19 ITERATION

Objectives

After studying this chapter you should

- understand the importance of graphical and numerical methods for the solution of equations;
- understand the principle of iteration;
- appreciate the need for convergence;
- be able to use several iterative methods including Newton's method.

19.0 Introduction

Before you begin studying this chapter you should be familiar with the basic algebraic and graph-plotting techniques covered in higher-level GCSE courses. You should also be able to differentiate at least simple algebraic functions: differentiation is covered in the Foundation Core. A number of examples and exercises involve trigonometry, but these are not essential and can be missed out if you have not covered that topic.

The solution of algebraic equations has always been a significant mathematical problem, and early Egyptian and Babylonian sources show how people of those civilizations were solving **linear**, **quadratic** and **cubic equations** more than three thousand years ago. The Egyptians often solved linear equations using an *aha* method (named after the Egyptian word for a heap, not because the answer came as a surprise!) in which they guessed an answer, tried it out, and then adjusted it; the Babylonians solved quadratic and cubic equations by using well-known algorithms together with written-out tables of values.

The spread of Greek mathematics, with its emphasis on elegance and precision, led to the disappearance of those early techniques among academic mathematicians, even if some of them survived among merchants, builders and other practical people. Instead, Mathematicians were more concerned to find general 'analytic' methods based on formulae for the exact solution of any equation of a particular type. Methods of solving quadratic equations were already known, but the first general method for solving a cubic equation was discovered by the Italian mathematician *Scipione del Ferro* in about 1500, and that for **quartics** by his compatriot *Ludovico Ferrari* some fifty years later.

At that point the process of discovery came to a stop, because no one was able to find a method for solving a general equation in x^5 or any higher power. For these equations, as they arose, people had to go back to the earlier trial-and-improvement methods, but the general slowness of those meant that the 'modern' analytic methods were much better. The development of electronic calculators and computers changed all this, however, so that nowadays it is often quicker to use a numerical method such as the Egyptians or Babylonians might have done than to spend time in developing a formula.

You will need a pocket calculator throughout the chapter; a graphic calculator will be particularly useful. Access to a computer with graph-plotting and/or programming facilities may be a further advantage.

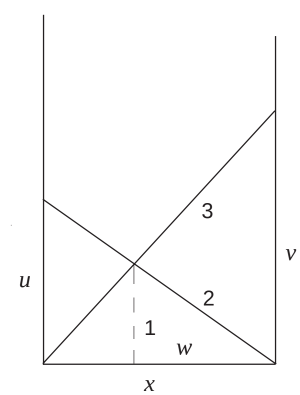
19.1 Crossing ladders

In a narrow passage between two walls there are two wooden ladders, a green one 3 m long and a red one 2 m long; the ground is horizontal and the walls are vertical. Each ladder has its foot at the bottom of one wall and its top resting against the other wall. The green ladder slopes up from left to right and the red ladder slopes up from right to left. The ladders cross 1 m above the ground. How wide is the passage?

The first step in solving most problems of this kind is the creation of a mathematical model - not a model made of cardboard and glue, but a set of equations and other relations describing the mathematically important features of the situation.

Stop and think what these important factors are.

The essential facts are summarised in the diagram opposite, which shows the approximate positions and lengths of the ladders and the position of their crossing point. It denotes by x metres the distance to be found, and by u m, v m and w m respectively three other lengths that may be important. The diagram says nothing about the fact that the ladders are made of wood or that they are differently coloured, since these facts are irrelevant to the particular problem to be solved.



The ladder problem - find the length x

From the diagram, various relationships can be deduced.

Using similar triangles, $\dfrac{w}{1} = \dfrac{x}{u}$

Again using similar triangles, $\dfrac{x-w}{1} = \dfrac{x}{v}$

Adding these equations, $$\frac{x}{1} = \frac{x}{u} + \frac{x}{v}$$

$$\Rightarrow \quad \frac{1}{u} + \frac{1}{v} = 1$$

Now by Pythagoras' theorem, $$u^2 + x^2 = 4$$

$$\Rightarrow \quad u = \sqrt{4 - x^2}$$

and similarly, $$v^2 + x^2 = 9$$

$$\Rightarrow \quad v = \sqrt{9 - x^2}$$

giving $$\frac{1}{\sqrt{4 - x^2}} + \frac{1}{\sqrt{9 - x^2}} = 1.$$

All that remains is to solve this equation, and the value obtained for x is the width of the passage in metres.

Consider how you might obtain a solution.

Several methods of solution are possible, but you should have realised almost at once that this is not the sort of equation that can be solved by a simple algorithm (such as, "Take all the x terms to one side and all the numbers to the other"), nor is there a formula such as the one commonly used for quadratic equations. This equation cannot in fact be solved by any such 'analytic' method. Instead we are going to resort to methods which will lead us to very accurate approximations to solutions.

There are two approaches which may work, one involving numerical substitution and the other based on graphs. If you can find a positive numerical value for x which satisfies the equation, then clearly this is a solution. Random guessing is likely to take a long time, however, so any approach must be based on some kind of systematic trial and improvement. Later in this chapter several numerical methods are examined in detail. The alternative graphical approach is the subject of the next section.

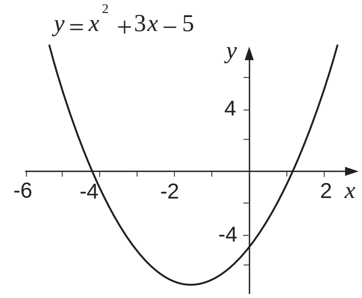
19.2 Graphical methods

You should already be familiar with the idea of solving an equation by means of a graph: an example will remind you of the method.

Example

Solve $x^2 + 3x - 5 = 0$.

Solution



The diagram shows the graph of $y = x^2 + 3x - 5$, plotted from a table of values in the usual way. It crosses the x-axis at the points $(-4.2, 0)$ and (1.2, 0) approximately - the graph certainly cannot be read to an accuracy greater than one decimal place - so the solutions of the equation are $x \approx -4.2$ and $x \approx 1.2$.

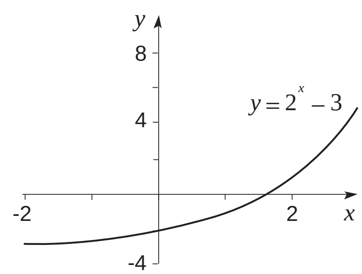
If you have the use of a graphic calculator or a computer with graph-drawing facilities, you can get the same result with much less effort. Draw the graph on the screen, and then use the computer mouse (or the <Trace> function on the calculator) to move the cursor to each of the crossing points in turn.

The same method can be used more generally to solve equations in higher powers of x. There are formulae (like the quadratic formula but much more complicated) for solving cubic and quartic equations, but the French mathematician *Evariste Galois* proved just under 200 years ago that no such formula can ever be found for general equations in powers of x higher than the fourth. As a bonus, the graphical method works for equations including sines, cosines, exponential and logarithmic functions, and so on. Look at some more examples.

Example

Solve the equation $2^x - 3 = 0$.

Solution



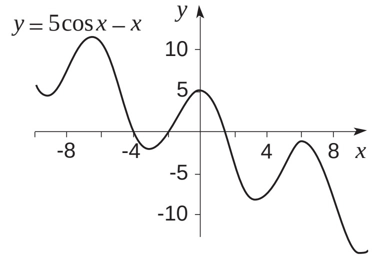
The diagram shows the graph of $y = 2^x - 3$, plotted from a table of values or drawn on a calculator or computer screen. The graph crosses the x-axis at (1.6, 0) approximately, and nowhere else, so $x \approx 1.6$ is the only solution of the equation.

Example

Find correct to one decimal place all the solutions of the equation $5\cos x - x = 0$, where x is expressed in radians.

Solution

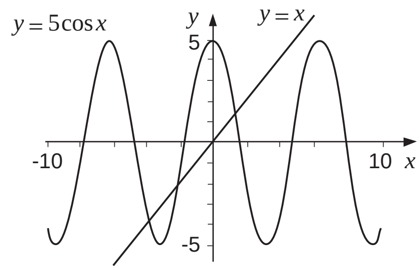
The diagram shows the graph of $y = 5\cos x - x$. It crosses the x-axis three times, at $(-3.8, 0)$, $(-2.0, 0)$ and $(1.3, 0)$, and so the equation has the three solutions, $x \approx -3.8$, $x \approx -2.0$, and $x \approx 1.3$.



The graph crosses the x-axis three times

Plotting two graphs

The equation in the last example could have been rewritten in the form $5\cos x = x$, so an alternative approach would have been to plot two graphs, the graph of $y = 5\cos x$ and the graph of $y = x$, and to find their points of intersection. The advantage of this method is that both these are well-known functions whose graphs should be familiar, making it quick and easy to draw them. The diagram shows these two graphs, and it is evident that the values of x at the points of intersection correspond to those already found.



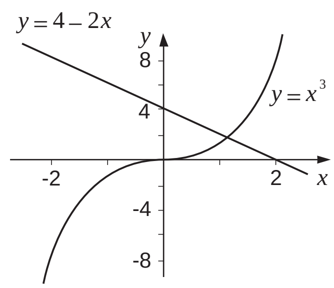
The two graphs intersect at three points

Example

By drawing two graphs, solve $x^3 + 2x - 4 = 0$.

Solution

The equation can be written as $x^3 = 4 - 2x$, and the diagram shows the graphs of $y = x^3$ and $y = 4 - 2x$. They intersect only once, at (1.2, 1.6), so the only solution of this cubic equation is $x \approx 1.2$.



The equation could have been solved in other ways. It could have been rearranged in the form $2x = 4 - x^3$, or even as $x^2 = \frac{4}{x} - 2$, and either of these would have given the same result. Often there is no one right way to solve an equation graphically, but a whole collection of ways, some of which may be easier than others.

A possible difficulty

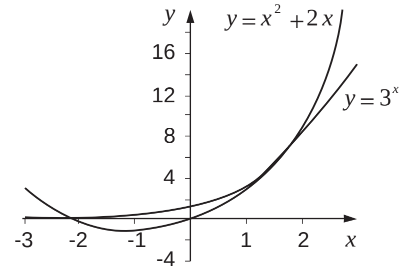
A particular difficulty arises when the graph is nearly flat at the point where it crosses the x-axis, or where two graphs are nearly parallel at their point of intersection. The next example provides an illustration.

Example

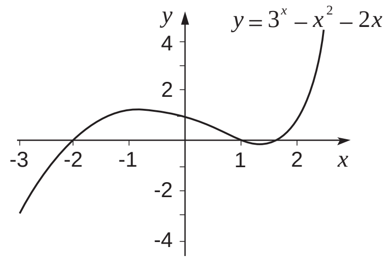
Solve the equation $3^x = x^2 + 2x$.

Solution

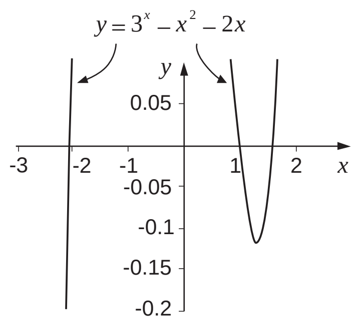
The first diagram opposite shows the graphs of $y = 3^x$ and $y = x^2 + 2x$, and while it is clear that there is one root of the equation at $x \approx -2.1$, the other (or others?) could be almost anywhere between 1.0 and 2.0.



The second diagram showing the graph of $y = 3^x - x^2 - 2x$ is not much more helpful.



It is only in the third diagram with its exaggerated vertical scale that the other two solutions can be identified as $x \approx 1.0$ ($x = 1$ is actually an exact solution) and $x \approx 1.6$.



Function on a larger scale

Exercise 19A

Use graphical methods to find approximate solutions of the following equations, giving answers correct to one decimal place.

1. $x^2 - 1 = \dfrac{1}{x}$
2. $x^3 - 6x^2 + 11x - 5 = 0$
3. $x^4 = 2x + 1$
4. $2^x - 5x = 0$

*5. $\sin x = \cos 2x$ (answers between 0 and 2π only)

19.3 Improving accuracy

In the previous section all the solutions were given correct to one decimal place, but this is not always good enough. How might you get a more accurate answer - to two or three decimal places, say?

Stop and think about how you could obtain a more accurate answer.

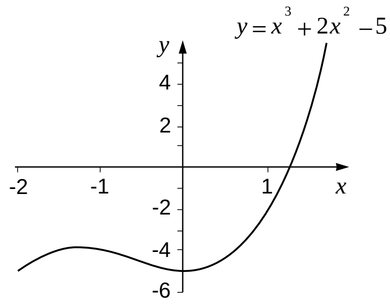
You may have found a hint in the last example of Section 19.2 – with graphical methods, it is usually possible to get a more accurate answer by redrawing the graph on a larger scale. The next example illustrates this.

Example

Solve $x^3 + 2x^2 - 5 = 0$, giving your answer correct to three decimal places.

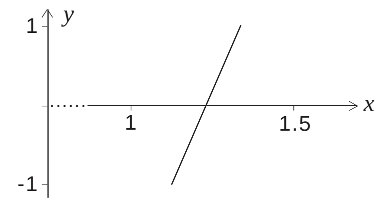
Solution

The first diagram shows the graph of $y = x^3 + 2x^2 - 5$, plotted for values of *x* between -2 and 2. The equation clearly has only one root, which lies between 1 and 2, and a graphical estimate might suggest $x \approx 1.2$ to one decimal place.



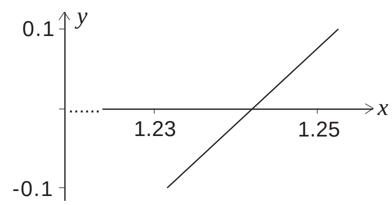
Graph of function for $-2 \le x \le 2$

The second diagram shows the same graph, but this time plotted only between $x = 1$ and $x = 1.5$ - notice that over this limited domain the graph is almost a straight line. On this larger scale it is possible to estimate the root more accurately, and to say that $x \approx 1.24$ to two decimal places.



The same graph for $1 \le x \le 1.5$

The third diagram increases the scale yet again, and shows the graph plotted over the domain $1.23 \le x \le 1.25$. Now the solution can be estimated even more accurately as $x \approx 1.242$ to three decimal places. Clearly there is no limit in theory to the accuracy that can be obtained by this method, but it is time-consuming.



The same again for $1.23 \le x \le 1.25$

Computers and calculators

The process can be carried out much more quickly with a computer graph package or a graphic calculator. Most graph-drawing software packages allow the user to change the scales without redrawing the whole graph, and by using this facility to 'zoom in' on the root the solution can be read quite easily to whatever level of accuracy is required.

Activity 1 *Graphic calculators*

The <Factor> command on the Casio *fx*-7000G is not widely used, but was designed for just this purpose. If you have such a calculator, try the following:

- <Range> -10, 10, 5, -10, 10, 5
- <Factor> 5 : <Graph> $Y = Xx^y 3 + 2X^2 - 5$ <EXE>
- <Trace> and use the $\Rightarrow$ and $\Leftarrow$ keys to move the flashing dot close to the point where the graph crosses the *x*-axis, then <EXE> again.
- Repeat the last instruction as often as necessary - probably

three or four times - until you are satisfied with the accuracy of the x-value given on the screen.

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*Accuracy

Although in theory there is no limit to the accuracy that can be obtained by a graphical method of this kind, it is difficult in practice to get solutions correct to more than five or six decimal places. Unless you are fond of very cumbersome pencil-and-paper arithmetic you will almost certainly use a calculator to work out the values to plot, and most calculators operate to no more than eight or ten significant figures at best. General-purpose computer packages have a similar limitation - six significant figures is not uncommon - making it impossible to obtain any more accurate result unless you are prepared to adjust the equation as well as the scales.

You should bear in mind too that many equations have **irrational** solutions - that is, the 'true' solutions are numbers that cannot be expressed exactly as fractions or decimals. Thus although it may be possible (in theory) to get as close to the true solution as you might wish, you may never be able to find its exact value. In real life this hardly matters - five or six decimal places is more than enough for any practical purpose - but a mathematician would be careful to distinguish a good decimal approximation from the 'exact' irrational solution.

Exercise 19B

Use a graphical method to solve each of the following equations correct to the stated level of accuracy:

1. $x^3 - 4x + 5 = 0$, to two decimal places
2. $x^5 - x^3 = 1$, to two decimal places
3. $2^x = x + 3$, to two decimal places (both solutions)
4. $2x^2 + 1 = \frac{1}{x}$, to three decimal places

*5. $x + \ln x = 0$, to three decimal places

19.4 Interval bisection

The graphical method of solving equations, as you will have realised, has two disadvantages. For one thing it tends to be quite time-consuming, though using a suitable calculator or computer can speed things up considerably. Secondly, however, it needs someone to read the graph, to estimate the position of the crossing point or move the cursor, and (if greater accuracy is required) to decide on the new range of values to be plotted.

These disadvantages are overcome, at least in part, by some of the algebraic methods discussed in the rest of the chapter. The chief benefit of such methods is that they can be expressed algorithmically in terms of yes/no decisions and routine operations, so eliminating the need for human intervention and making them suitable for programming into a computer or calculator.

Activity 2 Guess a number

Take a few minutes to play this game with another student. One of you thinks of a whole number between 1 and 100, and the other has to guess this number by asking no more than ten questions of a yes/no type. Play several rounds, taking it in turns to be the guesser, and try to find the most efficient strategy. How many questions do you really need?

In fact seven questions and a final 'guess' will do, as long as the questions are properly chosen. A skilful guesser might well have asked questions like the following:

Is your number more than 50?	Yes
Is it more than 75?	No
Is it more than 62?	Yes
Is it more than 69?	Yes
Is it more than 72?	No
Is it more than 70?	Yes
Is it more than 71?	Yes
The number is 72.	

At each stage, the guesser is roughly halving the number of possibilities. Initially the number could be anywhere between 1 and 100, but the first answer shows that it is actually between 51 and 100. Then it is between 51 and 75, then between 63 and 75, then between 70 and 75, then between 70 and 72, then between 71 and 72, and the final answer shows that it is 72.

If you played the game enough times you probably discovered this strategy (or something very similar) for yourself. If not, play two or three more rounds using this strategy, to be sure you understand how it works.

Locating a root

The same principle, known as **interval bisection** because at each stage the possible range of values is halved, can be applied to the solution of equations. If you know that a particular equation has a root between 2 and 3 (say), then you can ask whether the root is greater than 2.5, and so halve the interval in which it is to be found. By doing this repeatedly, you can eventually say that the root lies in an interval so small that you can give its value to whatever accuracy you want.

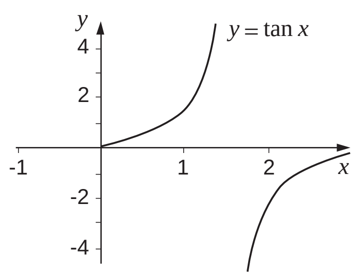
How do you find the first interval (i.e. between 2 and 3) ?

There are at least two practical methods of locating the root, either of which can be used alone but which are much better in combination. The first is to draw a quick rough graph; this does take a little time, but it shows how many roots there are altogether and helps you to avoid any of several possible traps.

The second method, which can be used on its own but which is much more reliable after you have sketched a graph, involves looking for a change of sign. If you find, for example, that $f(2)$ is negative and $f(3)$ positive, or vice versa, it follows that provided f is a continuous function then $f(x)$ is zero somewhere between $x = 2$ and $x = 3$.

The **change-of-sign method** is certainly more precise than the sketch graph in locating a root, but it does contain at least three possible traps. Firstly, it may well locate one root but miss another: unless you are very persevering (or know where to look) you are unlikely to find a root between (say) -11 and -10.

Secondly, the change-of-sign method will not work unless the graph of $y = f(x)$ is continuous over the interval in question. The diagram shows part of the graph of $y = \tan x$ for values of x (in radians) between 0 and 3. It is clear that $f(1) > 0$ and $f(2) < 0$, but equally clear that there is no root of the equation $f(x) = 0$ between 1 and 2.



$f(1) < 0$ and $f(2) < 2$ but there is no root between 1 and 2

Thirdly, the change-of-sign method will not show up repeated roots (where the graph just touches the x-axis without crossing it), nor two roots close together. For example, the equation $6x^2 - 29x + 35 = 0$ has solutions $x = 2\frac{1}{3}$ and $x = 2\frac{1}{2}$, as you can check by factorising, but $f(2)$ and $f(3)$ are both positive and so give no indication that these roots exist.

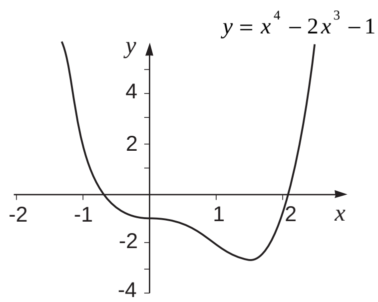
In spite of these three problems, the change-of-sign method of locating roots is very important, and useful too when properly applied in conjunction with a sketch. It forms the basis of the interval bisection and linear interpolation methods discussed in this section and the next, and is commonly used at least at the start of the more sophisticated methods of solution considered later in the chapter.

The method in practice

Example

Solve $x^4 - 2x^3 - 1 = 0$, correct to two decimal places.

Solution



The diagram shows the graph of $y = x^4 - 2x^3 - 1$, and there are evidently two roots, one negative and close to -1, and the other positive and close to 2.

$f(-1) = 2$, $f(0) = -1$ and the change of sign shows that there is a root between -1 and 0.

$f(2) = -1$, $f(3) = 26$ and the change of sign shows that the second root lies between 2 and 3.

Consider the positive root first.

$f(2.5) \approx 6.8$ so the change of sign is between 2 and 2.5.

$f(2.2) \approx 1.1$ so the change is between 2 and 2.2.

(This is not exactly the midpoint of the interval, but is near enough and keeps the calculation fairly simple.)

$f(2.1) \approx -0.1$ so the change is between 2.1 and 2.2.

$f(2.15) \approx 0.5$ so the change is between 2.1 and 2.15.

$f(2.12) \approx 0.1$ so the change is between 2.1 and 2.12.

$f(2.11) \approx 0.03$ so the change is between 2.1 and 2.11.

$f(2.105) \approx -0.02$ so the change is between 2.105 and 2.11.

This root is therefore 2.11 to two decimal places.

Similarly with the negative root,

$f(-0.5) \approx -0.7$ so the change is between -1 and -0.5.

$f(-0.7) \approx -0.1$ so the change is between -1 and -0.7.

$f(-0.85) \approx 0.8$ so the change is between -0.85 and -0.7.

$f(-0.78) \approx 0.3$ so the change is between -0.78 and -0.7.

$f(-0.74) \approx 0.1$ so the change is between -0.74 and -0.7.

$f(-0.72) \approx 0.02$ so the change is between -0.72 and -0.7.

$f(-0.71) \approx -0.03$ so the change is between -0.72 and -0.71.

$f(-0.715) \approx -0.01$ so the change is between -0.72 and -0.715.

So this root is -0.72 to two decimal places.

Exercise 19C

Use a sketch graph followed by a change-of-sign search to locate the roots of the equation $2^x - 2x - 3 = 0$. Then use an interval bisection method to find these solutions correct to two decimal places.

19.5 Rearrangement methods

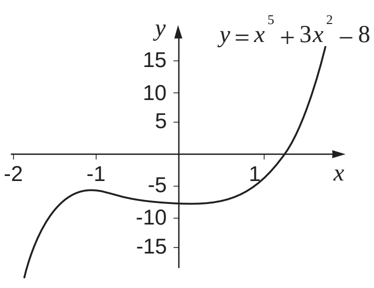
Interval bisection is a fairly straightforward iterative method for the solution of equations, but is not particularly efficient. You have seen in the examples and exercises that it may easily take six or eight iterations to get a solution accurate to even two decimal places, and in a world where time is at a premium this is not good enough. Other quicker methods must therefore be considered.

Example

Solve the equation $x^5 + 3x^2 - 8 = 0$.

Solution

Let $f(x) = x^5 + 3x^2 - 8$; the graph of $y = f(x)$ shows clearly that there is only one root. Since $f(1) = -4$ and $f(2) = 36$, and since f is a continuous function, the solution lies between 1 and 2, probably close to 1.



Now the equation can be rearranged as $x^5 = 8 - 3x^2$, and this in turn can be written in the form $x = \sqrt[5]{8 - 3x^2}$. This equation can then be used as the basis of an **iteration** formula; i.e. one where, given an approximation x_0, a new approximation x_1 can be calculated, and then a new approximation x_2 can be calculated, etc. In this case the formula is

$$x_{n+1} = \sqrt[5]{8 - 3x_n^2}.$$

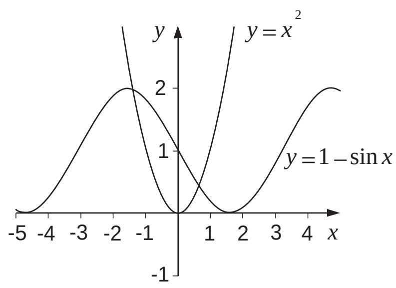
Substituting the first approximation $x_0 = 1$ gives $x_1 \approx 1.4$; substituting this result and then each of the others in turn gives $x_2 \approx 1.2$, $x_3 \approx 1.3$, $x_4 \approx 1.24$, $x_5 \approx 1.28$, $x_6 \approx 1.25$, $x_7 \approx 1.27$, $x_8 \approx 1.26$ and $x_9 \approx 1.26$ again. A quick check confirms that $f(1.255) < 0$ and $f(1.265) > 0$, so that the solution is $x \approx 1.26$ correct to two decimal places.

Although this has involved nine calculations (or iterations) and so is apparently no quicker than the previous methods, each iteration involves no more than the substitution of the previous result into a fairly simple formula. It is usually possible, in fact, to make the substitution using the value in the calculator directly, without the trouble of writing down each result and re-entering it: certainly a very simple program can be written for a programmable calculator or computer.

Example

Solve $x^2 + \sin x = 1$, with x in radians.

Solution



The equation can be rearranged as $x^2 = 1 - \sin x$, and from a sketch graph this has solutions at $x \approx 0.6$ and at $x \approx -1.4$. The rearrangement leads to an iteration formula

$$x_{n+1} = \sqrt{1 - \sin x_n}$$

and substituting $x_0 = 0.6$ gives in turn $x_1 \approx 0.66$, $x_2 \approx 0.622$, $x_3 \approx 0.646$, $x_4 \approx 0.631$, $x_5 \approx 0.640$, and $x_6 \approx 0.635$. Once again it is now easy to check that 0.635 and 0.645 give $x^2 + \sin x$ values respectively less than and greater than 1, so that the positive solution is $x \approx 0.64$ to two decimal places.

Substituting $x_0 = -1.4$, on the other hand, gives $x_1 \approx 1.41$ and $x_2 \approx 0.11$ and gradually moves in to the same solution as before. This is because the iteration formula has taken the positive square root, and so naturally gives only positive results. An alternative iteration formula with a negative square root would be equally valid:

$$x_{n+1} = -\sqrt{1 - \sin x_n}$$

and with $x_0 = -1.4$ this gives $x_1 \approx x_2 \approx -1.41$, which can be checked as correct in the usual way.

Exercise 19D

1. Show that the equation $x^3+2x-6=0$ has just one solution, and locate it. Show that the equation can lead to the iteration formula

$$x_{n+1}=\sqrt[3]{6-2x_n},$$

and use this formula to find the solution correct to two decimal places.

2. Write down the equation whose solution can be found by using the iteration formula

$$x_{n+1}=1+\frac{1}{x_n^2},$$

and use the formula to find this solution correct to two decimal places.

3. Show that the equation $x^4-3x+1=0$ has two solutions, and locate them approximately. Show that the equation can lead to two iteration formulae

$$x_{n+1}=\sqrt[4]{3x_n-1} \quad \text{and} \quad x_{n+1}=\frac{x_n^4+1}{3}$$

and show that each of these formulae leads to just one solution. Find each solution correct to two decimal places.

4. Find a suitable iteration formula for the equation $x^3-x^2-5=0$, and solve the equation correct to two decimal places.

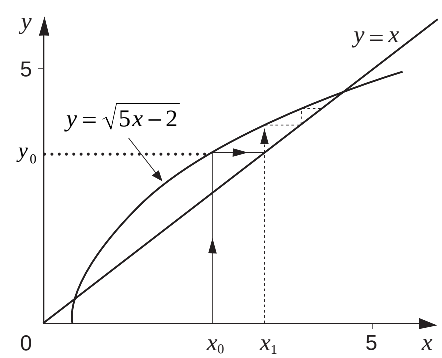
5. Use an iterative method to solve the equation $x^x=2$ correct to three decimal places.

19.6 Convergence

Exercise 19D should have started you thinking. Why is it, for example, that in Question 3 one iteration formula would work only for the smaller root and the other only for the greater? Why do some formulae seem to work faster than others? Why do some formulae not lead to a root at all, but give results that swing wildly from side to side?

These are all questions concerned with the **convergence** of the iterative process, and while it is useful for you to have a general understanding of the idea, you do not need to go deeply into the theory. If you want to study convergence more deeply than this section allows, look at an undergraduate textbook on numerical analysis.

Consider the equation $x^2-5x+2=0$, to be solved using the formula $x_{n+1}=\sqrt{5x_n-2}$. The diagram shows two graphs, those of $y=x$ and $y=\sqrt{5x-2}$ respectively; their points of intersection correspond to the solutions of the equation. The first approximation x_0 is substituted into the formula and gives a value - call it y_0 - which then becomes x_1 : this process is illustrated by the arrows.



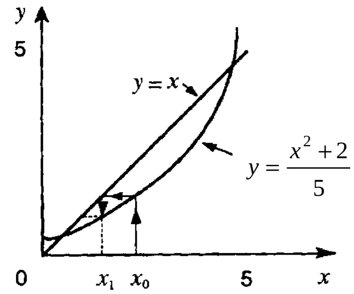
The principle of iteration

As the iteration is repeated, the arrows build into some sort of pattern. In this particular case, they are moving (slowly) closer to the right-hand point of intersection, and so will eventually lead to that solution.

If you try to apply the same method to find the left-hand point of intersection, you will fail no matter how hard you try - the arrows will either converge onto the right-hand point or diverge altogether. On the other hand, the iteration formula

$$x_{n+1} = \frac{x_n^2 + 2}{5}$$

(illustrated in a similar way in this diagram) converges quite neatly onto the left-hand point of intersection and cannot be persuaded to lead to the upper solution.



An alternative iteration formula for the lower root

Test for convergence

If you want to know whether or not a particular iteration formula will converge, it is often easiest in practice to apply the formula three or four times and look at the pattern of results. If the results are getting gradually closer together they are probably converging on a solution; if not, you are unlikely to find the solution using this formula. There is a more formal test, however, that you might want to use in a particular case.

Suppose that the iteration formula is $x_{n+1} = g(x_n)$, and that the derivative of $g(x)$ is $g'(x)$. Then it can be shown - the proof is not attempted here - that a **necessary** and **sufficient** condition for the formula to **converge** on the true solution λ is that

$$-1 < g'(\lambda) < 1.$$

This condition is all very well, except that the value of λ is what you are trying to find! In practice, therefore, it is usual to look for an x-interval containing both the unknown λ and the first approximation x_0 such that $|g'(x)| < 1$ throughout the interval. In the example above, if $g(x) = \sqrt{5x-2}$ then

$g'(x) = \frac{5}{2\sqrt{5x-2}}$. This has absolute value less than 1 if $x > 1.65$, and so can converge on the upper solution (given a suitable first approximation) but not the lower. On the other hand, if

$g(x) = \frac{x^2+2}{5}$ then $g'(x) = \frac{2x}{5}$, which lies between -1 and 1 only when $-2.5 < x < 2.5$; it thus converges only to the lower solution.

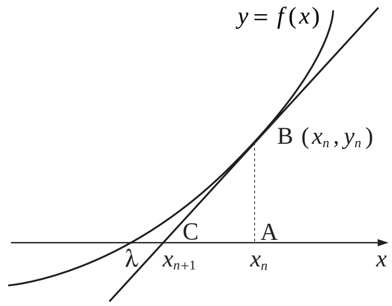
Exercise 19E

Use a graph to locate approximately the root or roots of each of the equations opposite. Rearrange each equation to give an iteration formula, and test each formula to determine whether it will lead to any or all of the roots. Repeat with another rearrangement if necessary, until all the roots are obtainable. Find each root correct to one decimal place.

1. $x^3 - 3x - 4 = 0$
2. $x^4 + 2x^3 = 5$.
3. $3^x = 3x + 2$.

19.7 Newton's method

The diagram shows part of the curve $y = f(x)$, where the equation to be solved is $f(x) = 0$. It also shows an approximate solution x_n and the tangent to the curve at the point (x_n, y_n). If this tangent cuts the x-axis at the point $(x_{n+1}, 0)$, then it is clear in this case that x_{n+1} is a better approximation than x_n to the true root λ.



Newton's method for solving $f(x) = 0$

Now the gradient of BC is $\dfrac{BA}{AC}$, which is $\dfrac{y_n}{(x_n - x_{n+1})}$.

But $y_n = f(x_n)$, and the gradient of the tangent is $f'(x_n)$,

so $$f'(x_n) = \frac{f(x_n)}{(x_n - x_{n+1})}.$$

Rearranging this gives **Newton's iteration** formula (sometimes known as the Newton-Raphson formula):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

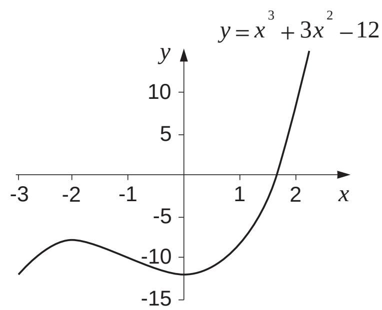
This can be used to obtain a sequence of results leading to a root in the same way as the iteration formulae discussed in Section 19.5; its main advantage over those formulae is that it tends to converge much more quickly.

Example

Solve the equation $x^3 + 3x^2 - 12 = 0$, correct to two decimal places.

Solution

From a graph, there is just one solution, which lies between 1 and 2. Take $x_0 = 1.5$ as a first approximation.



$f(x) = x^3 + 3x^2 - 12$, so $f'(x) = 3x^2 + 6x$.

Applying Newton's formula,

$$\begin{aligned} x_1 &= 1.5 - \frac{f(1.5)}{f'(1.5)} \\ &= 1.5 - \frac{-1.875}{15.75} \\ &= 1.62. \end{aligned}$$

Applying the formula again,

$$x_2 = 1.62 - \frac{f(1.62)}{f'(1.62)}$$
$$= 1.62 - \frac{0.125}{17.59}$$
$$= 1.613.$$

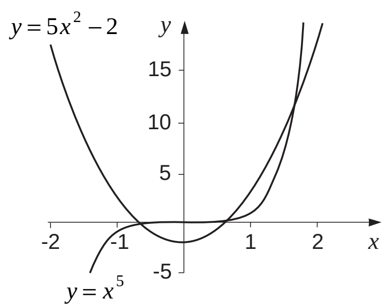
And it is not difficult now, after just two iterations, to check that $f(1.605) < 0$ and that $f(1.615) > 0$, so giving $x \approx 1.61$ correct to two decimal places.

Example

Solve the equation $x^5 = 5x^2 - 2$, giving your answers correct to three decimal places.

Solution

From a graph, there are solutions at $x \approx -0.6$, $x \approx 0.6$ and $x \approx 1.6$ respectively. Newton's method works only for equations in the form $f(x) = 0$, so a rearrangement gives $f(x) = x^5 - 5x^2 + 2$ and $f'(x) = 5x^4 - 10x$.



If $x_0 = -0.6$, then

$$x_1 = -0.6 - \frac{f(-0.6)}{f'(-0.6)}$$
$$= -0.6 - \frac{0.122}{6.648}$$
$$= -0.618.$$

If $x_1 = -0.618$, then

$$x_2 = -0.618 - \frac{f(-0.618)}{f'(-0.186)}$$
$$= -0.618 - \frac{0.00023}{6.909}$$
$$= -0.618.$$

$f'(-0.6185) < 0$ and $f'(-06175) > 0$, so $x \approx -0.618$ to three decimal places.

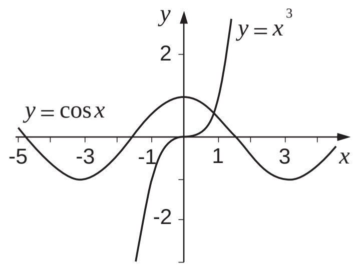
If $x_0 = 0.6$ then $x_1 = x_2 = 0.651$; and if $x_0 = 1.6$ then $x_1 = 1.619$ and $x_2 = 1.618$, which can similarly be confirmed as sufficiently accurate. Thus $x \approx -0.618$, 0.651 or 1.618 to three decimal places.

Example

Solve $\cos x = x^3$, where x is in radians, correct to three decimal places.

Solution

A sketch graph shows just one root, close to 0.9. Rearranging the equation gives $f(x) = \cos x - x^3$ and $f'(x) = -\sin x - 3x^2$.



If $x_0 = 0.9$, $x_1 = 0.9 - \dfrac{-0.107}{-3.213} = 0.867$

$$x_2 = 0.867 - \frac{-0.0046}{-3.017} = 0.866$$

$$x_3 = 0.866 - \frac{-0.0016}{-3.0116} = 0.865.$$

Since $f(0.8645) > 0$ and $f(0.8655) < 0$, $x \approx 0.865$ correct to three decimal places.

Evaluation of Newton's method

There is no doubt that Newton's method is a very useful one, and it does have certain advantages over the methods discussed earlier. It converges much faster than any of the previous methods and also the same formula can be used for each of the roots.

The method is not perfect, however, and it does have a number of disadvantages. The first of these is that $f(x)$ has to be differentiated, and your ability to do this will depend on your knowledge of calculus techniques. Then, the first approximation must normally be fairly close to the root you are trying to find, making a reasonable graph almost essential. Finally, the method can become unreliable if the graph of $y = f(x)$ has a turning point or inflexion close to the root - in such a case a different iterative method may prove more effective.

In general, however, you will find the most effective strategy for the solution of difficult equations to be the following:

1. Draw a graph or graphs to locate the root(s) approximately.
2. Use a change of sign and a single linear interpolation to get a good first approximation.
3. Apply Newton's method once or twice (or more).
4. Verify that your answer has the accuracy you require.

Exercise 19F

1. Show that the equation $x^3 - 2x^2 + 4 = 0$ has a root close to $x = -1$. Use Newton's method to find this root correct to two decimal places.
2. Find correct to three decimal places the smallest positive root of the equation $x^4 - 3x^3 + 5x^2 - 1 = 0$.
3. Use Newton's method to find both solutions of the equation $x = 3\ln x$ to three decimal places.
4. Find $\sqrt[3]{10}$ using $x_0 = 2$ and two applications of Newton's method, and calculate the percentage error from the 'true' value obtained from a calculator.

*5. Solve $x^x = 5$ correct to four decimal places.

19.8 The ladders again

With the techniques covered in this chapter it is possible to complete the solution of the ladder problem introduced in Section 19.1. The problem, you will recall, was as follows:

In a narrow passage between two walls there are two wooden ladders, a green one 3 m long and a red one 2 m long. Each ladder has its foot at the bottom of one wall and its top resting against the other wall. The green ladder slopes up from left to right and the red ladder slopes up from right to left. The ladders cross 1 m above the ground. How wide is the passage?

This problem led to the equation

$$\frac{1}{\sqrt{4-x^2}} + \frac{1}{\sqrt{9-x^2}} = 1$$

which was left unsolved at that time.

Although it is possible to attempt a graphical or numerical solution of the equation as it stands, it is probably better to simplify it by getting rid of the fractions and the roots. Multiplying the whole equation by both square roots,

$$\sqrt{9-x^2} + \sqrt{4-x^2} = \sqrt{4-x^2}\,\sqrt{9-x^2}\,.$$

Squaring,

$$\left(9-x^2\right) + 2\sqrt{9-x^2}\,\sqrt{4-x^2} + \left(4-x^2\right) = \left(4-x^2\right)\left(9-x^2\right).$$

Collecting terms,

$$\begin{aligned} 2\sqrt{9-x^2}\,\sqrt{4-x^2} &= \left(4-x^2\right)\left(9-x^2\right) - \left(9-x^2\right) - \left(4-x^2\right) \\ &= 23 - 11x^2 + x^4. \end{aligned}$$

Squaring again,

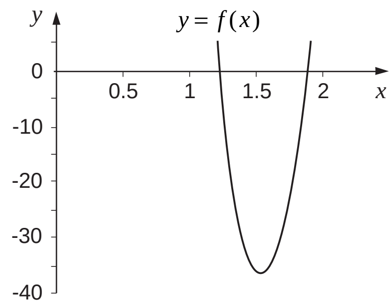
$$4\left(9-x^2\right)\left(4-x^2\right) = \left(23-11x^2+x^4\right)^2$$

$$144-52x^2+4x^4 = 529-506x^2+167x^4-22x^6+x^8$$

$$\therefore \quad x^8-22x^6+163x^4-454x^2+385=0.$$

Now let $f(x)=x^8-22x^6+163x^4-454x^2+385$ and draw the graph of $y=f(x)$ for $0 \le x \le 2$, since any valid solution must certainly lie within these bounds.

The diagram shows the result, and it is clear that there are two solutions, close to $x=1.2$ and $x=1.9$ respectively.



Graph of function for $0 \le x \le 2$

Now $f'(x)=8x^7-132x^5+652x^3-908x$: this is the step which perhaps justifies the algebra, because differentiating the original equation would have been very messy.

Applying the Newton-Raphson formula with $x_0=1.2$,

$$x_1 = 1.2-\frac{f(1.2)}{f'(1.2)} = 1.2-\frac{7.84}{-263} = 1.23 \text{ to two decimal places.}$$

Similarly, $x_2=1.23$ again.

It is easy to check that $f(1.225)>0$ and $f(1.235)<0$, confirming that this solution is accurate to the nearest centimetre.

Similarly, taking $x_0=1.9$, $x_1=1.87$ to two decimal places.

A second iteration gives 1.87 again and this solution of $f(x)=0$ can similarly be shown to have the necessary accuracy. However, a few moments of thought, possibly by sketching the positions of the ladders, will show you that the passage certainly cannot be 1.87 m wide. Sometimes squaring the original equation (as we have done in this case) leads to some phantom roots being introduced and so one ought to spend a little time checking that the answer obtained does solve the original problems.

The other root of $x \approx 1.23$ does work and so the passageway must be approximately 123 cm wide.

*Activity 3 Iterative chaos

Use a computer or a programmable calculator to investigate sequences given by the iteration formula $x_{n+1}=kx_n\left(1-x_n\right)$, with $x_0=0.7$, for different values of k between 1 and 4. The results could be quite chaotic!

19.9 Errors and error propagation

Errors can occur in a variety of ways, for example:

(a) **rounding errors** – in which it is convenient to work with a truncated decimal, such as 0.33 for $\frac{1}{3}$;

(b) **method errors** – in which the method used to solve a problem is only approximate, such as the Newton-Raphson method for finding the roots of an equation;

(c) **experimental errors** – in which inaccuracies in measurements occur, such as measuring lengths to the nearest millimetre;

(d) **human errors** – which we all make at times, but checking and cross-checking helps to reduce their impact.

Of particular concern here are errors arising from (a) and (c).

Absolute and relative errors

The **absolute error** is defined as

$$\boxed{e_{abs} = |X - x|} \qquad (= |\text{approx} - \text{exact}|)$$

where X is the approximate value of the exact value x.

Examples

(a) If $x = \frac{1}{3}$ and $X = 0.3$, then

$$e_{abs} = |0.3 - \tfrac{1}{3}| = 0.03333\ldots$$

(b) If $x = \frac{1}{3}$ and $X = 0.33$, then

$$e_{abs} = |0.33 - \tfrac{1}{3}| = 0.003333\ldots$$

(c) If x is measured to the nearest centimetre as 12 cm, then $X = 12$ cm, so the maximum and minimum possible values of x are 12.5 and 11.5 cm, respectively.

If $x = 12.5$ cm, then

$$e_{abs} = |12 - 12.5| = 0.5$$

and similarly, if $x = 11.5$, then

$$e_{abs} = |12 - 11.5| = 0.5,$$

so the **maximum** possible absolute error is 0.5.

The use of absolute errors as a measure of accuracy is fine if, for example, you are making comparisons of approximations to the exact value, as in (a) and (b) above. They are not so useful if you are making comparisons of approximations to different exact values or are comparing the accuracy of measurements of different sizes.

To overcome this difficulty, we use the concept of **relative error** defined by

$$e_{rel} = \frac{e_{abs}}{|x|}$$

Examples

(a) If $x = \frac{1}{3}$ and $X = 0.3$, then

$$e_{rel} = \frac{0.0333\ldots}{\frac{1}{3}} = 0.1.$$

(b) If $x = 5\frac{1}{3}$ and $X = 5.3$, then

$$e_{rel} = \frac{0.03333\ldots}{5\frac{1}{3}} = 0.00625.$$

(c) Measuring x as 12 cm, to the nearest centimetre, implies $11.5 \le x \le 12.5$, so

$$\text{maximum} \quad e_{abs} = 0.5$$

$$\text{giving maximum} \quad e_{rel} = \frac{\max e_{abs}}{\min |x|} = \frac{0.5}{11.5} \approx 0.043.$$

(d) Measuring x as 1020 cm, to the nearest centimetre, implies

$1019.5 \le x \le 1020.5$, so

$$\text{maximum} \quad e_{abs} = 0.5 \quad \text{(as in (c))}$$

$$\text{but maximum} \quad e_{rel} = \frac{0.5}{1019.5} \approx 0.00049.$$

So the relative error is a better concept to use when making comparisons

(i) between errors in approximations to different exact values,

(ii) between the accuracy of measurements of different sizes.

In some cases, the exact value of x is unknown, so $|x|$ is approximated by $|X|$ in the formula to give

$$\text{estimated relative error} = \frac{e_{abs}}{|X|}$$

For example, if x is measured as 12 cm, to the nearest centimetre, then

$$\text{estimated relative error} = \frac{0.5}{12} \approx 0.042.$$

Propagation of errors

When two numbers are multiplied together, the magnitude of the rounding error may be larger than the rounding errors in the two numbers. For example, if the two numbers, 4.2 and 1.1, are given correct to one decimal place, then the product is

$$4.2 \times 1.1 = 4.62.$$

However, the first number could be between 4.15 and 4.25, and the second between 1.05 and 1.15. So the maximum value of the product is

$$4.25 \times 1.15 = 4.8875$$

and the minimum value is

$$4.15 \times 1.05 = 4.3575.$$

So the error interval is now 4.3575 to 4.8875 – a range of 0.53, compared to the error range of 0.1 in each number.

Example

Find the maximum possible value of the volume of a cylinder when the radius and height are measured, correct to the nearest mm, as

$$r = 4.2 \text{ cm}, \quad h = 8.6 \text{ cm}.$$

Solution

The volume is given by

$$V = \pi r^2 h = \pi \times (4.2)^2 \times 8.6 \approx 476.6 \text{ cm}^3$$

whilst the maximum possible value of V is given by

$$V = \pi \times (4.25)^2 \times 8.65 \approx 490.8 \text{ cm}^3.$$

Example

Find the maximum possible value of x when evaluating the expression

$$x = \frac{(2.7 \times 3.6) - 4.2}{3.5 + 8.7} \quad (\approx 0.4525)$$

where each number is given correct to one decimal place. Also find an estimate for the percentage error.

Solution

The maximum possible value of x is given by

$$\frac{\text{maximum numerator}}{\text{minimum denominator}} = \frac{(2.75 \times 3.65) - 4.15}{3.45 + 8.65} \approx 0.4866$$

The percentage (%) error can be estimated as

$$\frac{(0.4866-0.4525)}{0.4525}\times 100 = 7.53\%$$

which assumes that the true value of x is 0.4525.

Activity 4

Also find the minimum possible value for the expression.

What is the percentage error using the expression if the true value is either the maximum or minimum value of the expression?

19.10 Miscellaneous Exercises

1. Show that the equation $x^3 - x - 2 = 0$ has only one real root, and that this root lies between 1 and 2. Use an iterative method to determine its value accurate to three decimal places.

2. Determine graphically the number of solutions of the equation
$$x^2 - 4 = \frac{1}{x},$$
and estimate their values.

3. Show that the equation $2^x = 3x + 2$ has two solutions, and find each of them correct to two decimal places.

4. Find correct to two decimal places all the solutions of the equation
$$x^4 + 2x^3 - 11x^2 - 12x + 21 = 0.$$

5. Find correct to four decimal places the smallest positive root of the equation
$$7x^3 - 19x^2 + 14x - 3 = 0.$$

6. Solve the equation $xe^x = 1$, correct to two decimal places.

7. Without using any calculator functions other than +, –, × and ÷, find $\sqrt{3}$ correct to six decimal places.

8. Find correct to two decimal places the coordinates of the points at which the circle $x^2 + y^2 = 16$ meets the rectangular hyperbola $x(y+1) = 9$.

9. Show that
$$x_{n+1} = (9x_{n-5})^{\frac{1}{3}}$$
is an iteration formula for the solution of the equation
$$x(x^2 - 9) + 5 = 0$$
Show that this equation has a root between – 4 and – 3, and use the given iteration formula with $x_0 = -3$ to find this root correct to 2 decimal places, showing that your answer has this accuracy. (AEB)

*10. In a circle whose radius is 10 cm, a segment of area 50 cm² is cut off by a chord AB. Show that AB subtends an angle θ radians at the centre of the circle, where $\theta - \sin\theta = 1$. Solve this equation and hence find the perimeter of the segment in cm correct to two decimal places.

11. Show that the equation $x^3 - x^2 - 2 = 0$ has a root α which lies between 1 and 2.

    (a) Using 1.5 as a first approximation for α, use the Newton-Raphson method once to obtain a second approximation for α, giving your answer to 3 decimal places.

    (b) Show that the equation $x^3 - x^2 - 2 = 0$ can be arranged in the form $x = \sqrt[3]{(f(x))}$ where $f(x)$ is a quadratic function.

    Use an iteration of the form $x_{n+1} = g(x_n)$ based on this rearrangement and with $x_1 = 1.5$ to find x_2 and x_3, giving your answers to 3 decimal places. (AEB)