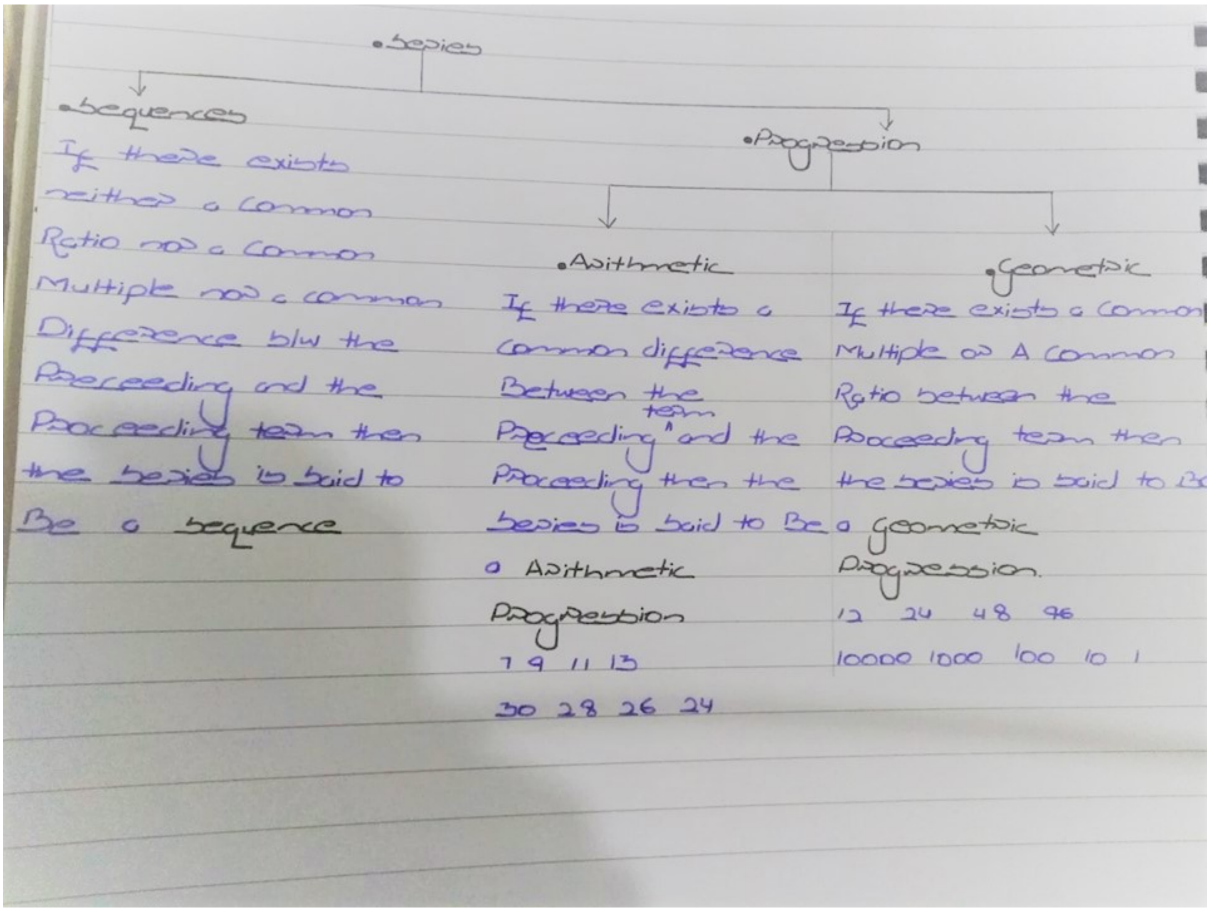
- recognise arithmetic and geometric progressions
- use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions

  Including knowledge that numbers a, b, c are 'in arithmetic progression' if $2b = a + c$ (or equivalent) and are 'in geometric progression' if $b^2 = ac$ (or equivalent).

  Questions may involve more than one progression.
- use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

Video lectures

https://www.youtube.com/watch?v=vlPHnGbKi9w&t=9s



The series of a sequence is the sum of the sequence to a certain number of terms. It is often written as Sn. So if the sequence is 2, 4, 6, 8, 10, ... , the sum to 3 terms = S3 = 2 + 4 + 6 = 12.

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Arithmetic Progressions

An arithmetic progression is a sequence where each term is a certain number larger than the previous term. The terms in the sequence are said to increase by a common difference, d.

For example: 3, 5, 7, 9, 11, is an arithmetic progression where d = 2. The nth term of this sequence is 2n + 1 .

In general, the nth term of an arithmetic progression, with first term a and common difference d, is: a + (n - 1)d . So for the sequence 3, 5, 7, 9, ... Un = 3 + 2(n - 1) = 2n + 1, which we already knew.

The sum to n terms of an arithmetic progression

This is given by: $S_n = \frac{1}{2} n [2a + (n - 1)d]$

You may need to be able to prove this formula. It is derived as follows:

The sum to n terms is given by: $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$ (1)

If we write this out backwards, we get: $S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots + a$ (2)

Now let's add (1) and (2): $2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d]$ So $S_n = \frac{1}{2} n [2a + (n - 1)d]$

Example Sum the first 20 terms of the sequence: 1, 3, 5, 7, 9, ... (i.e. the first 20 odd numbers).

$S_{20} = \frac{1}{2} (20) [2 \times 1 + (20 - 1)\times 2] = 10[2 + 19 \times 2] = 10[40] = 400$

Geometric Progressions

A geometric progression is a sequence where each term is r times larger than the previous term. r is known as the common ratio of the sequence. The nth term of a geometric progression, where a is the first term and r is the common ratio, is:

ar^{n-1} For example, in the following geometric progression, the first term is 1, and the common ratio is 2: 1, 2, 4, 8, 16, .

The nth term is therefore 2^{n-1}

The sum of a geometric progression

The sum of the first n terms of a geometric progression is:

$a(1 - r^n)/ 1 - r$ We can prove this as follows:

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ (1)

Multiplying by r: $rS_n = ar + ar^2 + \ldots + ar^n$ (2)

(1) – (2) gives us: $S_n(1 - r) = a - ar^n$ (since all the other terms cancel)

And so we get the formula above if we divide through by $1 - r$.

Example What is the sum of the first 5 terms of the following geometric progression: 2, 4, 8, 16, 32 ? $S_5 = 2(1 - 2^5) 1 - 2 = 2(1 - 32) -1 = 62$

The sum to infinity of a geometric progression

In geometric progressions where $|r| < 1$ (in other words where r is less than 1 and greater than –1), the sum of the sequence as n tends to infinity approaches a value. In other words, if you keep adding together the terms of the sequence forever, you will get a finite value. This value is equal to: a /1 – r

Example Find the sum to infinity of the following sequence:

1,1,1, 1,1,1,1,.... 2 4 8 16 32 64 Here, a = 1/2 and r = ½

Therefore, the sum to infinity is 0.5/0.5 = 1

So every time you add another term to the above sequence, the result gets closer and closer to 1.

Example The first, second and fifth terms of an arithmetic progression are the first three terms of a geometric progression. The third term of the arithmetic progression is 5. Find the 2 possible values for the fourth term of the geometric progression.

The first term of the arithmetic progression is: a The second term is: a + d The fifth term is: a + 4d So the first three terms of the geometric progression are a, a + d and a + 4d .

In a geometric progression, there is a common ratio. So the ratio of the second term to the first term is equal to the ratio of the third term to the second term.So: a + d/a = a + 4d/a + d

$(a + d)(a + d) = a(a + 4d)$ $a^2 + 2ad + d^2 = a^2 + 4ad$ $d^2 - 2ad = 0$ $d(d - 2a) = 0$ therefore d = 0 or d = 2a

The common ratio of the geometric progression, r, is equal to (a + d)/a Therefore, if d = 0, r = 1 If d = 2a, r = 3a/a = 3 So the common ratio of the geometric progression is either 1 or 3 .

We are told that the third term of the arithmetic progression is 5. So a + 2d = 5 . Therefore, when d = 0, a = 5 and when d = 2a, a = 1 . So the first term of the arithmetic progression (which is equal to the first term of the geometric progression) is either 5 or 1.

Therefore, when d = 0, a = 5 and r = 1. In this case, the geometric progression is 5, 5, 5, 5, ... and so the fourth term is 5.When d = 2a, r = 3 and a = 1, so the geometric progression is 1, 3, 9, 27, ... and so the fourth term is 27.

Questions from past papers

1: (a) The third and fourth terms of a geometric progression are 48 and 32 respectively. Find the sum to infinity of the progression. [3]
(b) Two schemes are proposed for increasing the amount of household waste that is recycled each week.
Scheme *A* is to increase the amount of waste recycled each month by 0.16 tonnes.
Scheme *B* is to increase the amount of waste recycled each month by 6% of the amount recycled in the previous month.
The proposal is to operate the scheme for a period of 24 months. The amount recycled in the
first month is 2.5 tonnes.
For each scheme, find the total amount of waste that would be recycled over the 24-month period.
[5]

2(a) In an arithmetic progression, the sum of the first ten terms is equal to the sum of the next five
terms. The first term is a.
(i) Show that the common difference of the progression is 1/3 a. [4]
(ii) Given that the tenth term is 36 more than the fourth term, find the value of a. [2]
(b) The sum to infinity of a geometric progression is 9 times the sum of the first four terms.
Given that the first term is 12, find the value of the fifth term. [4]

3(a) Two heavyweight boxers decide that they would be more successful if they competed in a lower
weight class. For each boxer this would require a total weight loss of 13 kg. At the end of week 1
they have each recorded a weight loss of 1 kg and they both find that in each of the following weeks
their weight loss is slightly less than the week before.
Boxer *A*'s weight loss in week 2 is 0.98 kg. It is given that his weekly weight loss follows an arithmetic
progression.
(i) Write down an expression for his total weight loss after x weeks. [1]
(ii) He reaches his 13 kg target during week n. Use your answer to part **(i)** to find the value of n. [2]
Boxer *B*'s weight loss in week 2 is 0.92 kg and it is given that his weekly weight loss follows a geometric progression.
(iii) Calculate his total weight loss after 20 weeks and show that he can never reach his target. [4]

4**(a) (i)** The first and second terms of a geometric progression are p and $2p$ respectively, where p is a

positive constant. The sum of the first n terms is greater than 1000p. Show that $2^n > 1001$. [2]
(ii) In another case, p and $2p$ are the first and second terms respectively of an arithmetic progression.
The nth term is 336 and the sum of the first n terms is 7224. Write down two equations in n and p and hence find the values of n and p. [5]

5 The first term of a series is 6 and the second term is 2.
(i) For the case where the series is an arithmetic progression, find the sum of the first 80 terms. [3]
(ii) For the case where the series is a geometric progression, find the sum to infinity. [2]

6(I) The first three terms of an arithmetic progression are 4, x and y respectively. The first three terms of
a geometric progression are x, y and 18 respectively. It is given that both x and y are positive.
(i) Find the value of x and the value of y. [4]
(ii) Find the fourth term of each progression. [3]

7In an arithmetic progression the first term is a and the common difference is 3. The nth term is 94
and the sum of the first n terms is 1420. Find n and a. [6]

8The first term of a geometric progression is 81 and the fourth term is 24. Find
(i) the common ratio of the progression, [2]
(ii) the sum to infinity of the progression. [2]
The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.
(iii) Find the sum of the first ten terms of the arithmetic progression. [3]

9The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has n terms and the sum of all the terms is 90. Find the value of n. [4]

10An arithmetic progression has first term a and common difference d. It is given that the sum of the first 200 terms is 4 times the sum of the first 100 terms.
(i) Find d in terms of a. [3]
(ii) Find the 100th term in terms of a. [2]

11(a) The first and last terms of an arithmetic progression are 12 and 48 respectively. The sum of the first four terms is 57. Find the number of terms in the progression. [4]
(b) The third term of a geometric progression is four times the first term. The sum of the first six terms is k times the first term. Find the possible values of k. [4]

12(a) In a geometric progression, the sum to infinity is equal to eight times the first term. Find the common ratio. [2]
(b) In an arithmetic progression, the fifth term is 197 and the sum of the first ten terms is 2040. Find the common difference. [4]

13(a) An athlete runs the first mile of a marathon in 5 minutes. His speed reduces in such a way that each mile takes 12 seconds longer than the preceding mile.
(i) Given that the nth mile takes 9 minutes, find the value of n. [2]
(ii) Assuming that the length of the marathon is 26 miles, find the total time, in hours and minutes, to complete the marathon. [2]

(b) The second and third terms of a geometric progression are 48 and 32 respectively. Find the sum to infinity of the progression. [4]

14(a) The third and fourth terms of a geometric progression are 1/3 and 2/9 respectively. Find the sum to
infinity of the progression. [4]**(b)** A circle is divided into 5 sectors in such a way that the angles of the sectors are in arithmeticprogression. Given that the angle of the largest sector is 4 times the angle of the smallest sector find the angle of the largest sector. [4]
15 A runner who is training for a long-distance race plans to run increasing distances each day for 21 days.
She will run x km on day 1, and on each subsequent day she will increase the distance by 10% of the previous day's distance. On day 21 she will run 20 km.
(i) Find the distance she must run on day 1 in order to achieve this. Give your answer in km correct to 1 decimal place. [3]
(ii) Find the total distance she runs over the 21 days. [2]

16 (a) Over a 21-day period an athlete prepares for a marathon by increasing the distance she runs each
day by 1.2 km. On the first day she runs 13 km.
(i) Find the distance she runs on the last day of the 21-day period. [1]

17 The first, second and third terms of a geometric progression are 3k, 5k – 6 and 6k – 4, respectively.
(i) Show that k satisfies the equation 7k^2 – 48k + 36 = 0. [2]
(ii) Find, showing all necessary working, the exact values of the common ratio corresponding to each of the possible values of k. [4]

18 (a) A geometric progression has a second term of 12 and a sum to infinity of 54. Find the possible
values of the first term of the progression. [4]
(b) The nth term of a progression is $p + qn$, where p and q are constants, and S_n is the sum of the
first n terms.
(i) Find an expression, in terms of p, q and n, for S_n. [3]
(ii) Given that $S_4 = 40$ and $S_6 = 72$, find the values of p and q. [2]

19 The ninth term of an arithmetic progression is 22 and the sum of the first 4 terms is 49.
(i) Find the first term of the progression and the common difference. [4]
The nth term of the progression is 46.
(ii) Find the value of n. [2]

20a) An athlete runs the first mile of a marathon in 5 minutes. His speed reduces in such a way that each mile takes 12 seconds longer than the preceding mile.
(i) Given that the nth mile takes 9 minutes, find the value of n. [2]
(ii) Assuming that the length of the marathon is 26 miles, find the total time, in hours and minutes, to complete the marathon. [2

21 A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression
in each of the following cases:

(i) the progression is arithmetic, [3]
(ii) the progression is geometric with a positive common ratio. [3]

22 The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has *n* terms
and the sum of all the terms is 90. Find the value of *n*. [4]

23 (a) Two convergent geometric progressions, *P* and *Q*, have the same sum to infinity. The first and second terms of *P* are 6 and 6*r* respectively. The first and second terms of *Q* are 12 and −12*r* respectively. Find the value of the common sum to infinity. [3]
(b) The first term of an arithmetic progression is cos 1 and the second term is cos 1 + sin2 1, where 0 ≤ 1 ≤ 0. The sum of the first 13 terms is 52. Find the possible values of 1. [5]

24 The sum of the 1st and 2nd terms of a geometric progression is 50 and the sum of the 2nd and 3rd terms is 30. Find the sum to infinity. [6]

25 The 1st, 3rd and 13th terms of an arithmetic progression are also the 1st, 2nd and 3rd terms respectively of a geometric progression. The first term of each progression is 3. Find the common difference of the arithmetic progression and the common ratio of the geometric progression. [5]

26 The 12th term of an arithmetic progression is 17 and the sum of the first 31 terms is 1023. Find the 31st term. [5]

A sequence of numbers is called a harmonic progression if the reciprocal of the terms are in AP.

In simple terms, a,b,c,d,e,f are in HP if 1/a, 1/b, 1/c, 1/d, 1/e, 1/f are in AP. For two terms ‘a’ and ‘b’, Harmonic Mean = (2 a b) / (a + b) For two numbers, if A, G and H are respectively the arithmetic, geometric and harmonic means, then A ≥ G ≥ H A H = G2, i.e., A, G, H are in GP Sample Problems

Question 1 : Find the nth term for the AP : 11, 17, 23, 29, … Solution : Here, a = 11, d = 17 – 11 = 23 – 17 = 29 – 23 = 6 We know that nth term of an AP is a + (n – 1) d => nth term for the given AP = 11 + (n – 1) 6 => nth term for the given AP = 5 + 6 n We can verify the answer by putting values of 'n'. => n = 1 -> First term = 5 + 6 = 11 => n = 2 -> Second term = 5 + 12 = 17 => n = 3 -> Third term = 5 + 18 = 23 and so on …

Question 2 : Find the sum of the AP in the above question till first 10 terms. Solution : From the above question, => nth term for the given AP = 5 + 6 n => First term = 5 + 6 = 11 => Tenth term = 5 + 60 = 65 => Sum of 10 terms of the AP = 0.5 n (first term + last term) = 0.5 x 10 (11 + 65) => Sum of 10 terms of the AP = 5 x 76 = 380

Question 3 : For the elements 4 and 6, verify that A ≥ G ≥ H. Solution : A = Arithmetic Mean = (4 + 6) / 2 = 5 G = Geometric Mean = \sqrt{{4}\times{6}} = 4.8989 H = Harmonic Mean = (2 x 4 x 6) / (4 + 6) = 48 / 10 = 4.8 Therefore, A ≥ G ≥ H

Question 4 : Find the sum of the series 32, 16, 8, 4, … upto infinity. Solution : First term, a = 32 Common ratio, r = 16 / 32 = 8 / 16 = 4 / 8 = 1 / 2 = 0.5 We know that for an infinite GP, Sum of terms = a / (1 – r) => Sum of terms of the GP = 32 / (1 – 0.5) = 32 / 0.5 = 64

Question 5 : The sum of three numbers in a GP is 26 and their product is 216. ind the numbers.
Solution :
Let the numbers be a/r, a, ar. => (a / r) + a + a r = 26 => a (1 + r + r2) / r = 26
Also, it is given that product = 216 => (a / r) x (a) x (a r) = 216 => a3 = 216 => a = 6 => 6 (1 + r + r2) / r = 26 => (1 + r + r2) / r = 26 / 6 = 13 / 3 => 3 + 3 r + 3 r2 = 13 r => 3 r2 – 10 r + 3 = 0 => (r – 3) (r – (1 / 3)) = 0 => r = 3 or r = 1 / 3 Thus, the required numbers are 2, 6 and 18