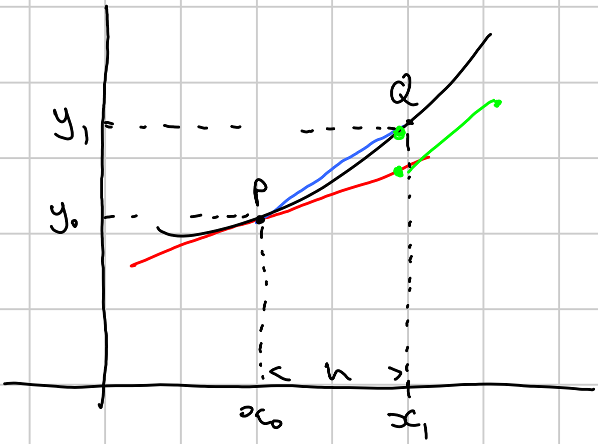
Chapter 5 – Step-by-step methods for DEs

Given a DE and some initial conditions (x_0, y_0) which lie on the curve, these methods generate a sequence of points (x_1, y_1), (x_2, y_2) ... which lie approximately on the curve. The further we go from the initial conditions, the less reliable the approximation.

First Order DEs



Given a DE which defines a curve, a point (x_0, y_0) on the curve and a step value h, we wish to find an approximation for y_1 (which is $f(x_1)$) at the point x_1 (which is $x_0 + h$)

Now the gradient of PQ can be approximated by the gradient of the curve at P, which can be written as $f'(x_0)$ or $\left(\frac{dy}{dx}\right)_0$

ie/ $$\boxed{\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}}$$

* These will be given when required but their derivation could be asked for.

Rearranging gives

$$y_1 \approx y_0 + h\left(\frac{dy}{dx}\right)_0$$

or, in a form suitable for iteration:

$$y_{n+1} \approx y_n + h\left(\frac{dy}{dx}\right)_n$$

Example Given the DE $\frac{dy}{dx} = \frac{2x-y}{x+1}$, with $y = 0.5$ when $x = 1$, use the above method to find the approximate value of y at $x = 1.2$, 1.4 and 1.6

We have

$$x_0 = 1, \quad y_0 = 0.5, \quad h = 0.2$$

So $\left(\frac{dy}{dx}\right)_0 = \frac{2 - 0.5}{1+1} = 0.75$

So $x_1 = 1.2$, $y_1 = 0.5 + 0.2 \times 0.75$
$= 0.65$

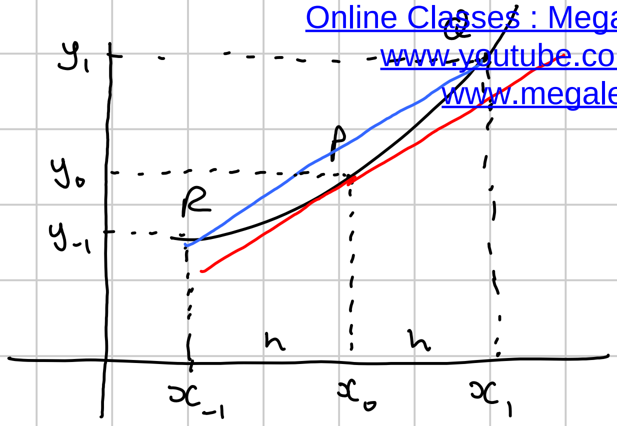
$$\left(\frac{dy}{dx}\right)_1 = \frac{2 \times 1.2 - 0.65}{1.2 + 1} = 0.795\ldots$$

So $x_2 = 1.4$, $y_2 = 0.65 + 0.2 \times 0.795\ldots$
$= 0.809\ldots$

$$\left(\frac{dy}{dx}\right)_2 = \frac{2 \times 1.4 - 0.809\ldots}{1.4 + 1} = 0.829\ldots$$

So $x_3 = 1.6$, $y_3 = 0.809\ldots + 0.2 \times 0.829\ldots$
$= \underline{\underline{0.975}}$

The above method can be improved as follows:



Gradient at P $\approx$ Gradient of RQ

$$\boxed{\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}} \quad *$$

Rearranging, $\quad y_1 \approx y_{-1} + 2h\left(\frac{dy}{dx}\right)_0$

or as an iterative formula

$$y_{n+1} \approx y_{n-1} + 2h\left(\frac{dy}{dx}\right)_n$$

To use this we need to know two points. So we need to use the previous method once to get (x_1, y_1) – then we can switch to this method.

Example Apply this 'improved' method to the previous example. $\left(\frac{dy}{dx} = \frac{2x-y}{x+1} \text{ with } y = 0.5 \text{ when } x = 1\right)$

We have $x_0 = 1$, $y_0 = 0.5$

and we use the previous method once to find $x_1 = 1.2$, $y_1 = 0.65$

Now $x_2 = 1.4$, and $y_2 \approx y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$$\approx 0.5 + 0.4 \times 0.795 \ldots$$

$\approx 0.8181\ldots$

So $\left(\frac{dy}{dx}\right)_2 = \frac{2\times 1.4 - 0.8181\ldots}{1.4+1} = 1.09\ldots$

And $x_3 = 1.6$, $\quad y_3 \approx y_1 + 2h\left(\frac{dy}{dx}\right)_2$

$\approx 0.65 + 0.4 \times 0.8257\ldots$

$\approx \underline{\underline{0.980}}$

[To solve this DE exactly,

$$\frac{dy}{dx} = \frac{2x-y}{x+1}$$

$$\frac{dy}{dx} + \frac{1}{x+1}y = \frac{2x}{x+1}$$

$$IF = e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x+1$$

$$(x+1)\frac{dy}{dx} + y = 2x$$

$$\frac{d}{dx}\left(y(x+1)\right) = 2x$$

$$y(x+1) = x^2 + c$$

$$y = \frac{x^2+c}{x+1}$$

When $x = 1$, $y = 0.5$ $\Rightarrow$ $0.5 = \frac{1+c}{2}$

$\Rightarrow$ $c = 0$

Hence $\quad y = \frac{x^2}{x+1}$

When $x = 1.6$, $\quad y = \frac{2.56}{2.6}$

$= \underline{\underline{0.985}}$]

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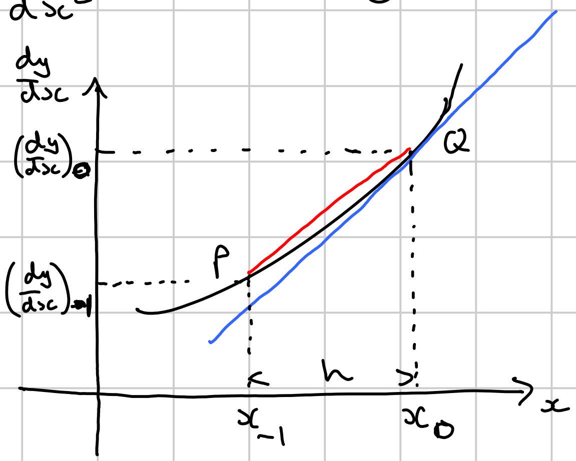
Second Order DEs

$\frac{d^2y}{dx^2}$ is the gradient of $\frac{dy}{dx}$



gradient of tangent at Q $\approx$ gradient of chord PQ.

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{\left(\frac{dy}{dx}\right)_0 - \left(\frac{dy}{dx}\right)_{-1}}{h}$$

(use the first form of approximation above)
$$\approx \frac{\frac{y_1 - y_0}{h} - \frac{y_0 - y_{-1}}{h}}{h}$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}}$$ *

Examples

① Given the DE $\frac{d^2y}{dx^2} = x - y$, with $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$, find the approximate value of y at $x = 0.1$ and $x = 0.2$

We know $h = 0.1$

$x_0 = 0$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 2$

We can find $\left(\frac{d^2y}{dx^2}\right)_0 = 0 - 1 = -1$

So in the formula above

$$-1 = \frac{y_1 - 2 + y_{-1}}{0.01} \quad \text{—} ①$$

But also, using $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_{-1}}{2h}$

$$2 = \frac{y_1 - y_{-1}}{0.2} \quad \text{—} ②$$

$$\left.\begin{array}{l} ① \Rightarrow \quad y_1 + y_{-1} = 1.99 \\ ② \Rightarrow \quad y_1 - y_{-1} = 0.4 \end{array}\right\}$$

So $y_1 = \frac{2.39}{2} = \underline{\underline{1.195}}$ (at $x_1 = 0.1$)

Now we can use $\left(\frac{d^2y}{dx^2}\right)_1 = \frac{y_2 - 2y_1 + y_0}{h^2}$

and $\left(\frac{d^2y}{dx^2}\right)_1 = x_1 - y_1$

$= 0.1 - 1.195$

$= -1.095$

So $-1.095 = \frac{y_2 - 2 \times 1.195 + 1}{0.01}$

$y_2 = 2.39 - 1 - 0.01095$

$= \underline{\underline{1.37905}}$

(see notes on Series approximations, which gave $y = 1.379$ at $x = 0.2$)

② Given $\frac{d^2y}{dx^2} - \frac{1}{2}y\frac{dy}{dx} - 1 = 0$, and $y = 2$ when $x = 0$, $y = 2.1$ when $x = 0.1$, find y when $x = 0.2$.

We have $x_{-1} = 0$, $y_{-1} = 2$ $\qquad h = 0.1$

$$x_0 = 0.1, \quad y_0 = 2.1$$

$$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{y_1 - 2 \times 2.1 + 2}{0.01}$$

$$\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - 2}{0.2}$$

(using the "better" approximation)

Sub into DE (using subscript 0 throughout)

$$\frac{y_1 - 2.2}{0.01} - \frac{1}{2}(2.1)\left(\frac{y_1 - 2}{0.2}\right) - 1 = 0$$

$$100y_1 - 220 - \frac{5}{2} \times 2.1(y_1 - 2) - 1 = 0$$

$$200y_1 - 440 - 10.5y_1 + 21 - 2 = 0$$

$$189.5y_1 = 421$$

$$\underline{\underline{y_1 = 2.222}} \quad (3\text{dp})$$

(at $x_1 = 0.2$)

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[Example 1 again:—

$$\frac{d^2y}{dx^2} + y = x$$

AE $\qquad \alpha^2 + 1 = 0$

$$\alpha = \pm i$$

CF $\qquad y = R\cos x + S\sin x$

Trial PI $\qquad y = ax + b \Rightarrow \frac{dy}{dx} = a, \quad \frac{d^2y}{dx^2} = 0$

$$0 + ax + b = x \Rightarrow a = 1, \; b = 0$$

General solution $\qquad y = R\cos x + S\sin x + x$

$\frac{dy}{dx} = S\cos x - R\sin x + 1.$

$y = 1$ when $x = 0 \Rightarrow 1 = R$

$\frac{dy}{dx} = 2$ when $x = 0 \Rightarrow 2 = S + 1 \Rightarrow S = 1$

So $\underline{\underline{y = \cos x + \sin x + x.}}$

When $x = 0.2$, $y = \cos 0.2 + \sin 0.2 + 0.2$

$= 1.3787\ldots$

$= \underline{\underline{1.379}}$ (3 dp)