7 STRAIGHT LINES

Objectives

After studying this chapter you should

- be familiar with the equation of a straight line;
- understand what information is needed to define a straight line;
- appreciate the significance of the gradient of a straight line;
- be able to solve simple linear inequality problems;
- be able to find the distance between two points in the *xy*-plane and the coordinates of the midpoint.

7.0 Introduction

In solving problems, a great deal of effort is often made to find and use a function which 'models' the situation being studied. The easiest type of functions to use and work with are those whose graphs are straight lines. These are called **linear** functions.

You will see how linear models can be constructed and then used for predicting in the following activity. For this activity you will need:

spring, stand and set of weights.

Activity 1 Spring extensions

The extension of a spring held vertically depends on the weight fixed on the free end. Measure the extension, x, in cm, of the spring beyond its natural length for a variety of weights, w, in grams.

(a) Plot the data points on a graph of x against w.

(b) Draw a straight line as accurately as possible through the data points.

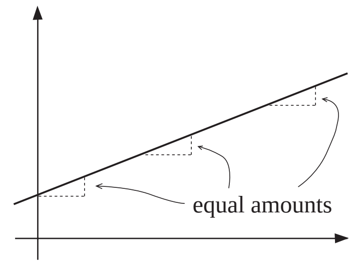
(c) Assuming that the straight line has an equation of the form

$$x = \alpha w$$

for some constant α, find a point on your line and use it to find the value of α.

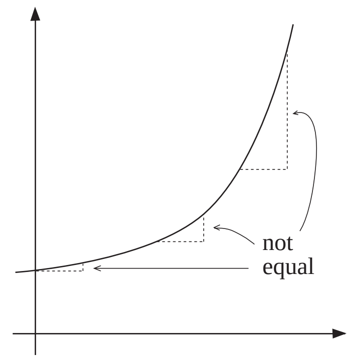
(d) Use your model to predict the extension for various weights; test your model by obtaining further experimental data.

7.1 Gradients



The figure shows a straight line - the graph increases its 'heights' by equal amounts for equal increases of the horizontal variable. This is because the slope or gradient is constant. Remember the gradient is defined as

$$\text{gradient} = \frac{\text{increase in } y}{\text{increase in } x}$$



When the graph is curved as shown in the figure opposite, then the gradient is no longer constant.

Example

A company offers a personal life insurance so that your life is insured against accidental death for

£250 000 for a payment of £50 per year

or

£100 000 for a payment of £25 per year.

If other rates are 'pro-rata' (in proportion) determine

(a) the amount of life insurance that can be obtained for a yearly payment of
(i) £10 (ii) £80

(b) the yearly payment for life insurance of amount
(i) £160 000 (ii) £550 000

Solution

Define

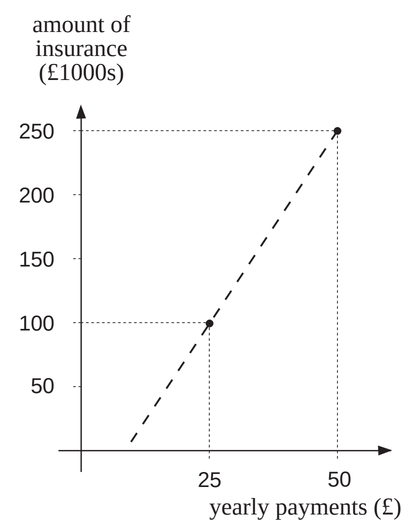
x = yearly payment in £'s

y = amount of insurance in £1000's

The two pieces of information can now be written as

$y = 250$ when $x = 50$,

$y = 100$ when $x = 25$.



These are illustrated on the graph opposite.

A graphical approach would be to draw the straight line between the two data points, extending in each direction beyond the points. The line could then be used to find corresponding values of yearly payment and amount of insurance.

Algebraically, the relationship between x and y can be written as

$$y = mx + c.$$

Since the two points, (25, 100) and (50, 250), satisfy this equation

$$250 = 50\,m + c \qquad (1)$$

$$100 = 25\,m + c. \qquad (2)$$

You can solve these equations for m and c by first subtracting (2) from (1).

$$250 - 100 = 50m + c - (25m + c)$$

$$= 50m - 25m + (c - c)$$

$$\Rightarrow \quad 150 = 25m$$

$$\Rightarrow \quad m = 6.$$

Substituting m in (1) now gives

$$c = 250 - 50 \times 6 = -50.$$

(You should now check that (2) is in fact satisfied by $m = 6$ and $c = -50$)

So the life insurance system can be modelled by the equation

$$\boxed{y = 6x - 50} \qquad (3)$$

(a) (i) $x = 10 \Rightarrow y = 6 \times 10 - 50 = 10$ or £10 000 life insurance.

(ii) $x = 80 \Rightarrow y = 6 \times 80 - 50 = 430$ or £430 000 life insurance.

(b) It is easier to first make x the subject of equation (3).

Now $6x = y + 50$

$$\Rightarrow \quad x = \frac{1}{6}(y + 50).$$

(i) $y = 160 \Rightarrow x = \frac{1}{6}(160 + 50) = 35$

(ii) $y = 550 \Rightarrow x = \frac{1}{6}(550 + 50) = 100.$

In the example above, it is worth noting that the value of m, namely 6, represents the gradient of the line. It shows that any increase of £1 in the yearly payment results in an extra insurance cover of £6000.

Can you suggest why $x \neq 0$ when $y = 0$?

The real bonus of this algebraic approach is that the model equation, $y = 6x - 50$, can be used to solve any problem related to this life insurance system.

Activity 2 Ski-passes

At a particular ski resort in Switzerland, ski-passes are advertised at the following two rates:

6 day pass for 38 Swiss Francs
13 day pass for 80 Swiss Francs.

Assuming that other days are charged at the pro-rata rate, find using algebraic methods a linear model to describe this situation. Use the model to find the cost of a ski pass for

(a) 3 days (b) 30 days.

Exercise 7A

1. The volume of 20 g of the metal Lithium was measured by a chemist and was found to be 37 cm^3. Use the fact that 0 g of Lithium must have a volume of 0 cm^3 to draw a graph of weight on the vertical axis against volume on the horizontal axis. Find the gradient of the graph, and hence state the density of Lithium in g cm^{-3}.

2. A car overtakes a lorry. At the start of the manoeuvre, the car is travelling at a speed of 13.4 ms^{-1} (30 mph). Five seconds later, after passing the lorry, it is travelling at a speed of 22 ms^{-1} (50 mph). Draw a graph of the speed, v, of the car against the time, t. Put v on the vertical axis, using metres per second, and t on the horizontal axis, using seconds. You may assume the graph is a straight line. Find the gradient of the graph, and so find the rate at which the speed of the car has increased, in metres per second per second.

3. The speed of a train pulling into a station decreases from 11.2 ms^{-1} to 0 ms^{-1} in 15 seconds. Draw a straight line graph showing this information, with the speed on the vertical axis. By finding the gradient of the line, find the rate at which the train decelerates, in ms^{-2}.

4. A petrol pump works at a rate of 20 litres per minute. Draw a graph showing the volume of petrol pumped on the vertical axis against the time taken in seconds on the horizontal axis. You should choose scales from 0 to 40 litres and from 0 to 2 minutes. Use your graph to find the time required to pump 35 litres.

5. Boats are hired at the following rates

   2 hours for £11 5 hours for £20

   Assuming charges for other times are pro-rata, develop a linear model to describe this relationship and use it to find the hire charges for

   (a) $1\frac{1}{2}$ hours (b) 3 hours (c) 12 hours.

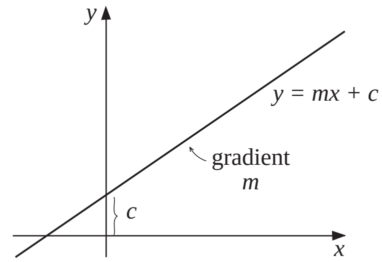
7.2 Equation of a straight line

One of the advantages that linear functions have is that their equations can all be written in a simple form. When the equation of a straight line is known, it is possible to find its gradient and the point it crosses the vertical axis immediately, without drawing. Also, it is possible to write down the equation of the line once the gradient and any single point on the line is known.

Remember that the equation of any straight line can be written in the form

$$\boxed{y = mx + c}$$

where m and c are constant.



This is illustrated opposite; m is the value of the gradient of the line, whilst c is the length of the intercept on the y-axis (since when $x = 0$, $y = m \times 0 + c = c$).

Example

A straight line with a gradient of -2 passes through the point $(4, \ -1)$. Find the equation of the line, and draw its graph.

Solution

As the graph is a straight line, its equation can be written in the form

$$y = mx + c.$$

As the gradient is -2, m in this equation must be -2, and

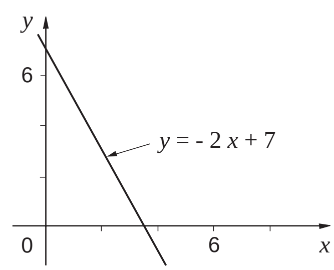
$$y = -2x + c.$$

The point $(4, -1)$ lies on this line, so its x and y coordinates must satisfy the equation. That is, the coordinates $x = 4$ and $y = -1$ can be substituted to make the equation true.
So

$$-1 = -2 \times 4 + c$$
$$-1 = -8 + c$$
$$\Rightarrow \quad 7 = c$$

So the equation is given by

$$y = -2x + 7.$$

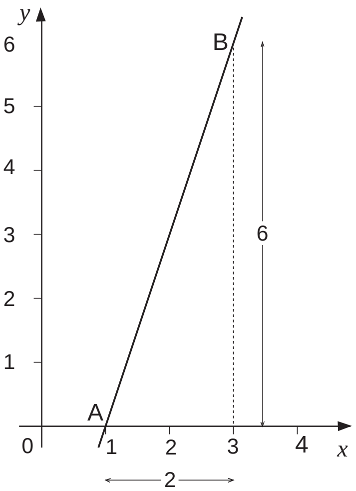


Example

A straight line passes through the points A, (1, 0) and B, (3, 6). Find the gradient of the line, and its equation.

Solution

The figure opposite shows a sketch of the line. The gradient can be calculated using this formula:



$$\text{gradient} = \frac{y_B - y_A}{x_B - x_A} \quad \left(\frac{\text{differences in } y\text{'s}}{\text{differences in } x\text{'s}}\right)$$

$$= \frac{6-0}{3-1} \quad \text{since } y_B = 6,\ y_A = 0,\ x_B = 3 \text{ and } x_A = 1$$

$$= \frac{6}{2} = 3$$

As the graph is a straight line, the equation can be written in the form $y = mx + c$. The gradient is 3, so this is the value of m :

$$y = 3x + c.$$

To find c, the coordinates of a point on the line must be substituted into the equation. The coordinates of point A $(x = 1,\ y = 0)$ are used here ;

$$0 = 3 \times 1 + c$$

$$0 = 3 + c$$

$$\Rightarrow \quad c = -3.$$

The equation is $y = 3x - 3$.

Example

The graph $4x + 3y - 6 = 0$ is a straight line. Find its gradient, and the intercept with the y-axis.

Solution

To read the gradient and intercept of a line from its equation it must be written in the form $y = mx + c$ first. Now

$$4x + 3y = 6$$

$$\Rightarrow \quad 3y = -4x + 6$$

$$\Rightarrow \quad y = -\frac{4}{3}x + \frac{6}{3}$$

$$= -\frac{4}{3}x + 2$$

If this equation is compared with $y = mx + c$, it can be seen that the gradient is $-\frac{4}{3}$ and the intercept with the y-axis is 2.

Activity 3

Draw the following lines on the same axes

(a) $2y + x = 0$ (b) $4y = 1 - 2x$ (c) $6y + 3x = 1$.

What do you notice about these lines?

Exercise 7B

1. Write down the gradient and its intercept with the y-axis of each of these lines

   (a) $y = 3x - 1$ (b) $y = -4x - 3$

   (c) $y = \frac{1}{2}x + 5$ (d) $y = -\frac{6}{5}x - \frac{1}{2}$

   (e) $y = 4x$ (f) $y = x$

   (g) $y = -x$ (h) $y = 5$.

2. Find the gradient and intercept with the y-axis of each of the lines without drawing the graphs.

   (a) $4x + y = 9$ (b) $x + 2y = 6$

   (c) $3x - 2y = 4$ (d) $4y - 2x + 6 = 0$

3. A line passes through the point (5, 1) and has a gradient of 3. Find the equation of the line.

4. A line with gradient $-\frac{1}{3}$ passes through the point (4, 6). Find the equation of the line, leaving fractions in your answer.

5. Find the equation of the straight line which passes through the points (2, -1) and (6, 7).

6. A line which is parallel to $y = 2x$ passes through the point (3, -2). Find the equation of this line.

7. A line which is parallel to $4x + 3y - 6 = 0$ passes through the origin. Find its equation.

7.3 Perpendicular lines

There is an important result that connects the gradients of perpendicular lines.

Activity 4

(a) Accurately construct a number of pairs of perpendicular lines. Measure the gradients of each pair of lines. What do you notice?

(b) Draw on the same graph

(i) $y = 2x$, $y = -\dfrac{1}{2}x$

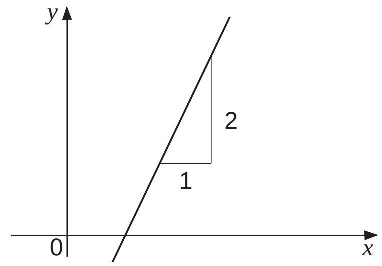
(ii) $y + x = 1$, $y = 1 + x$

What can you conjecture about the gradients of perpendicular lines?

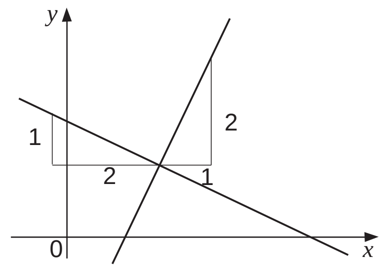
Suppose you have two lines with gradients m_1 and m_2 - remember that the gradient can be positive or negative. Then the two lines are **perpendicular** if and only if

$$\boxed{m_1\, m_2 = -1}$$

You can easily see this from the geometry of the situation. Take, for example, a line with gradient 2, as shown opposite. This means that for a unit increase in the x direction, there will be an increase of 2 in the y direction.



Now consider a line perpendicular to this line as llustrated opposite.



Since the angle between them is $90°$, then the triangles shown are congruent, and it can be seen that the gradient of the perpendicular line is given by

$$\frac{1}{-2} = -\frac{1}{2}$$

So in this case $m_1 = 2, \quad m_2 = -\frac{1}{2}$ and

$$m_1 \, m_2 = -1$$

This is of course not a proof of this result only a verification.

Activity 5

Prove the result, $m_1 \, m_2 = -1$, for perpendicular lines by generalising the method above.

Example

Find the equation of the line that passes through the point (1,2) and is perpendicular to $y = 1 + x$.

Solution

Since the gradient of $y = 1 + x$ is 1, the gradient of the perpendicular line is -1. Hence its equation is of the form

$$y = -x + c$$

To pass through the point (1,2) requires

$$2 = -1 + c$$
$$\Rightarrow \quad c = 3$$
$$\Rightarrow \quad y = 3 - x$$

Exercise 7C

1. Find the equation of the straight line which passes through the point (1,1) and is perpendicular to the line $2y+x=1$.

2. Find the equation of the straight line which passes through the origin, and is perpendicular to the line joining the points (1,4) and (4,1).

3. Show that the lines $y+2x=3$ and $2y-x=1$ are perpendicular. At what point do they intersect?

4. A straight line passes through the intersection of the two lines with equation

$$y+x=2$$

$$2y-x=7$$

and is perpendicular to the line with equation $y=2x$. Find its equation.

7.4 Linear inequalities

Often real life problems can have more than one solution, but some solutions may be better than others.

Activity 6

A club is organising a trip, and needs to transport at least 210 people using mini buses, coaches or some of each. A coach holds 45 people and costs £120 to hire, whilst a minibus holds 15 and costs £60. The club only has £600 to spend on the transport. Find a possible solution to this problem. Is it unique?

Find the most economic way to arrange the transport.

To solve the problem in Activity 6 in a logical and precise way requires the use of inequalities. Another similar problem is given below.

A farmer has 100 hectares available on which to sow 2 crops, wheat and sugar beet. Each hectare of wheat is expected to produce a profit of £20, whilst each hectare of sugar beet should produce a profit of £30. However EEC quota regulations will not allow the farmers to grow more than 50 acres of sugar beet and 70 hectares of wheat. The time taken in soil preparation, seeding and tendering is estimated to be 5 man hours per hectare of wheat and 10 man hours per hectare of sugar beet. The available man power is up to 700 hours.

Can you find the optimum solution - that is a solution which satifies all the conditions and maximises the profit?

You can use intelligent trial and error methods, but an algebraic approach is ideal.

Let $x =$ no. of hectares sown with wheat

$y =$ no. of hectares sown with sugar beet.

The land restriction can be written as

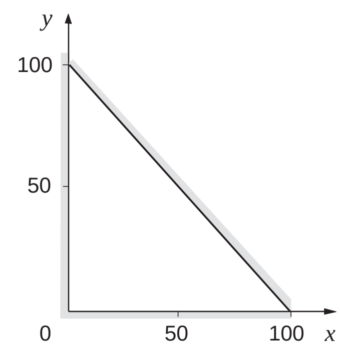
$$x + y \le 100.$$

It should also be noted that

$$x \ge 0$$

and $y \ge 0$.

These three inequalities can be illustrated on a graph.



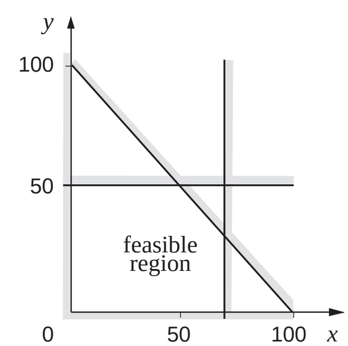
Note that the actual lines shown are $x + y = 100$, $x = 0$, $y = 0$ and that, for example, all the region to the left of the line $x + y = 100$ satisfies the inequality

$$x + y \le 100$$

(for example, the point $x = y = 0$ satisfies this inequality).

So to find the 'solution' of an inequality, you first draw the equality, and then identify the allowable region.

The region to be excluded, is shown partially shaded.



There are further inequalities to be satisfied, because of the regulations; namely

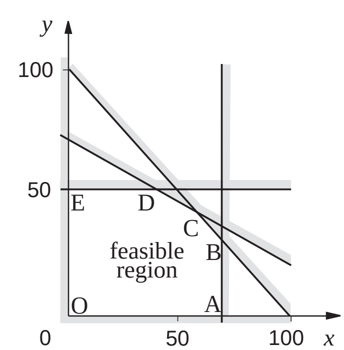
$$x \le 70$$
$$y \le 50$$

These can be shown on the graph so that the allowable (or feasible) region is further restricted.

There is yet one more inequality to consider; that is the available number of man-hours. This gives

$$5x + 10y \le 700.$$

This is added to the graph as shown opposite.



The feasible region, which contains all possible solutions, is a convex polygon with vertices at O, A, B, C, D, and E.

Any point inside this polygon is a possible solution, but the optimum solution is the one which gives rise to a maximum value of

$$P = 20x + 30y$$

Why should P obtain its maximum value at a vertex of the polygon?

In this case the optimum solution will occur at one of the vertices, and the table below evaluates P at each vertex.

Point	Coordinates	Profit	
0	$x=0,\ y=0$	$P=0$	
A	$x=70,\ y=0$	$P=20\times70+30\times0=1400$	
B	$x=70,\ y=30$	$P=20\times70+30\times30=2300$	
C	$x=60,\ y=40$	$P=20\times60+30\times40=2400 \leftarrow$	maximum value
D	$x=40,\ y=50$	$P=20\times40+30\times50=2300$	
E	$x=0,\ y=50$	$P=20\times0+30\times50=1500$	

So the optimum solution occurs at $x=60,\ y=40$ showing that the farmer should sow 60 hectares of wheat and 40 hectares of sugar beet.

This example illustrates the technique of **linear programming**, which has been used extensively in business and commerce.

The complete theory is beyond the scope of this text (it is, in fact, dealt with in detail in the Decision Mathematics text), but it does indicate the importance of being able to illustrate inequalities.

Activity 7

Return to the problem in Activity 6 and use graphical analysis of the type shown above to find the most economic way of organising the transport.

Exercise 7D

1. Show graphically the region defined by

   $x+y\le2,\ y-2x\le0,\ y\ge-1$

2. Show that the region defined by

   $x+y\le1,\ x-y\le1,\ y\le1$

   is finite.

3. Determine the region defined by

   $x+2y\le4,\ y\ge x,\ y\le2x$

   If, in addition, $x\le\frac{1}{2}$, are there any values of x and y which satisfy all the inequalities?

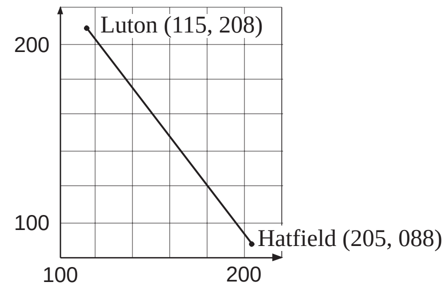
7.5 Cartesian coordinates

The x and y axes frequently used in mathematics form part of a system called Cartesian coordinates, named after a French mathematician and philosopher, *René Descartes*. It has been found to be one of the most convenient ways of describing how things are related in space.

As an example of the usefulness of such a system, consider the grid reference system which in effect is a set of Cartesian coordinates, each 100 m standing for a unit on a typical OS (Ordnance Survey) map.

Example

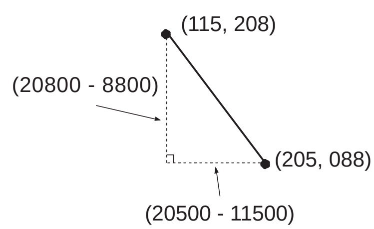
A helicopter pilot is told to fly from grid reference (115, 208) to (205, 088) (Luton Airport to Hatfield Aerodrome). The figure opposite illustrates the journey.



The pilot needs to know the distance in order to estimate his time of arrival. Using Pythagoras' theorem, calculate the distance to be flown.

Solution

The figure shows the right angled triangle to be used. Note that the coordinates are given in 100 m units. By Pythagoras' theorem, the distance d in metres can be calculated from



$$d^2 = (20800 - 8800)^2 + (20500 - 11500)^2$$

$$\Rightarrow \quad d^2 = 12000^2 + 9000^2$$

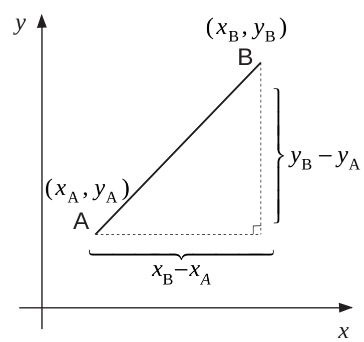
$$\Rightarrow \quad d^2 = 225000000\,.$$

Hence $\quad d = \sqrt{225\,000\,000}$

$$\Rightarrow \quad = 15000 \text{ m}$$

$$= 15 \text{ km}\,.$$

This example illustrates the method for calculating the distance between any two points on a Cartesian coordinate grid.



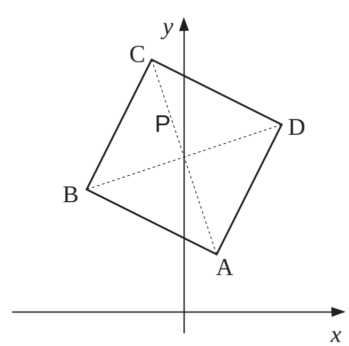
If a point A has coordinates (x_A, y_A), and point B is (x_B, y_B), then, using Pythagoras' theorem, the distance AB is given by

$$\boxed{AB = \sqrt{(y_B - y_A)^2 + (x_B - x_A)^2}}$$

The formula applies even when coordinates are negative.

Example

The points A(1, 1), B(–3, 3), C(–1, 7) and D(3, 5) form a square. Find the area of the square and the point P at which its diagonals cross.



Solution

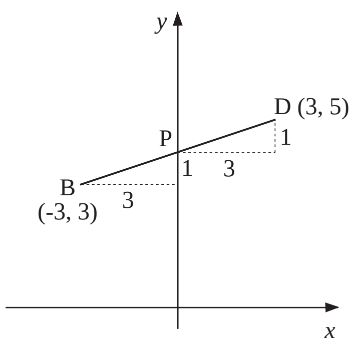
The figure opposite shows the square ABCD. The area of ABCD is simply the square of the length of one of its sides. The length of any of the four sides can be found using Pythagoras' theorem, as above.

For example

$$\begin{aligned}
AB &= \sqrt{(y_A - y_B)^2 + (x_A - x_B)^2} \\
&= \sqrt{(1-3)^2 + (1-(-3))^2} \\
&= \sqrt{(-2)^2 + (4)^2} \\
&= \sqrt{4+16} \\
&= \sqrt{20}
\end{aligned}$$

So the area of $ABCD = (AB)^2 = \left(\sqrt{20}\right)^2 = 20 \text{ units}^2$.

Now P is the midpoint of either diagonal. One of the diagonals, BD, is shown opposite. Since P is the midpoint, its coordinates will be the average of those of B and D.



$$x: \quad \frac{1}{2}(-3+3) = \frac{1}{2} \times 0 = 0$$

$$y: \quad \frac{1}{2}(3+5) = \frac{1}{2} \times 8 = 4$$

So P has coordinates (0, 4), which can be verified from the figure.

In general, the coordinates of the midpoint of a line AB are found by averaging the coordinates of A and B, to give the midpoint coordinates as

$$\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right).$$

Exercise 7E

1. Find the distances between these pairs of points using Pythagoras's theorem

   (a) $(0, 0)$ and $(3, 4)$ (b) $(-1, 3)$ and $(11, 2)$

   (c) $(4, 1)$ and $(-2, -5)$ (d) $(0, 6)$ and $(3, -8)$

   Give answers to three significant figures where necessary.

2. Find the midpoints of the lines joining the pairs of points in Question 1.

3. Calculate the area of the right angled triangle formed by the points A(1, 3), B(2, 5) and C(5, 1). The right angle is formed at the point A.

4. Show that the quadrilateral PQRS is a square, where P is the point (–2, 5), Q is (–5, 1), R is (–1, –2) and S is (2, 2).

7.6 Miscellaneous Exercises

1. A bank charges £10 for 850 Danish kroner and £20 for 1900 Danish kroner. Plot these data on a graph, with pounds on the horizontal axis, and draw a straight line graph through the points. Find the gradient of the graph, stating its units and its meaning.

2. A car entering a town decelerates from 26.8 ms^{-1} to 13.4 ms^{-1} s in 8 seconds. Draw a graph showing the velocity, v, on the vertical axis and time t, on the horizontal. Find the gradient of the line joining the points, stating the units and the meaning of the gradient.

3. Find the gradient of the lines joining these pairs of points :

   (a) $(2, 1)$ and $(8, 7)$ (b) $(-3, 6)$ and $(1, 12)$

   (c) $(1, 9)$ and $(5, 3)$ (d) $(3, -6)$ and $(12, -20)$

4. Without drawing any of the graphs for the equations below, state the gradient and intercept with the y-axis for each of these lines :

   (a) $y=5x+3$ (b) $y=-x+1$

   (c) $y=\frac{1}{2}x$ (d) $3y=x+6$

   (e) $5x-2y-11=0$ (f) $3x+4y+1=0$

5. A line has a gradient of 2 and passes through the point $(5, -1)$ Find its equation.

6. Find the equation of the line parallel to the line $y=4x-1$ which passes through the point $(-3, 9)$

7. Find the equation of the straight line which passes through (0, 5) and (3, 1).

8. An experiment was carried out to see the extension, e cm, produced when different weights were hung from a spring. The results are shown in this table.

Weight, w (grams)	0	100	200	300	400
Extension, e (cm)	0	5	11	14.5	21

   Plot these data on a graph, with e on the vertical axis, and draw a line of best fit. Find the gradient of the line and the intercept with the vertical axis, and hence find an equation relating e to w. Use your equation to estimate the extension produced when the weight on the spring is 1 kg.

9. Find the equation of the line which crosses the line $y=4x+2$ at the point (3, 14) at right angles.

10. Find the equation of the line which is perpendicular to the line $3y-2x+5=0$ and which passes through the point (1, 1).

11. Find the equation of the line parallel to $y=5x-1$ which passes through the point (4, 0).

12. Two sides of a square are formed by the lines $y=3x$ and $3y+x-6=0$. Find the coordinates of the corner of the square at which these sides meet using an algebraic method.

13. Find the equation of the straight line passing through the point (2, 3) which is perpendicular to the line with equation $5x+3y=0$. (AEB)

14. The points P, Q and R have coordinates (2, 4), (7, –2) and (6, 2) respectively. Find the equation of the straight line l which is perpendicular to the line PQ and which passes through the midpoint of PR. (AEB)