is denoted by R. Find the volume generated when R is rotated through $360°$ about

(a) the x-axis, (b) the y-axis.

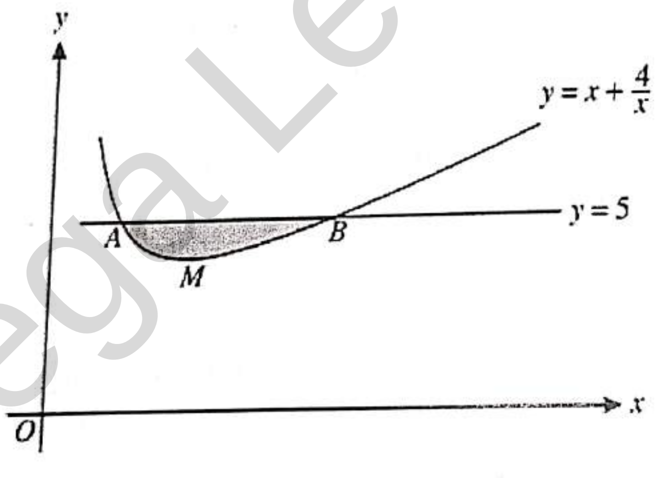
3 The region enclosed between the graphs of $y=4x$ and $y=x^2$ is denoted by R. Find the volume generated when R is rotated through $360°$ about

(a) the x-axis, (b) the y-axis.

4 The region enclosed between the graphs of $y=\sqrt{x}$ and $y=x^2$ is denoted by R. Find the volume generated when R is rotated through $360°$ about

(a) the x-axis, (b) the y-axis.

5



The diagram shows part of the curve $y = x + \frac{4}{x}$ which has a minimum point at M. The line $y = 5$ intersects the curve at the points A and B.

(i) Find the coordinates of A, B and M. [5]

(ii) Find the volume obtained when the shaded region is rotated through $360°$ about the x-axis. [6]

Answers

1 (a) $\frac{512}{15}\pi$ (b) $\frac{16}{15}\pi$

(c) $\frac{1}{30}\pi$ (d) $\frac{81}{10}\pi$

2 (a) $\frac{2}{15}\pi$ (b) $\frac{1}{6}\pi$

3 (a) $\frac{2048}{15}\pi$ (b) $\frac{128}{3}\pi$

4 (a) $\frac{3}{10}\pi$ (b) $\frac{3}{10}\pi$

5

$y = x + \frac{4}{x}$

(i) $x + \frac{4}{x} = 5 \rightarrow A(1, 5), B(4, 5)$

$\frac{dy}{dx} = 1 - \frac{4}{x^2}$

$= 0$ when $x = 2$, $M(2, 4)$.

(ii) Vol of cylinder $= \pi 5^2.3$

Vol under curve $= \pi \int y^2 \, dx$

Integral $= \frac{x^3}{3} - \frac{16}{x} + 8x$

Uses his limits "1 to 4"

$\rightarrow 75\pi - 57\pi = 18\pi$

Past Paper Questions

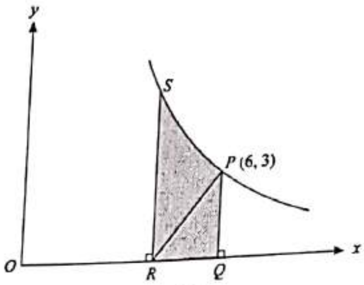
Integration

1 (a) Differentiate $4x + \frac{6}{x^2}$ with respect to x. [2]

(b) Find $\int \left(4x + \frac{6}{x^2}\right) dx$. [3]

2 Evaluate $\int_0^1 \sqrt{(3x + 1)}\, dx$. [4]

3



The diagram shows part of the graph of $y = \frac{18}{x}$ and the normal to the curve at $P(6, 3)$. This normal meets the x-axis at R. The point Q on the x-axis and the point S on the curve are such that PQ and SR are parallel to the y-axis.

(i) Find the equation of the normal at P and show that R is the point $(4\frac{1}{2}, 0)$. [5]

(ii) Show that the volume of the solid obtained when the shaded region $PQRS$ is rotated through 360° about the x-axis is 18π. [4]

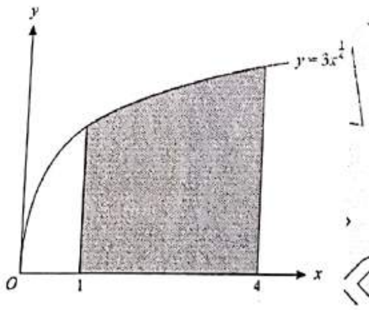
4 A curve is such that $\frac{dy}{dx} = 2x^2 - 5$. Given that the point (3, 8) lies on the curve, find the equation of the curve. [4]

5 A curve is such that $\frac{dy}{dx} = \frac{4}{\sqrt{(6 - 2x)}}$, and $P(1, 8)$ is a point on the curve.

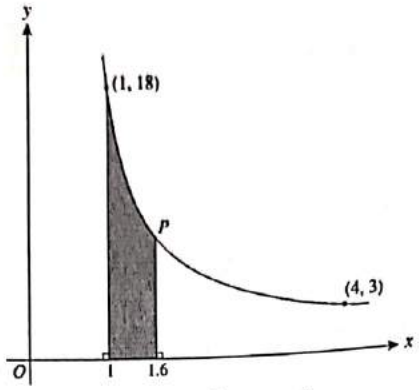
(i) The normal to the curve at the point P meets the coordinate axes at Q and at R. Find the coordinates of the mid-point of QR. [5]

(ii) Find the equation of the curve. [4]

6



The diagram shows the curve $y = 3x^{\frac{1}{4}}$. The shaded region is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$. Find the volume of the solid obtained when this shaded region is rotated completely about the x-axis, giving your answer in terms of π. [4]

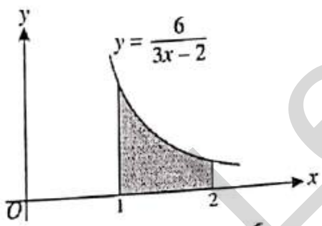
7 

The diagram shows a curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where k is a constant. The curve passes through the points (1, 18) and (4, 3).

(i) Show, by integration, that the equation of the curve is $y = \frac{16}{x^2} + 2$. [4]

The point P lies on the curve and has x-coordinate 1.6.

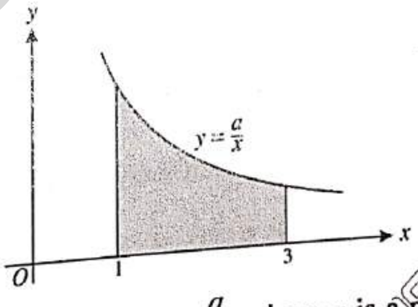
(ii) Find the area of the shaded region. [4]

8 

The diagram shows part of the curve $y = \frac{6}{3x-2}$.

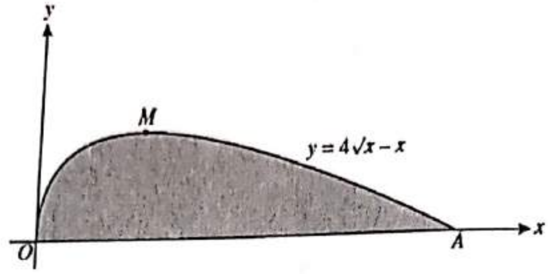
(i) Find the gradient of the curve at the point where $x = 2$. [3]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis, giving your answer in terms of π. [5]

9 

The diagram shows part of the curve $y = \frac{a}{x}$, where a is a positive constant. Given that the volume obtained when the shaded region is rotated through 360° about the x-axis is 24π, find the value of a. [4]

10

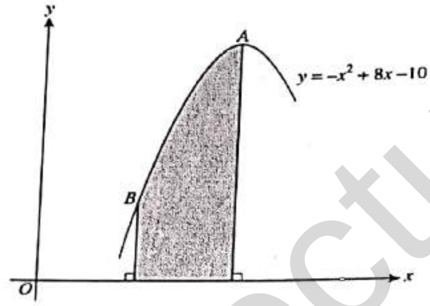


The diagram shows part of the curve $y = 4\sqrt{x} - x$. The curve has a maximum point at M and meets the x-axis at O and A.

(i) Find the coordinates of A and M. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis, giving your answer in terms of π. [6]

11



The diagram shows part of the curve $y = -x^2 + 8x - 10$ which passes through the points A and B. The curve has a maximum point at A and the gradient of the line BA is 2.

(i) Find the coordinates of A and B. [7]

(ii) Find $\int y\,dx$ and hence evaluate the area of the shaded region. [4]

12 The volume of a solid circular cylinder of radius r cm is 250π cm^3.

(i) Show that the total surface area, S cm^2, of the cylinder is given by

$$S = 2\pi r^2 + \frac{500\pi}{r}.$$ [2]

(ii) Given that r can vary, find the stationary value of S. [4]

(iii) Determine the nature of this stationary value. [2]

13 The equation of a curve is such that $\frac{d^2y}{dx^2} = 2x - 1$. Given that the curve has a minimum point at $(3, -10)$, find the coordinates of the maximum point. [8]

14 The equation of a curve is $y = \frac{4}{2x - 1}$.

(i) Find, showing all necessary working, the volume obtained when the region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis. [4]

(ii) Given that the line $2y = x + c$ is a normal to the curve, find the possible values of the constant c. [6]

15 A curve is such that $\frac{dy}{dx} = \frac{8}{(5-2x)^2}$. Given that the curve passes through (2, 7), find the equation of the curve. [4]

16 The gradient at any point (x, y) on a curve is $\sqrt{(1+2x)}$. The curve passes through the point (4, 11). Find

(i) the equation of the curve, [4]

(ii) the point at which the curve intersects the y-axis. [2]

17 A curve is such that $\frac{dy}{dx} = 3x^2 - 4x + 1$. The curve passes through the point (1, 5).

(i) Find the equation of the curve. [3]

(ii) Find the set of values of x for which the gradient of the curve is positive. [3]

18 A curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(4x-3)}}$ and P (3, 3) is a point on the curve.

(i) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by = c$. [3]

(ii) Find the equation of the curve. [4]

19 A curve is such that $\frac{dy}{dx} = \frac{16}{x^3}$, and (1, 4) is a point on the curve.

(i) Find the equation of the curve. [4]

(ii) A line with gradient $-\frac{1}{2}$ is a normal to the curve. Find the equation of this normal, giving your answer in the form $ax + by = c$. [4]

(iii) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$. [4]

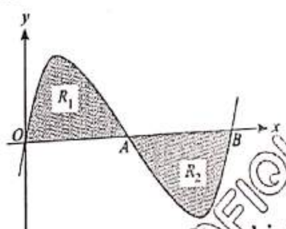
19 A curve is such that $\frac{dy}{dx} = \frac{16}{x^3}$, and (1, 4) is a point on the curve.

(i) Find the equation of the curve. [4]

(ii) A line with gradient $-\frac{1}{2}$ is a normal to the curve. Find the equation of this normal, giving your answer in the form $ax + by = c$. [4]

(iii) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$. [4]

20



The diagram shows the curve $y = x(x-1)(x-2)$, which crosses the x-axis at the points O (0, 0), A (1, 0) and B (2, 0).

(i) The tangents to the curve at the points A and B meet at the point C. Find the x-coordinate of C. [5]

(ii) Show by integration that the area of the shaded region R_1 is the same as the area of the shaded region R_2. [4]

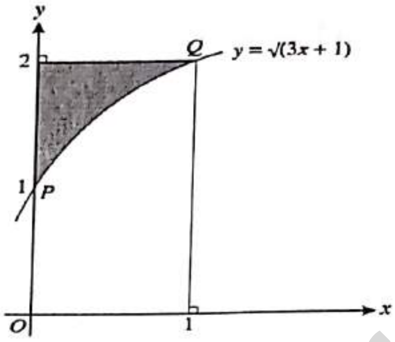
21 A curve is such that $\frac{dy}{dx} = 4 - x$ and the point $P(2, 9)$ lies on the curve. The normal to the curve at P meets the curve again at Q. Find

(i) the equation of the curve, [3]

(ii) the equation of the normal to the curve at P, [3]

(iii) the coordinates of Q. [3]

22



The diagram shows the curve $y = \sqrt{(3x + 1)}$ and the points $P(0, 1)$ and $Q(1, 2)$ on the curve. The shaded region is bounded by the curve, the y-axis and the line $y = 2$.

(i) Find the area of the shaded region. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Tangents are drawn to the curve at the points P and Q.

(iii) Find the acute angle, in degrees correct to 1 decimal place, between the two tangents. [4]

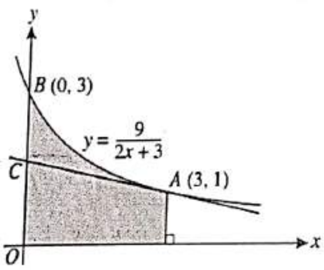
23 The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$. Given that the curve passes through the point $(4, 6)$, find the equation of the curve. [4]

24 A curve is such that $\frac{dy}{dx} = 5 - \frac{8}{x^2}$. The line $3y + x = 17$ is the normal to the curve at the point P on the curve. Given that the x-coordinate of P is positive, find

(i) the coordinates of P, [4]

(ii) the equation of the curve. [4]

25



The diagram shows part of the curve $y = \frac{9}{2x + 3}$, crossing the y-axis at the point $B(0, 3)$. The point A on the curve has coordinates $(3, 1)$ and the tangent to the curve at A crosses the y-axis at C.

(i) Find the equation of the tangent to the curve at A. [4]

(ii) Determine, showing all necessary working, whether C is nearer to B or to O. [1]

(iii) Find, showing all necessary working, the exact volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Answers

1 (a) $4-\frac{12}{c^3}$ (b) $2x^2-\frac{6}{x}+K$

2 $1\frac{5}{9}$

3 (i) $y=2x-9$

4 $y=\frac{2}{3}x^3-5x+5$

5 (i) $\left(\frac{17}{2}, \frac{17}{4}\right)$ (ii) $y=16-4\sqrt{6-2x}$

6 42π cube units

7 (ii) 7.2 units^2

8 (i) $-\frac{9}{8}$ (ii) 9π unit^3

9 $a=6$

10 (i) A are (16, 0) M are (4, 4) (ii) $136\frac{8}{15}\pi$

11 (i) $A(4, 6)B(2, 2)$ (ii) $9\frac{1}{3}$ sq.units

12 (ii) 471 cm^2 (ii) stationary value is a minimum

13 $(-2, 10\frac{5}{6})$

14 (i) $\frac{16}{3}\pi$ units^3 (ii) $c=\frac{5}{2}$ or $-\frac{7}{2}$

15 $y=\frac{4}{(5-2x)}+3$

16 (i) $y=\frac{1}{3}(1+2x)^{\frac{3}{2}}+2$ (ii) $\left(0, \frac{7}{3}\right)$

17 (i) $y=x^3-2x^2+x+5$ (ii) $x<\frac{1}{3}$ or $x>1$

18 (i) $2y+x=9$ (ii) $y=3\sqrt{(4x-3)}-6$

19 (i) $y=-\frac{8}{x^2}+12$ (ii) $x+2y=22$ (iii) 8 sq. units

20 (i) x-coordinate of C is $\frac{5}{3}$ (ii) $\frac{1}{4}$

21 (i) $y=4x-\frac{x^2}{2}+3$ (ii) $2y+x=20$ (iii) (7, 6.5)

22 (i) $\frac{4}{9}$ (ii) $\frac{3}{2}\pi$ units^3 (iii) $\theta=19.4°$

23 $y=6\sqrt{x}-\frac{1}{2}x^2+2$

24 (i) (2, 5) (ii) $y=5x+\frac{8}{x}-9$

25 (i) $2x+9y=15$ (ii) C is nearer to B. (iii) 9π units^3