

HYPERBOLIC FUNCTIONS

We know the power series for $\sin x$, $\cos x$ and e^x :

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

From this we can confirm that

$$\begin{aligned} \frac{d}{dx} (\sin x) &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots \\ &= \cos x \end{aligned}$$

$$\text{and } \frac{d}{dx} (\cos x) = -\sin x$$

$$\text{and } \frac{d}{dx} (e^x) = e^x$$

If we replace the alternating signs in the power series for $\sin x$ and $\cos x$, we obtain two new functions called hyperbolic sine and cosine:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

These functions have many similarities with trig functions, but also with exponential functions.

We can see that

$$\frac{d}{dx} (\sinh x) = \cosh x$$

$$\text{AND } \frac{d}{dx} (\cosh x) = \sinh x$$

Also
and

$$\cosh x + \sinh x = e^x$$

$$\cosh x - \sinh x = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - \dots$$

$$= e^{-x}$$

By adding these ~~two~~ ~~two~~ equations we obtain www.youtube.com/megalecture
www.megalecture.com

$$2 \cosh x = e^x + e^{-x}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

DEFINITIONS
 - use in proofs

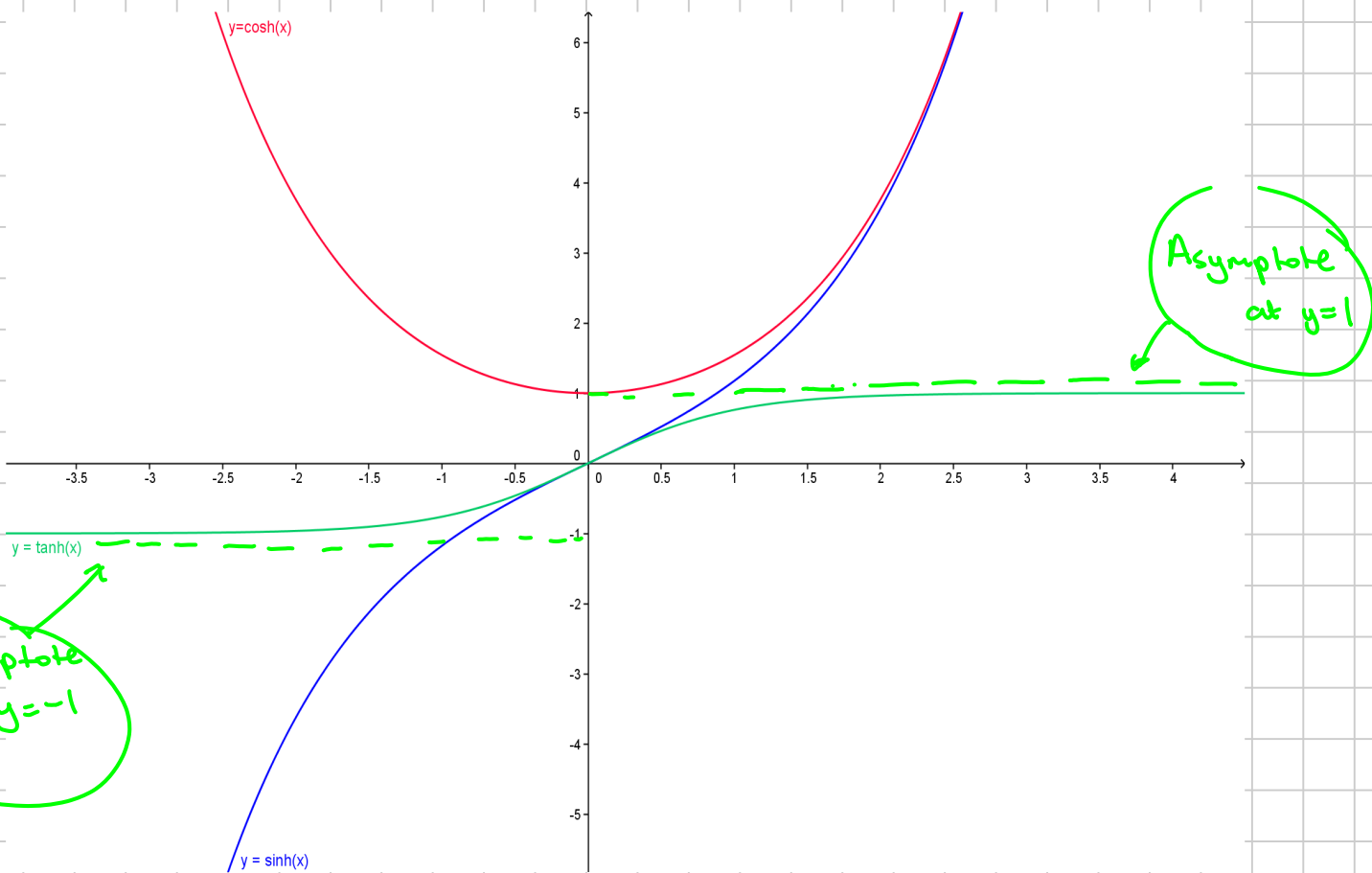
and by subtracting,

In the same way as for trig functions, we can define

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\operatorname{coth} x$ as well.

Graphs



Unlike the trig functions, hyperbolic functions are not PERIODIC. But there are some similarities:

$$\cosh 0 = \cos 0 = 1$$

and $\cosh x$, like $\cos x$, is SYMMETRICAL about the y-axis.

$$\sinh 0 = \sin 0 = 0$$

and $\sinh x$, like $\sin x$, has ROTATIONAL SYMMETRY

These look like trig identities, but to PROVE them use the definitions in terms of e^x

e.g ① Prove that $\sinh 2x = 2 \sinh x \cosh x$

$$\begin{aligned} \text{RHS} &= \cancel{2} \frac{(e^x - e^{-x})}{\cancel{2}} \frac{(e^x + e^{-x})}{2} \\ &= \frac{1}{2} (e^{2x} + 1 - 1 - e^{-2x}) \\ &= \text{LHS} \end{aligned}$$

② Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$$\begin{aligned} \text{RHS} &= \frac{(e^x + e^{-x})}{2} \frac{(e^y + e^{-y})}{2} + \frac{(e^x - e^{-x})}{2} \frac{(e^y - e^{-y})}{2} \\ &= \frac{1}{4} [e^x e^y + \cancel{e^x e^{-y}} + \cancel{e^{-x} e^y} + e^x e^y - \cancel{e^x e^{-y}} - \cancel{e^{-x} e^y} + e^{-x} e^{-y}] \\ &= \frac{1}{4} [2e^{x+y} + 2e^{-(x+y)}] \\ &= \frac{1}{2} (e^{x+y} + e^{-(x+y)}) \\ &= \text{LHS} \end{aligned}$$

Note that ① is identical to the corresponding trig identity, but in ② the sign in the middle is changed. This illustrates OSBORN'S RULE, which says that

Hyperbolic identities exactly match the trig equivalents, EXCEPT that if a term contains a product of two sines, the sign of that term is changed.

Notes:

- ① The product of sines may be $\sin^2 x$, or concealed as $\tan^2 x$ ($= \frac{\sin^2 x}{\cos^2 x}$)
- ② Of course Osborn's rule doesn't PROVE identities - just helps to remember them.
- ③ For the reason behind Osborn's rule, see notes on Complex Numbers II (later this term).

Equations

We can use identities to simplify, then convert to e^x form.

e.g. $3 \sinh 2x - 2 \cosh^2 x = 2 \cosh x$

$$6 \sinh x \cosh x - 2 \cosh^2 x - 2 \cosh x = 0$$

$$\cosh x (3 \sinh x - \cosh x - 1) = 0$$

either $\cosh x = 0$ (not possible)

or $\frac{3}{2} (e^x - e^{-x}) - \frac{1}{2} (e^x + e^{-x}) - 1 = 0$

$$2e^x - 4e^{-x} - 2 = 0$$

(multiply by e^x)

$$2e^{2x} - 4 - 2e^x = 0$$

(let $y = e^x$)

$$2y^2 - 2y - 4 = 0$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

$$e^{2x} = 2 \quad \text{or} \quad e^x = -1 \quad (\text{not possible})$$

$$\underline{\underline{x = \ln 2}}$$

More derivatives

We have seen that $\frac{d}{dx} (\sinh x) = \cosh x$

$$\frac{d}{dx} (\cosh x) = \sinh x$$

Also, $\frac{d}{dx} (\tanh x) = \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right)$ (by quotient rule)

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \quad (\text{but } \cosh^2 x - \sinh^2 x = 1)$$

like $\cos^2 x + \sin^2 x = 1$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x$$

Similarly, $\frac{d}{dx} (\coth x) = -\operatorname{cosech}^2 x$.

$$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx} (\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

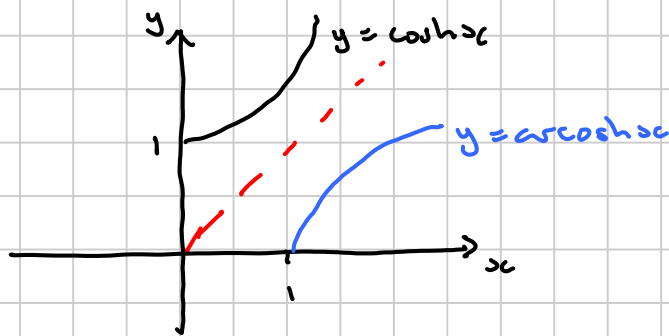
} Same as trig except for one - sign.
No need to learn these - if you need one, work it out.

Inverse hyperbolic functions

These are given the prefix "ar", rather than "arc" as for trig functions, e.g. $y = \sinh x \Leftrightarrow x = \operatorname{arsinh} y$
(or $x = \sinh^{-1} y$)

$\sinh x$ is a one-to-one function with domain \mathbb{R} and range \mathbb{R} , so $\operatorname{arsinh} x$ is likewise

$\cosh x$ is many-to-one, so we need to restrict its domain to $x \geq 0$; the range is $\cosh x \geq 1$.
So $\operatorname{arcosh} x$ has domain $x \geq 1$, and range $\operatorname{arcosh} x \geq 0$



$\tanh x$ is one-to-one with domain \mathbb{R} and range $-1 < \tanh x < 1$.

So $\operatorname{artanh} x$ has domain $-1 < x < 1$ and range \mathbb{R}

Just as hyperbolic functions can be written in terms of e^x , their inverse can be written in terms of $\ln x$.

$$\begin{aligned} \text{e.g. } y = \operatorname{arcosh} x &\Rightarrow x = \cosh y \\ &\Rightarrow x = \frac{1}{2}(e^y + e^{-y}) \end{aligned}$$

$$\Rightarrow e^y - 2x + e^{-y} = 0$$

$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow (e^y - x)^2 - x^2 + 1 = 0$$

$$\Rightarrow e^y - x = \pm \sqrt{x^2 - 1}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

(multiply by e^y)
(complete the square)

But $x - \sqrt{x^2 - 1} < 1$, so this would make y negative. But we have defined $\operatorname{arcosh} x$ to be positive, so

$$\Rightarrow y = \ln(x + \sqrt{x^2 - 1})$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

Similarly we can show that $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$ ($x \in \mathbb{R}$)

and $\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ ($-1 < x < 1$)

Example Solve $\cosh^2 x - 3 \cosh x - 10 = 0$, giving the answer as a natural logarithm.

[We could convert $\cosh x$ into $\frac{1}{2}(e^x + e^{-x})$ as in the example above, but that would take longer.]

$$(\cosh x - 5)(\cosh x + 2) = 0$$

$$\cosh x = 5 \quad \text{or} \quad \cosh x = -2 \quad (\text{not possible})$$

$$x = \operatorname{arcosh} 5$$

$$= \ln(5 + \sqrt{24})$$

Derivatives of inverse trig and hyperbolic functions

To find these, use the rule

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

e.g.

$$y = \operatorname{arcsin} x \Rightarrow x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

(but we want $\frac{dy}{dx}$ in terms of x ,

so use $\cos^2 y + \sin^2 y = 1$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Similarly, we can show that

$$\frac{d}{dx} (\operatorname{arccos} x) = \frac{-1}{\sqrt{1 - x^2}}$$

and (using $\sec^2 y = 1 + \tan^2 y$)

$$\frac{d}{dx} (\operatorname{artanh} x) = \frac{1}{1 - x^2}$$

e.g $y = \operatorname{arsinh} x \Rightarrow x = \sinh y$
 $\Rightarrow \frac{dx}{dy} = \cosh y$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

But $\cosh^2 y - \sinh^2 y = 1$
 $\Rightarrow \cosh y = \sqrt{1 + \sinh^2 y}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$

Similarly, $\frac{d}{dx} (\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2 - 1}}$

$$\frac{d}{dx} (\operatorname{artanh} x) = \frac{1}{1 - x^2}$$

[These are mainly useful in integration - see "Further Integration"]