9 PHYSICAL STRUCTURES

Objectives

After studying this chapter you should

- be able to analyse the forces and turning effects on simple structures and frameworks;
- be able to find the centre of mass of simple and composite bodies;
- be able to analyse the stability and toppling conditions of a simple body.

9.0 Introduction

The objects in the list opposite may not at first seem to have much in common but they are all examples of structures. They either hold up or support something, reach across or span a distance or contain or protect something. They all, however, in some way support a load, so a structure can be thought of as an assembly of materials so arranged that loads can be supported.

a chair and occupant;
a dam;
the Severn Road Bridge;
a cable-stayed roof;
a ballet dancer balancing on one foot;
a snow-laden tree;
a spider's web with a fly trapped in it.

The chair both holds you up and supports your weight. The dam contains and supports the water behind it. The bridge spans the River Severn and supports both its own weight and that of the traffic. The roof protects the inside of the building and has its weight supported. The ballet dancer's weight is supported by one foot. The tree trunk supports the branches as well as the extra load of snow. The spider's web contains and supports the fly.

The weight of the load the structure supports needs to be transferred to the ground in some way. The chair supports your weight but the ground supports both your weight and the chair's, this load being transferred to the ground via the chair legs.

Your body's skeleton is a structure. What are its purposes?

Activity 1 Investigating structures

What are the purposes of the structures listed opposite?

What are the loads on them?

How are these loads supported?

a tower crane;
a suspension bridge;
a spider's web;
a diving board;
a North Sea oil platform;
a wheelbarrow;
a trellis with creeper.

As the examples in Activity 1 show, structures are very widespread. Some are natural, some man-made. They are made from a variety of materials: stone, wood, iron, steel, brick, reinforced concrete, bone and tissue being common ones.

The book of Genesis in the Bible describes an early structure of brick, the Tower of Babel. The pyramids in Egypt, the Parthenon in Athens, St Peter's in Rome, the Taj Mahal in India, great cathedrals like Durham, York and Salisbury and castles like Stirling and Harlech, the Iron Bridge in Shropshire, are all structures built many years ago and still standing today. Structures like these satisfy the basic requirement that they must not break or fail.

Not all structures succeed in this. Trees can break under the weight of snow on them. Bridges such as the old Tay Railway Bridge in Scotland and the Tacoma Narrows Bridge in the United States have collapsed. The roofs of medieval cathedrals did sometimes fall in. Babylonian houses collapsed and the Code of Hammarabic in about 2000 BC made the builder pay with his life if the collapse killed the owner.

Structures fail when a part breaks or is permanently disturbed in some way. This can happen because the forces on the structure become too great for it to bear. The forces may be due to the weight of whatever the structure is carrying or supporting or to its own weight. They may also be due to external forces such as wind and snow or the impact of some object on the structure. Wind causes the Eiffel Tower in Paris to sway up to 12.7 cm and trees are blown over in strong gales. Cars in collision are damaged.

Since it is important that a structure achieves its purpose and does not fail, man-made structures need to be designed carefully. However, the designer needs to take account not only of safety but also of the cost and method of construction and the appearance of the structure. A bridge built from the opposite banks of a river must meet in the middle. A skyscraper is out of place in the countryside.

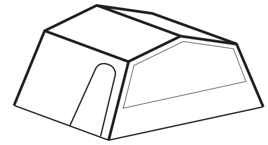
Mathematics plays an important part in the design of structures, especially in their mechanics. The aim of this chapter is to find out something of how mathematics is applied via mechanics in the design of structures and how it can also be used to help understand why man-made and natural structures are the way they are.

9.1 Introducing frameworks

A frame tent consists of a frame and a covering. The frame is usually made from aluminium poles which are strong but extremely lightweight. The complete frame is very light compared to the loads it supports. The load is not simply the force on the frame due to the weight of the covering, but also the forces due to the wind and the rain to which the tent might be subjected.

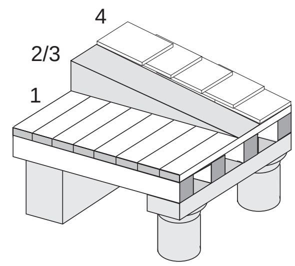
A framework is a structure which consists of a system of connected members.

In the case of the frame tent, the system of connected members is the frame of poles. A framework is designed to support a weight or load, and generally it is very light compared to the load it supports.



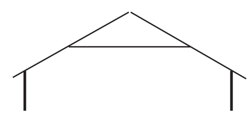
A second example is the roof truss, the framework which supports a roof. In fact, roof trusses have not always been used to put pitched roofs on to buildings. The following method was used by the ancient Greeks to build their temples:-

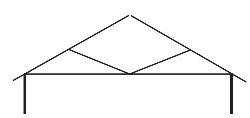
1. Put a flat timber roof over the walls and a large number of supporting pillars.
2. Heap thousands of tonnes of earth on to this.
3. Shape the earth to the desired pitch.
4. Lay tiles directly on to this earth.



This crude but interesting method was superseded by the development of the roof truss.

The A-shaped roof truss is a very simple example of a framework. It is light, yet rigid and strong enough to support the roof covering. Several A-shaped roof trusses spanning the walls of a building could support a thatched or tiled roof on horizontally attached laths. The modern roof truss is very little altered from its early counterpart, although a considerable variety of designs are now used.





Not all frameworks are man-made. There are plenty of examples of natural frameworks such as animals' skeletons and spiders' webs.

Activity 2   Finding frameworks

Discuss and list other examples of frameworks.

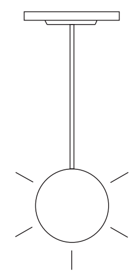
Look for small examples such as shelf brackets as well as larger ones such as electricity pylons.

Try also to find buildings in which frameworks have been made into features rather than hidden. Naturally occurring frameworks are less obvious, but try to find some of these also.

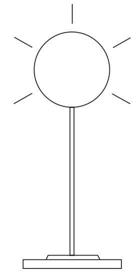
Many of the frameworks you find in Activity 2 are likely to be three dimensional as are the frame tent and skeleton examples. In this section you analyse only two dimensional frameworks but should be aware that three dimensional ones are probably more common.

Ties and struts

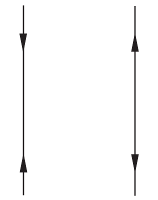
The frameworks you have considered so far each consist of rigid members joined at their ends to form a structure which does not collapse or move. The members used are often metal bars or wooden struts. It is important to know how these behave when the framework is loaded.



The first diagram opposite shows a light fitting suspended from the ceiling by a metal rod. It could be replaced by something flexible like a chain or string and still support the light fitting. The force in the rod supporting the light fitting is a **tension**.



The second diagram shows the same light fitting made into a standard lamp. The weight of the light fitting now tends to compress the rod. In this case it could not be replaced by something flexible and still support the light fitting. The force in the rod supporting the light fitting in this case is a **thrust**.



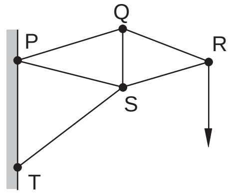
When a rod is in **tension** it is said to be a **tie**.

When a rod is in **thrust** it is said to be a **strut**.

When any framework is subjected to a load, some members will be in tension and act as ties; others will be in thrust and act as struts.

Example

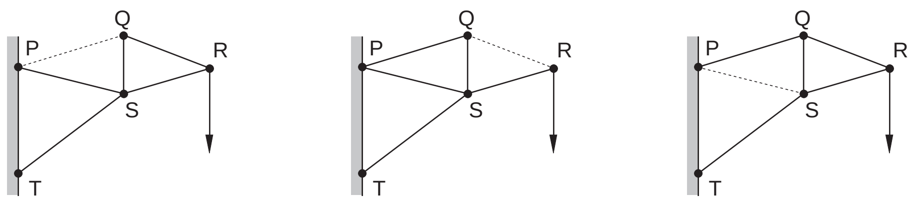
For the framework PQRST, say whether the light rods are ties or struts.



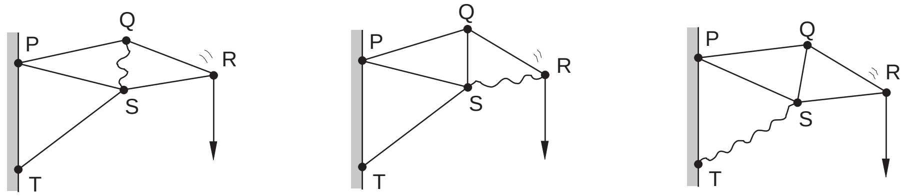
Solution

A good strategy is to consider each of the rods in the framework removed in turn.

If the rod could be successfully replaced by string, then it is in tension, a tie. If its replacement by string would result in collapse of the framework, then the rod is in thrust, a strut.



The rods PQ, QR and PS could all be successfully replaced by string.

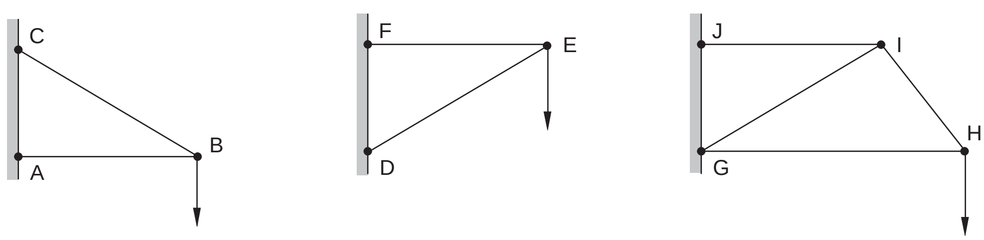


The rods QS, SR and ST could not be successfully replaced by string.

PQ, QR and PS are ties. QS, SR and ST are struts.

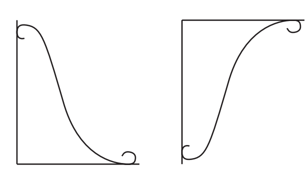
Activity 3 Ties and struts

For these simple frameworks, say whether the light rods are ties or struts.

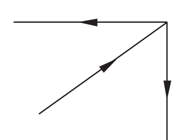


Compare the two simple frameworks of hanging basket brackets shown opposite. Discuss how they differ in terms of which members are ties and which are struts.

Look for other simple frameworks to analyse in the same way.



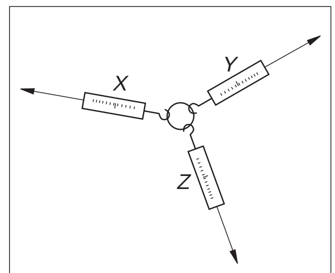
Even the simplest of frameworks often have joints at which there are three or more rods connected.



Forces in equilibrium

When a framework does not collapse or move supporting a load, then the forces in the rods acting at its joints must be in equilibrium. If a framework is to be designed to support a load, it is necessary to find the forces which act in its members. This is known as **analysing the framework**. In order to analyse a framework, you need to be able to solve problems concerning the equilibrium of forces. One condition for a system of forces to be in equilibrium is that their resultant is zero. This idea was first discussed in Section 4.5 and is now explored. Often for frameworks the forces act at a point.

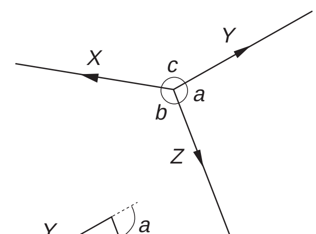
Activity 4 Equilibrium of forces at a point



You will need 3 newton meters, a metal ring (for instance, a key ring) and paper.

Hook each newton meter into the metal ring as shown opposite and pull on the end of each.

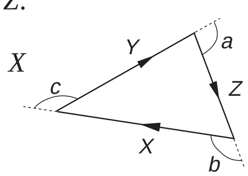
When the ring is in equilibrium, that is, it does not move, mark the positions of the centre of the ring and the handle end of each newton meter.



Note also the readings X, Y and Z on the newton meters.

Remove the newton meters and use the marks on the paper to find the angles a, b and c between the forces X, Y and Z.

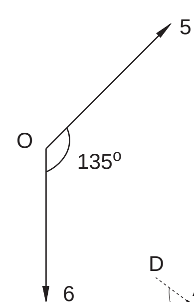
Draw a scale diagram of the forces X followed by Y followed by Z.



What do you notice about the positions of the beginning of force X and end of force Z on your scale diagram?

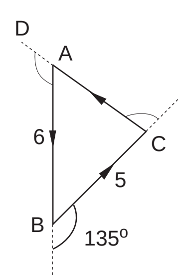
Can you explain the result in terms of the triangle of forces?

Example



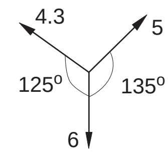
Two forces, in newtons, act at the point O as shown in the diagram opposite. Find, by scale diagram, the magnitude and direction of the third force acting at O which maintains equilibrium.

Solution



Choose a scale so that the diagram fits comfortably on the page; 1 cm for 1 N is satisfactory in this case. Draw the two given forces, one followed by the other.

The third force at O required to maintain equilibrium can be measured on your scale drawing and is $4.3\text{ N at }125°$ to the 6 N force.



There are a number of alternative methods of solving problems such as those in the last example.

You know how to resolve a force into two components at right angles using the parallelogram law (see Sections 4.2 and 4.3).

This can be used as the basis of one alternative method for solving problems involving the equilibrium of forces acting at a point. The last example shown is reconsidered using this method.

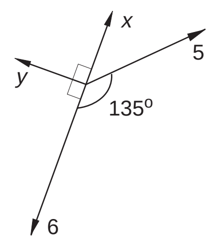
Alternative solution

Let the required force have components X and Y opposite to and at right angles to the 6 N force.

The sum of the forces in the direction of X is zero,

giving

$$X + 5\cos 45° - 6 = 0$$
$$X = 6 - 5(0.707)$$
$$X = 2.46.$$



The sum of the forces in the direction of Y is zero,

giving

$$Y - 5\cos 45° = 0$$
$$Y = 5(0.707)$$
$$Y = 3.54.$$

The required force is then

$$\sqrt{X^2 + Y^2}$$
$$= 4.31\text{ N}.$$

Its angle to the 6 N force is

$$180 - \tan^{-1}\left(\frac{Y}{X}\right)$$
$$= 180 - \tan^{-1}(1.43)$$
$$= 180 - 55.1$$
$$= 124.9°$$

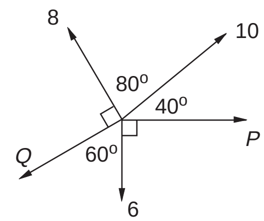
The technique used in the above example relies upon forces in equilibrium at a point having zero resultant. It then follows that:

If forces are in equilibrium at a point, then the sum of the components of the forces in any one direction is zero.

By using this result in two directions at right angles, it is possible to set up two equations and solve for two unknowns.

Example

The forces P, Q, 6, 8 and 10 N in the diagram opposite are in equilibrium. Find forces P and Q.

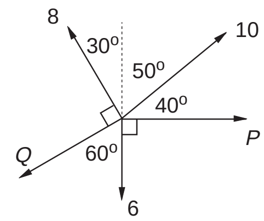


Solution

The sum of the forces in the direction at right angles to P is zero, giving

$$6+Q\cos 60°-10\cos 50°-8\cos 30°=0.$$

This is known as **resolving** in the direction at right angles to P. The reason for starting with this direction is that it leads to an equation in Q only. It simplifies to

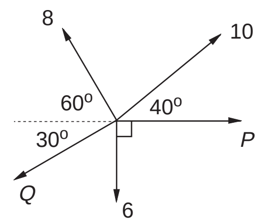


$$0.5Q=10\cos 50°+8\cos 30°-6$$

giving $\quad 0.5Q=7.36$

so that $\quad Q=14.72$.

The sum of the forces in the direction of P is zero and so resolving in this direction gives



$$P+10\cos 40°-Q\cos 30°-8\cos 60°=0.$$

Substituting the value of Q gives

$$P=14.72\cos30°+8\cos60°-10\cos40°$$

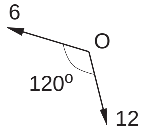
or $\quad P=9.08.$

The forces are $\quad P=9.08$ N and $Q=14.72$ N.

Exercise 9A

1. In each case, find by scale diagram the force at O required to maintain equilibrium. Forces are in newtons.

(a)



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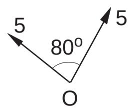
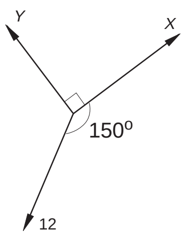


2. If each set of forces is in equilibrium, find the unknown forces. Forces are in newtons.

(a)



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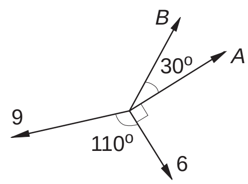

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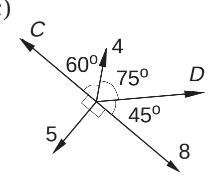

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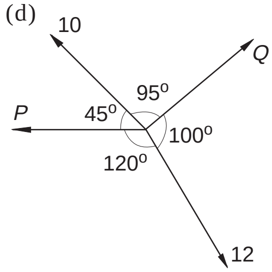
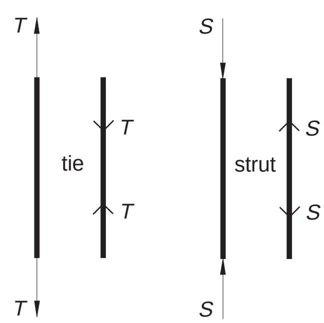


9.2 Analysis of a simple framework

The analysis of a simple framework involves finding the forces acting in each of its members when the framework is subject to a load. This would then allow a suitable material and cross-section to be chosen for each member so that it could withstand the calculated forces. So this analysis is part of the design process for the framework.



Some assumptions have to be made in the force analysis of a framework. Firstly, it is assumed that all members are two-force members. That is, each is in equilibrium under the action of two equal and opposite forces applied at its ends.

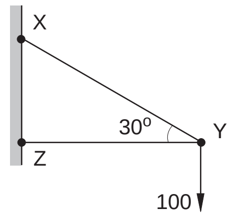
These two forces are tensions *T* in the case of a tie, or thrusts *S*, in the case of a strut. Equal and opposite forces to those applied at its ends act within the tie or strut to maintain equilibrium.

Secondly, it is assumed that the weight of the framework is so small compared to the load it supports that this weight can be neglected. For individual members of the framework, this means that the weight of a member is so small that it can be neglected compared to the force it supports.

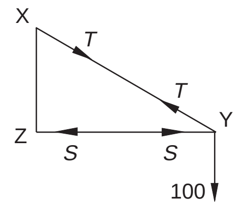
Thirdly, it is assumed that whether the members of the framework are bolted, riveted or welded to each other at joints, these joints are points in equilibrium due to the action of the forces in those members. It is for this reason that the method of resolving at a point can be used to find the forces in the members of a framework.

Example

For the simple framework XYZ supporting a load of 100 N at Y, find the forces in its members XY and YZ. Find also the forces exerted by the framework on the wall at X and Z.

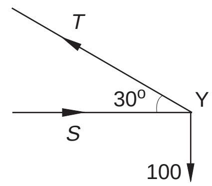


Solution



By inspection, there is a tension, T, in XY and a thrust, S, in YZ.

Since the point Y is in equilibrium under the action of the three forces T, S and 100 N, resolving vertically gives



$$T\cos 60° - 100 = 0$$

so that

$$0.5T = 100$$

$$T = 200$$

and resolving horizontally gives

$$S - T\cos 30° = 0$$

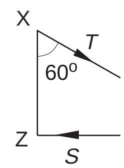
so that

$$S = 200(0.866)$$

$$S = 173.$$

The member XY is a tie supporting a tension of 200 N

and YZ is a strut supporting a thrust of 173 N.



The force exerted by the framework on the wall at X is T, that is 200 N acting away from the wall at 60° to the downward vertical.

The force exerted by the framework on the wall at Z is S, that is 173 N acting towards the wall at 90° to it.

Note: If the directions for S and T are chosen incorrectly in the first part of this solution, it does not matter. The values of S and T would work out negative from the equations indicating that the directions are opposite to those chosen.

Exercise 9B

1. Find the forces in the members PQ and QR in each case and the forces exerted by the framework on the wall at P and R. Forces are in newtons.

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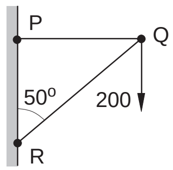

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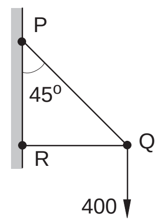

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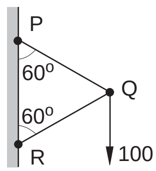
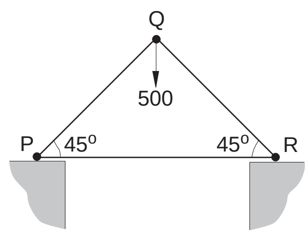
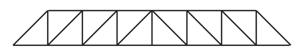


2. Framework PQR rests on the supports P and R as shown. It is not fixed. Find the forces in the members PQ, QR and PR and the normal contact forces exerted by the supports at P and R.



9.3 More complex frameworks

More complex frameworks are often found in bridge trusses as well as in roof trusses. They are often named after the engineers who first designed them. When the railroads pushed westwards in North America, wooden truss bridges were used to span the wide rivers of the West and a considerable number of American engineers invented trusses. Two bridge trusses and two roof trusses are shown on the right.



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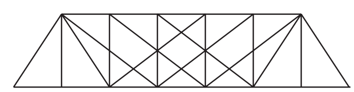
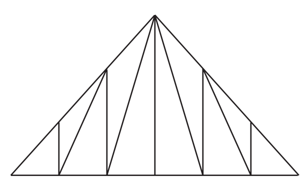
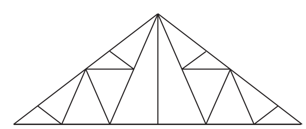


Whipple - Murphey



Pratt



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Activity 5 Bridge and roof trusses

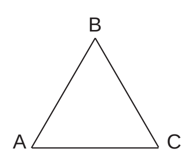
You will find bridge trusses on railway and foot bridges. Many modern buildings have their roof trusses painted as a feature, rather than hidden.

Find and sketch some roof and bridge trusses different from the four named ones.

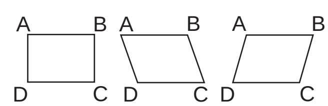
See if you can also research the names of the extra trusses you find.

Perfect trusses

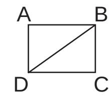
You should realise that a triangle of rods is a rigid structure, whereas a square is not.



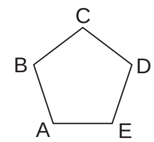
ABC cannot move.



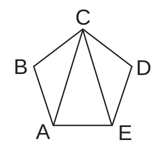
If one extra rod is added to ABCD it becomes rigid as shown opposite.



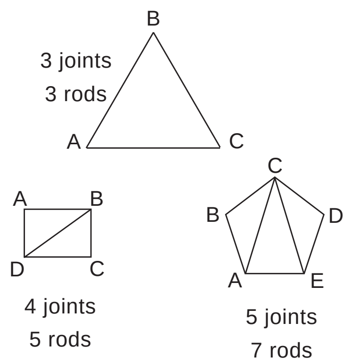
A pentagon of rods can move.



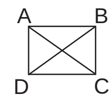
If two extra rods are added to ABCDE, it becomes rigid as shown opposite.



The three examples shown opposite are of **correctly triangulated trusses**. They have just enough rods so that they are rigid. They are also known as **perfect trusses**.



If two extra rods had been added to the square, it would have been rigid, but one rod would have been unnecessary. It would not have been a perfect truss.



Activity 6 The perfect truss

You can use geostrips and connectors to try out the ideas in this activity, or simply treat it as a pencil and paper activity.

Investigate the least number of rods necessary for a perfect truss of 3, 4, 5, 6, 7 ..., joints.

Can you generalise the result and find a formula for the least number of rods necessary for a perfect truss of *n* joints?

Look at the four named trusses and others you have sketched. Are they perfect trusses?

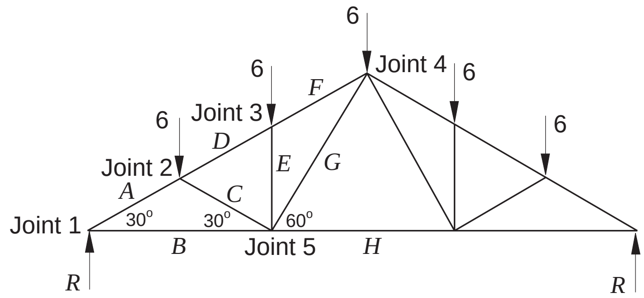
Suggest some reasons why some of the roof and bridge trusses you have found are not perfect trusses.

9.4 Method of joints

The method used to analyse frameworks in Exercise 9B is a simple case of the method of joints. This method can be used to analyse the forces in more complicated frameworks.

Example

Find the forces in all the members of the roof truss shown in the diagram.



The load due to the roof takes the form of five separate 6 kN forces acting vertically as shown. The truss is supported by two vertical contact forces.

Solution

The framework and the loads acting upon it are symmetric, so the contact forces are equal, to R say, and only the joints on one side of the line of symmetry need to be considered.

The complete roof truss is in equilibrium and so resolving vertically gives

$$2R - 30 = 0$$

so that $$R = 15.$$

Since the framework is in equilibrium, the forces acting at each joint are in equilibrium. Joints 1, 2, 3, 4 and 5 are taken in turn to find the forces A, B, C, D, E, F, G and H in the members as shown.

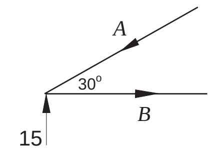
The directions of A and B are reasonably obvious here and so are inserted as shown in the first diagram on the next page.

If the direction of a force is not obvious, simply choose a direction and find its value. If it turns out to be negative, you know the direction is opposite to that chosen.

Joint 1

Resolving vertically gives

Joint 1



$$A \sin 30° = 15$$

so that $\quad A = 30.$

Resolving horizontally gives

$$B = A \cos 30°$$

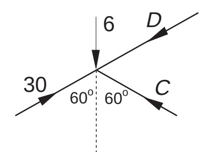
so $\quad B = 30(0.866)$

and $\quad B = 26.0.$

Joint 2

Resolving vertically gives

Joint 2



$$6 + D \cos 60° = 30 \cos 60° + C \cos 60°$$

so $\quad 12 + D = 30 + C$

or $\quad D - C = 18. \qquad (1)$

Resolving horizontally gives

$$30 \cos 30° = D \cos 30° + C \cos 30°$$

so $\quad D + C = 30. \qquad (2)$

Solving equations (1) and (2) gives

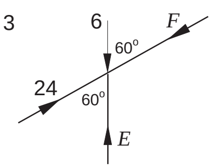
$$D = 24$$

and $\quad C = 6.$

Joint 3

Resolving vertically gives

Joint 3



$$6 + F \cos 60° = E + 24 \cos 60°$$

so $\quad 12 + F = 2E + 24$

and $\quad F - 2E = 12. \qquad (3)$

Resolving horizontally gives

$$24 \cos 30° = F \cos 30°$$

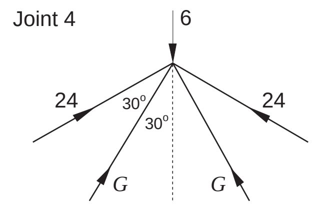
so $\quad F = 24$

and from equation (3)

$$E = 6.$$

Joint 4

The direction of G is not obvious but is inserted as shown.



Resolving vertically gives

$$6 = 2(24\cos 60) + 2G\cos 30$$

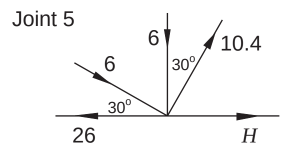
so $$6 = 24 + 1.73G$$

and $$G = -10.4.$$

This shows that G is in the opposite direction to that on the diagram.

Joint 5

Resolving horizontally gives



$$H + 10.4\cos 60 + 6\cos 30 = 26$$

so $$H + 5.2 + 5.2 = 26$$

and $$H = 15.6.$$

So the forces in the members are:

$$A = 30,\quad B = -26.0,\quad C = 6,\quad D = 24,\quad E = 6,$$
$$F = 24,\quad G = -10.4 \text{ and } H = 15.6,\quad \text{all in kN.}$$

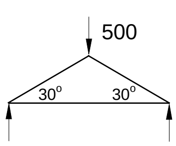
Negative signs indicate tensions in ties.
All others are thrusts in struts.

Exercise 9C

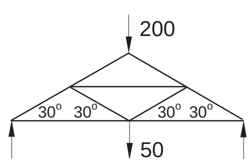
In Questions 1 to 4, calculate the force in each rod of the framework together with any vertical contact forces indicated. All forces are in newtons and each framework can be assumed to be in a vertical plane.

Questions 3 and 4 represent parts of a truss as used in some bridges.

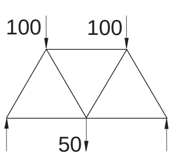
1.



2.

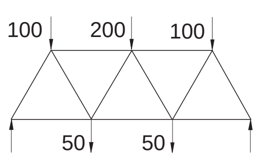


3.



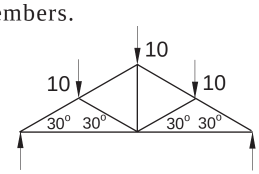
rods all of equal length

4.



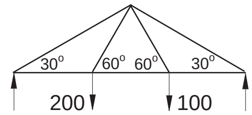
rods all of equal length

5. The roof truss in the diagram below supports a roof which can be considered to act as three separate loads of 10 kN as shown. Find the two vertical contact forces and the forces in all the members.



6. The framework shown below is subject to the loads 200 N and 100 N. Two vertical contact forces support the framework. Find the contact forces and the force in each member of the framework.

(Hint: Although the framework is symmetric, it is not loaded symmetrically. All the jointsin the framework will need to be considered.)



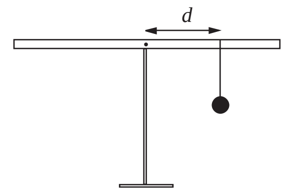
9.5 Lifting devices

The list opposite gives examples of lifting devices, some of which you meet in this section.

a see-saw;
a wheelbarrow being pushed;
a pair of scissors;
a fishing rod;
the arm of a shot putter;
a discus thrower;
a road barrier;
a crane.

Activity 7   Balancing weights

You need a stand, a metre rule with a hole at 50 cm, Blu-tack, masses (3×10 g, 50 g, 100 g) and cotton to suspend masses.



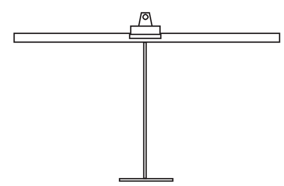
Pivot the rule at its centre on the stand, and if necessary use Blu-tack to balance it.

Suspend one mass on the left hand side of the pivot and two masses on the right hand side, so that the rule balances.

Use your data to deduce a rule which relates weights of the masses and distances from the pivot.

Verify your rule by suspending different combinations of masses on either side of the pivot.

Note the weights and the distances of their points of suspension from the pivot.



[If a metre rule with a hole in it is not available, use a bull-dog clip and stand. Take measurements from the centre of the rule. This is the point referred to as the pivot in the text.]

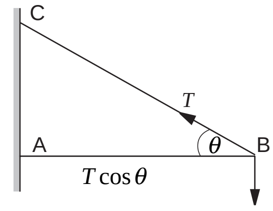
You should have found that the number

$$\text{weight} \times \text{distance from the pivot}$$

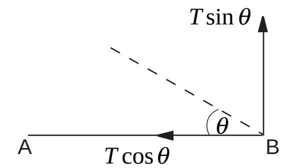
is important in this balance problem. This is called the **moment** of the weight about the pivot and measures the turning effect the weight produces about that point. Weights on opposite sides of the pivot balance if their moments about the pivot are equal.

9.6 Moment of a force

In Activity 7 the line from the pivot to the point of suspension of each mass is perpendicular to the direction in which its weight is acting. In the framework shown opposite, the weight suspended at B is acting vertically downwards and so its direction is perpendicular to the rod AB. The moment about A is (weight) × AB. The tension, T, in the tie BC acts at an angle, θ, to AB. What is its moment about A?



The tension, T, can be resolved into its components $T\sin\theta$ perpendicular to AB and $T\cos\theta$ along BA. Since the component $T\cos\theta$ acts along BA, it does not have any turning effect about A. The component $T\sin\theta$ does, however, have a turning effect about A. It is measured by its moment



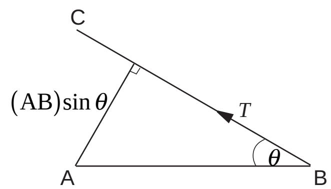
$$(T\sin\theta)\times(\text{AB}).$$

The earlier definition is therefore modified to

Moment of a force acting at point B about point A	=	(component of force perpendicular to AB) × (distance AB)

Since

$$(T\sin\theta)\times\text{AB} = T\times(\text{AB}\sin\theta)$$
$$= T\times\begin{pmatrix}\text{perpendicular distance from A to}\\ \text{the line BC along which } T \text{ acts}\end{pmatrix},$$



an alternative definition of moment is

The moment of a force acting at point B about point A is equal to the force times the **perpendicular** distance from A to the line along which the force acts.

Since

$$\text{moment} = (\text{force}) \times (\text{distance})$$

moments are measured in newton metres (Nm).

In Activity 7 two other forces act: the weight of the rule and the normal contact force at the pivot.

Why do these forces not come into the earlier calculations?

The weight of the rule acts through its midpoint. You can check this by balancing the rule about its midpoint on your finger. The only forces on the rule are its weight and the force on it due to your finger. Since the force due to your finger acts at the midpoint, for these forces to balance, the weight must also act there and so its moment about the midpoint, that is the pivot, is zero. The normal contact force also acts at the pivot. Its moment about that point is zero and so neither of these forces contributes to the earlier calculations.

The point in a body through which the weight acts is called the **centre of gravity**. This idea is developed further in Section 9.10. For now, note that

The centre of gravity of a uniform rod is at its midpoint.

Moments arise in many situations. When you open a door you apply a force to the knob or handle and there is a turning effect and so a moment about the hinge. If you have two spanners, one longer than the other, and you apply the same force to both, you achieve a larger turning effect with the longer spanner. Some of you may have been boating on the canals and had to manoeuvre lock gates. Here you usually push against the gates so that you are perpendicular to the bars and walk in a circular path. In this way you achieve the maximum turning effect.

In the human body through muscle contraction, forces are applied at points where tendons are attached to the bones and moments arise about the joints.

To put a screw into a piece of wood you turn the screwdriver clockwise, whereas to take it out you turn the screwdriver anticlockwise. This suggests that moments are either in a clockwise or an anticlockwise sense.

Moments are taken to be **positive** when in the **anticlockwise** sense and **negative** when in the **clockwise** sense.

In each case the force, F, has a turning effect about a line through P, perpendicular to the plane of the paper. The direction of this line is associated with the moment to give a vector quantity.

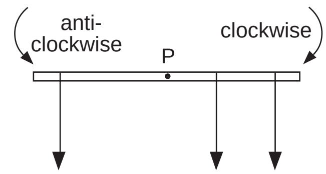
Activity 8 Moments

It is obvious why the door handle is as far as possible from the hinge and which line the door turns about.

List other situations in which moments arise.

Identify the line about which there is a turning effect.

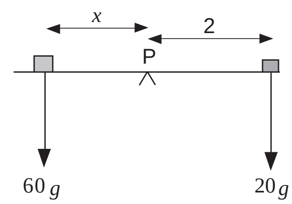
Principle of moments



In Activity 7 you found that the rule only balances when the moments of the weights suspended on one side of the pivot are equal to those on the other side. The weights on the right hand side of P have negative moment whereas those on the left hand side have positive moment and so an equivalent statement is that the ruler balances at P when the algebraic sum of the moments of the weights about P is zero. This is an example of the **principle of moments**, one form of which is

> For a body in equilibrium under a system of forces acting at different points in the body, the algebraic sum of moments of the forces about **any** point is zero.

Example



A see-saw pivoted at its centre rests in a horizontal position. A boy of mass 20 kg sits on it 2 m from the pivot. How far from the pivot should his father of mass 60 kg sit if the see-saw is to balance?

Solution

The boy has a negative moment about P

$$= -20g \times 2 = -40g.$$

His father has a positive moment about P

$$= 60g \times x = 60gx.$$

By the principle of moments

$$-40g + 60gx = 0,$$

so that

$$40g = 60gx$$

giving

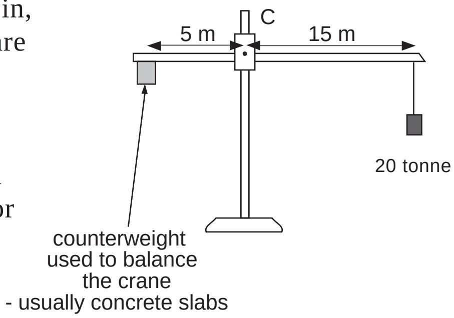
$$x = \frac{40}{60} = \frac{2}{3} \text{ m.}$$

What other forces act on the see-saw? Why are they not considered?

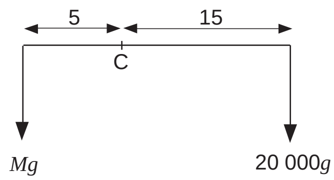
Example

In the tower crane shown opposite, the distances from the cabin, C, of the suspended mass of 20 tonne and the counterweight are as shown. Assuming that the mass of the crane can be neglected compared with the suspended mass and the counterweight, determine the mass of the counterweight necessary for the crane to be in equilibrium. If the suspended mass is now trebled, determine where it must be positioned for the crane to be in equilibrium with the same counterweight.

[1 tonne $\equiv$ 1000 kg]



Solution



If the counterweight has mass M, applying the principle of moments about C gives

$$-15 \times 20 \times 1000g + 5Mg = 0,$$

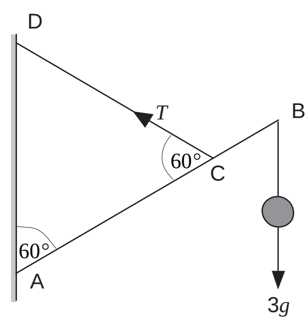
so that

$$M = 60 \times 10^3 \text{ kg}$$
$$= 60 \text{ t.}$$

If the suspended mass is now 60 t, the same as the counterweight, for equilibrium it must be positioned 5 m to the right of the cabin for the crane to be in equilibrium.

Example

A 40 cm long rod, AB, has a 3 kg mass hanging from B. It is hinged at A and is supported by a chain CD. If the masses of the rod and chain can be neglected compared with the suspended mass, find the tension in the chain if ACD is an equilateral triangle of side 30 cm and D is vertically above A.



Solution

In Section 9.2 you solved this type of problem using resolution of forces. Here is an alternative method using the principle of moments.

The forces acting on the rod are the tension, T, in the chain, the suspended weight and the force exerted by the hinge at A. Applying the principle of moments at A, the force at A does not come into the calculation since it has zero moment about A. Neither T nor the suspended weight is perpendicular to the rod, but the components of these two forces perpendicular to the rod are $T\cos 30$ and $3g\cos 30$ respectively.

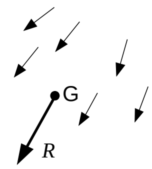
Applying the principle of moments about A gives

$$T\cos 30 \times 30 - 3g\cos 30° \times 40 = 0,$$

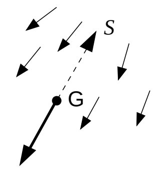
so that

$$T = 4g = 40\,\text{N},$$

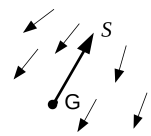
putting $g = 10\ \text{ms}^{-2}$.



A useful alternative form of the principle of moments is described in the next unit.



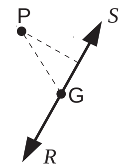
Suppose that the resultant of the system of forces has magnitude R and acts at G. Now at G introduce a force S which has magnitude R and acts in the opposite direction to the resultant. The force S cancels out the resultant force and so the original system of forces together with S is in equilibrium. By the earlier statement, the algebraic sum of the moments of all these forces about any point P is zero, so that



$$(\text{moment of } S \text{ about P}) + \left(\begin{array}{c}\text{algebraic sum of the moments of the} \\ \text{original system of forces about P}\end{array}\right) = 0$$

But

$$(\text{moment of } S \text{ about P}) = -(\text{moment of } R \text{ about P})$$



and so, combining these two statements gives

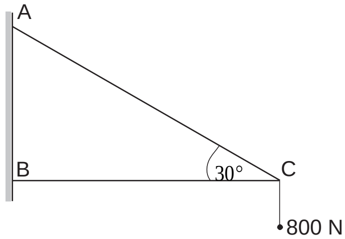
$$\text{moment of } R \text{ about P} \;=\; \begin{array}{l}\text{moments of the original} \\ \text{system of forces about P.}\end{array}$$

> The moment of the resultant of a system of forces about any point is equal to the algebraic sum of the moments of the forces about that point.

Exercise 9D

1. A see-saw of length 4 m, pivoted at its centre, rests in a horizontal position. John, who weighs 30 kg, sits on one end. Where should his friend James, who weighs 40 kg, sit if the see-saw is to balance?

2. The diagram shows two thin rods of negligible weight jointed together at C and anchored to a vertical wall at A and B. Rod BC is horizontal and rod AC makes an angle of $30°$ to the horizontal. A load of 800 N hangs from the joint at C.

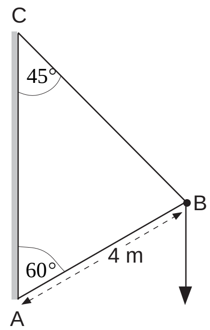


(a) Find the tension or thrust in each of the rods.

(b) Which of the rods could be replaced by a sufficiently strong length of cable without otherwise altering the structure or causing it to move?

(c) Determine the force exerted by the rod BC on the wall.

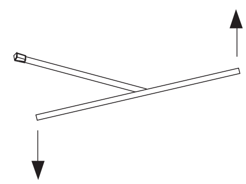
3. The figure shows a rod AB, whose weight can be neglected, hinged at A and connected to C by a cable BC. A mass of 4 tonnes is suspended from B. Calculate the tension in the cable BC. Find the thrust in the rod AB.



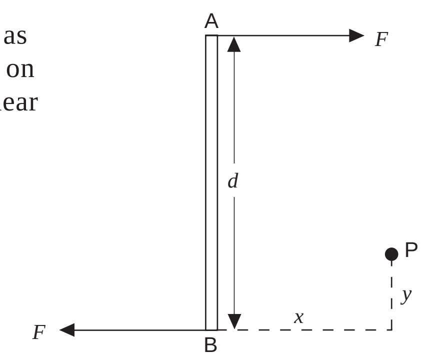
4. Repeat Question 3 with the mass suspended 3 m along the rod from A. Also determine the force exerted by the cable on the wall.

5. Susan and Alison, who weigh 30 kg and 36 kg, sit on a see-saw 1.8 m and 1.6 m respectively on the left of the pivot. The see-saw is pivoted at its centre, is 4 m long, and when unoccupied, rests in a horizontal position. Their father, who weighs 70 kg, sits to the right of the pivot.
   (a) Where should he sit for the see-saw to balance?
   (b) Does the father's position have to change significantly if the children change places with each other?
   (c) Can their father always balance the see-saw wherever the children sit on the left hand side?
   (d) Is this the case if their mother, who weighs 60 kg, changes places with their father?
   (e) The girls sit on the left of the pivot with Alison 1 m from the pivot and their mother on the opposite side. If their mother sits x m and Susan y m from the pivot, what is the relation between x and y for the see-saw to balance?

9.7 Couples



A special spanner is sometimes used to remove the nuts which hold on a car wheel. Force is applied to the arms of a T-shaped spanner and there is a turning effect on the nut; the longer the arms of the 'T', the greater the turning effect.



If two equal and parallel forces, F, act in opposite directions as shown on the diagram opposite, then there is a turning effect on the rod AB. Such a pair of forces is called a **couple**. The linear resultant in any direction is zero and there is no translational effect.

Taking moments about A gives

$$M = -Fd$$

and taking moments about B also gives

$$M = -Fd.$$

Furthermore, taking moments about P gives

$$\begin{aligned} M &= -F(d-y) - Fy \\ &= -Fd. \end{aligned}$$

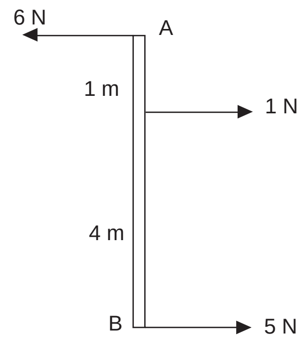
These three results show that the moment of the couple is not zero and is independent of the position of the point about which moments are taken.

To show that a couple exists, therefore, you need to establish that

- the linear resultant of the couple is zero
- the resultant moment of the couple is not zero.

Example

Show that the forces on the rod AB, shown opposite, form a couple and find the moment of this couple.



Solution

Resolving the forces perpendicular to AB gives

$$1+5-6=0,$$

thus there is no linear resultant of the forces and the forces form a couple since they do not act at a point.

Taking moments about A gives

$$\begin{aligned} M &= 1\times 1+5\times 5 \\ &= 26 \text{ Nm.} \end{aligned}$$

Alternatively, taking moments about B gives

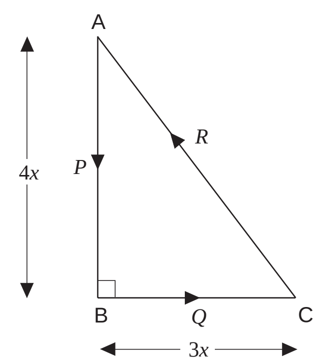
$$\begin{aligned} M &= 6\times 5-1\times 4 \\ &= 26 \text{ Nm.} \end{aligned}$$

The forces therefore form a couple with a resultant moment of 26 Nm anticlockwise.

The examples considered so far have systems of parallel forces but other frameworks with non parallel forces can be such that the forces form a couple.

Example

The diagram opposite shows a system of forces P, Q and R acting along the sides of a right-angled triangle ABC. Find the ratio $P:Q:R$ if the system of forces is equivalent to a couple.



Solution

By Pythagoras' theorem, $\text{AC}=5x$.

If the system of forces is a couple, the linear resultant is zero.

Resolving vertically,

$$R\sin \text{A}\hat{\text{C}}\text{B}-P=0$$

$$R\frac{4x}{5x}-P=0$$

$$P=\frac{4R}{5}.$$

Resolving horizontally,

$$R\cos A\hat{C}B - Q = 0$$

$$R\frac{3x}{5x} - Q = 0$$

$$Q = \frac{3R}{5},$$

so $\quad Q : R = 3 : 5.$

Also $\quad P : Q = \frac{4R}{5} : \frac{3R}{5}$

$$= 4 : 3,$$

so $\quad P : Q : R = 4 : 3 : 5.$

Exercise 9E

1. A rectangle PQRS has forces of 3 N acting along the sides PQ, QR, RS and SP respectively. If $PQ = 5$ m and $QR = 2$ m, show that a couple is acting and find the moment of the couple.
2. A regular hexagon ABCDEF with side length l has forces of $7a$, $6a$, $4a$, $4a$, $3a$ and a acting along the sides AB, CB, CD, DE, FE and FA respectively. Show that the forces are equivalent to a couple and find the magnitude of the couple.
3. Three forces given by the vectors $(-4\mathbf{i} + \mathbf{j})$, $(3\mathbf{i} - 2\mathbf{j})$ and $(\mathbf{i} + \mathbf{j})$ act through the points $(-\mathbf{i} + 4\mathbf{j})$, $(4\mathbf{i} - 2\mathbf{j})$ and $(2\mathbf{i} + 2\mathbf{j})$ respectively. Show that the forces are equivalent to a couple and find the moment of the couple.

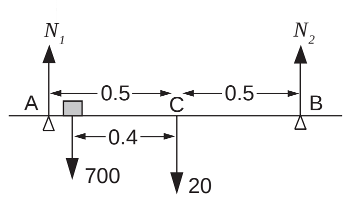
9.8 Contact forces

Before the next example, recall the following:

> For a body to be in equilibrium under a system of forces acting at different points in the body:
>
> 1. the resultant force must be zero;
>
> 2. the algebraic sum of moments about **any** point is zero.

Example

A plank of length 2 m and mass 2 kg rests horizontally on smooth supports positioned 0.5 m either side of its centre. John, who weighs 70 kg, is standing on the plank 0.4 m from the centre. Find the forces exerted on the plank by the supports.



Solution

Let N_1 and N_2 be the normal contact forces (in newtons) exerted on the plank by the supports.

Since the plank is in equilibrium, the resultant force is zero, so that

$$N_1 + N_2 = 720.$$

The sum of moments about **any** point is also zero.

To simplify the calculation, take moments about either A or B as this gives an equation with only one unknown.

Taking moments about A gives

$$-700 \times (0.1) - 20 \times (0.5) + N_2 = 0$$

and so

$$N_2 = 70 + 10 = 80.$$

Also

$$N_1 = 720 - N_2 = 640.$$

The forces exerted by the supports at A and B are 640 N and 80 N respectively.

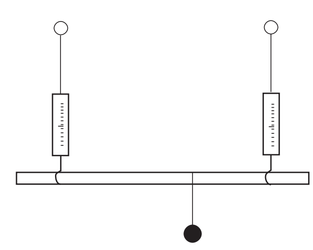
In this example, since all the forces are in the vertical direction, it is only necessary to consider forces in one direction and take moments about one point. In situations in which the forces do not act in one direction, it is necessary to consider the components of the forces in two directions and equal to zero. In many problems these directions may be horizontal and vertical but these may not always be the most appropriate directions.

Activity 9 Finding the contact forces

You need a metre rule, 2 newton meters, masses and some string loops.

Weigh the metre rule.

Place a known mass on the rule at a known distance from one end.



Attach the newton meters to the metre rule using loops of string.

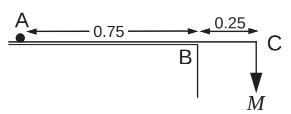
Holding the newton meters, read off the contact forces.

Calculate the theoretical values of the contact forces. Do these agree with your readings?

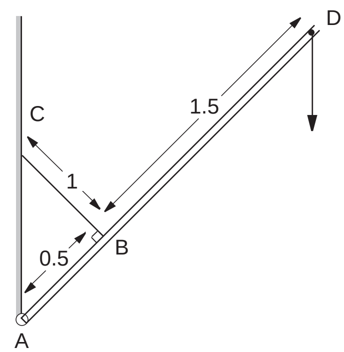
Repeat for different positions and with different masses.

Exercise 9F

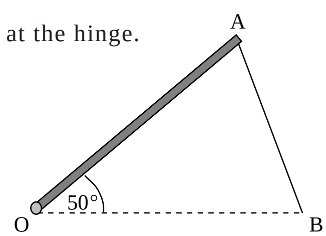
1. A 50 g mass and a 100 g mass are suspended from opposite ends of a metre rule which has mass 100 g. If the rule and the masses are attached to a string and hang vertically with the rule in a horizontal position, find
   (a) the tension in the string;
   (b) the position on the rule at which the string should be attached.
2. The rule and masses from Question 1 are now suspended by two strings instead of one. If the strings are attached 30 cms either side of the centre of gravity of the metre rule, find the tensions in the strings.
3. A plank of length 1.6 m and mass 4 kg rests on two supports which are 0.3 m from each end of the plank. A mass is attached to one end of the plank. If the normal contact force on the support nearest to this load is twice the normal contact force on the other support, determine the mass attached.
4. A metre rule is pivoted 20 cm from one end A and is balanced in a horizontal position by hanging a mass of 180 g at A. What is the mass of the rule? What additional mass should be hung from A if the pivot is moved 10 cm nearer to A?
5. A metre rule of mass 100 g is placed on the edge of a table as shown with a 200 g mass at A. A mass *M* grammes is attached at C.



   (a) When the rule is just on the point of overturning, where does the normal contact force act?
   (b) Determine the maximum value of *M* for which the rule will not overturn. Set up this experimental arrangement and test your predictions.
6. A plank of length 2 m and mass 6 kg is suspended in a horizontal position by two vertical ropes, one at each end. A 2 kg mass is placed on the plank at a variable point P. If either rope snaps when the tension in it exceeds 42 N, find the section of the plank in which P can be.
7. A uniform plank of length 2 m and mass 5 kg is connected to a vertical wall by a pin joint at A and a wire CB as shown. If a 10 kg mass is attached to D, find:
   (a) the tension in the wire;
   (b) the reaction at the pin joint A.



8. A loft door OA of weight 100 N is propped open at 50° to the horizontal by a strut AB. The door is hinged at O; OA = OB = 1 m. Assuming that the mass of the strut can be neglected compared to the mass of the door and that the weight of the door acts through the midpoint of OA, find:
   (a) the force in the strut;
   (b) the reaction at the hinge.



9.9 Stability

You may wonder what the items opposite have in common. All of them can be regarded as structures which must not collapse either by toppling or sliding.

dam; bridge; skyscraper; cathedral; gymnast; dog; insect; tree.

Not all structures succeed in this! (For example: *the Leaning Tower of Pisa; a collapsed bridge; an athlete fallen on the track.*)

Structures which do succeed are called **stable**, ones which do not, **unstable**.

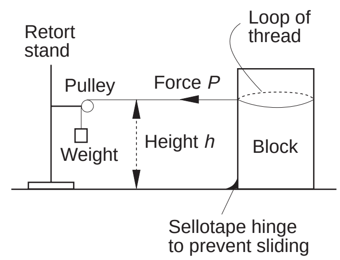
What causes a structure to become unstable?

All structures have external forces acting on them. For example, high winds produce significant forces on animals including human beings as well as on tall buildings. Dams are acted on by the forces due to the water they hold back. All structures are acted on by the Earth's gravitational field, both through their own weight and the weight of whatever load they carry. When these forces are too large or act in the wrong places, the structure falls. Structures need to be designed to withstand the loads they are likely to encounter though exceptionally strong forces can still cause failure. Some structures, such as a hurdle on an athletics track, are deliberately designed to fail when the force on them exceeds a certain amount.

High winds can produce such large forces on people that they are blown over. A practical simulation of this uses a rectangular block or box.

Activity 10 Toppling a tower

You need a block of wood, thread, pulley, retort stand, masses, Sellotape and sandpaper.



Vary the height, h, at which the thread is looped round the block by varying the height of the pulley. (The retort stand and clamp are useful here.)

For different h find the force, P, which just causes one end of the block to lift off the ground. Investigate the relation between P and h.

Why does the block not lift off the ground for smaller values of P?

Now remove the Sellotape and, for different heights, h, investigate whether the block topples or slides as P is increased from zero.

Place a sheet of sandpaper under the block and investigate how this affects it sliding or toppling.

Keep your data for use in Activity 14. Weigh the block since this weight is also needed.

Why do trees not topple over in high winds? What does happen to them?

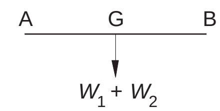
9.10 The centre of gravity of a body

In Section 9.6 you met the centre of gravity of a rule, the fixed point of the rule through which its weight acts. All bodies and systems of particles have centres of gravity and these are important in stability.

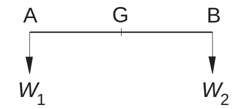
The centre of gravity of two particles

The weights W_1 and W_2 of two particles at A and B have a resultant $W_1 + W_2$ parallel to W_1 and W_2. The line along which this resultant acts meets the line AB at a point G whose position can be found using the principle of moments.

From the second form of the principle the moment of the resultant $W_1 + W_2$ about any point is the sum of the moments of W_1 and W_2 about that point. Since the moment of the resultant about G is zero, the sum of moments of W_1 and W_2 about G is zero, so that G is a convenient point about which to take moments. This gives



$$W_1 \times (\mathrm{AG}) - W_2 \times (\mathrm{BG}) = 0$$



or $$W_1 \times (\mathrm{AG}) - W_2 \times (\mathrm{AB} - \mathrm{AG}) = 0,$$

$$W_1 \mathrm{AG} + W_2 \mathrm{AG} - W_2 AB = 0$$

$$\mathrm{AG}(W_1 + W_2) = W_2 \mathrm{AB}$$

so that $$\mathrm{AG} = \frac{W_2}{W_1 + W_2} \times (\mathrm{AB}).$$

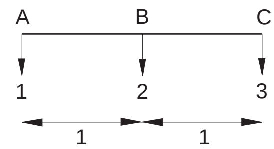
The weights W_1 and W_2 of the two particles at A and B are equivalent to a single weight $W_1 + W_2$ at G. The point G is the centre of gravity of the two particles.

For particles of equal weight $\mathrm{AG} = \frac{1}{2}\mathrm{AB}$ and so the centre of gravity is at the midpoint of AB.

The centre of gravity of any number of particles on the same horizontal straight line can be found using the principle of moments.

Example

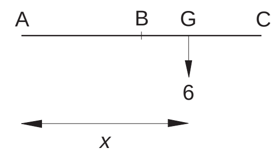
Three particles A, B, C of weights 1 N, 2 N, 3 N respectively, lie on the same horizontal line as shown with $AB = BC = 1$ m. Find the distance of their centre of gravity, G, from A.



Solution

The resultant of the three weights is a weight 6 N acting through G.

The point G is again a convenient point about which to take moments since the moment of the resultant about G is zero and so the sum of the moments of the three weights about G is zero.

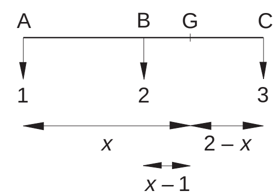


This gives

$$1 \times (AG) + 2 \times (BG) - 3 \times (CG) = 0,$$

or, with $AG = x$,

$$1.\,x + 2(x-1) - 3(2-x) = 0,$$



so that $6x - 8 = 0$

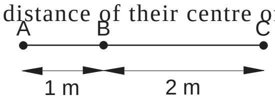
or $x = \dfrac{4}{3}.$

This gives $AG = 1.33$ m.

Exercise 9G

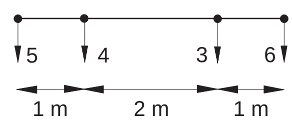
1. Find the centres of gravity of two particles
   (a) of weights 3 N and 5 N a distance of 1 m apart;
   (b) of weights 4 N and 7 N a distance of 2 m apart;
   (c) of weights 5 N and 10 N a distance of 9 m apart.

2. Three particles A, B,C, of weights 5 N, 3 N, 4 N, respectively, lie on the same horizontal line as shown. Find the distance of their centre of gravity from A.



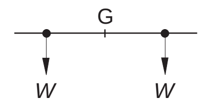
   The particle A is removed and replaced by a new particle. What is the greatest value of its weight if the centre of gravity of the three particles is to lie in BC?

3. Four particles of weights 5 N, 4 N, 3 N and 6 N lie on the same horizontal line as shown. How far is their centre of gravity from the 5 N particle?



9.11 The centres of gravity of some simple bodies

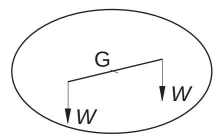
The result that the centre of gravity of two particles of equal weight is at the midpoint of the line joining the particles can be used to find the centre of gravity of simple bodies.



Since a uniform rod, such as a rule, is symmetric about its midpoint, G, it can be regarded as made up of pairs of particles of equal weight equidistant from G. Since the centre of gravity of each pair of particles is at G, the centre of gravity of the rod is at G.

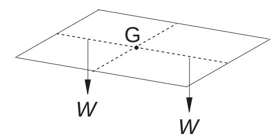
The same argument shows that:

the centre of gravity of a uniform circular disc is at its centre;



the centre of gravity of a rectangle is at the point where the lines forming the midpoints of opposite sides intersect;

the centre of gravity of a rectangular block is at the point where the diagonals forming opposite vertices intersect.

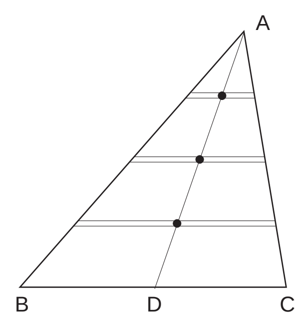


In each case and generally

> The weight of the body can be regarded as acting at its centre of gravity.

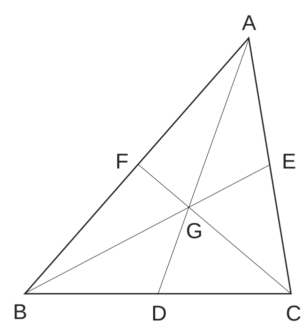
The centre of gravity of a triangle

A triangle can be thought of as made up of rods parallel to one of its sides BC. Since the centre of gravity of each rod is at its midpoint, the centre of gravity G of the triangle lies on the line joining A to the midpoint D of BC, the median AD.



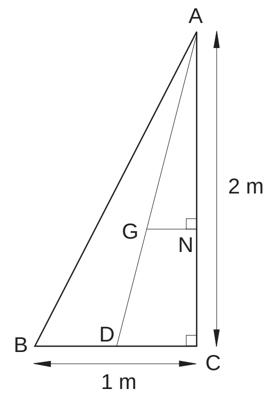
Repeating this argument for the other two sides shows that G is at the point of intersection of the medians AD, BE and CF, where E and F are the midpoints of AC and AB. From geometry

$$AG = \frac{2}{3}AD, \quad BG = \frac{2}{3}BE, \quad CG = \frac{2}{3}CF.$$



Example

Find the distances of the centre of gravity G from BC and AC in the triangle shown opposite.



Solution

Triangles AGN and ADC are similar, so

$$\frac{AN}{AC} = \frac{GN}{DC} = \frac{AG}{AD} = \frac{2}{3},$$

which gives $AN = \frac{2}{3}AC$ and $GN = \frac{2}{3}DC$.

The distance of G from $BC = NC = AC - AN = \frac{1}{3}AC$

$$= \frac{2}{3}$$

and the distance of G from $AC = GN = \frac{2}{3}DC = \frac{1}{3}BC$

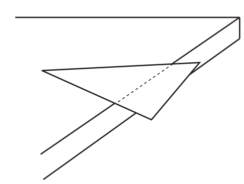
$$= \frac{1}{3}.$$

The distances of G from BC and AC are $\frac{2}{3}$ m and $\frac{1}{3}$ m respectively.

Activity 11 Finding the centre of gravity of a triangle

You need cardboard, scissors, thread and a table.

Cut a triangle out of cardboard.



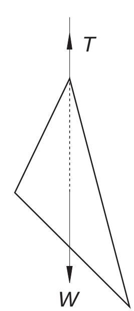
Place the triangle so as to partly overhang the edge of the table.

Adjust it so that it is just on the point of toppling and draw a line on it to indicate the edge of the table.

Why is the centre of gravity of the triangle on this line?

Orientate the triangle another way and so find a second line on which the centre of gravity lies.

The intersection of the two lines gives the centre of gravity of the triangle.



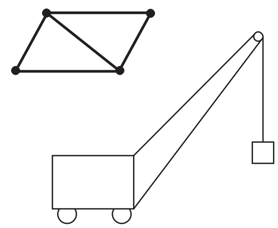
Repeat with the triangle orientated a third way and check that the third line also passes through G.

Draw the medians of your triangle and see if G is at their point of intersection.

As a further check, suspend the triangle from different points by a thread. Since the triangle is in equilibrium, the tension, T, in the thread and the weight, W, of the triangle must act along the same line, so the centre of gravity lies on the vertical line through the point of suspension.

This technique is used by biologists to determine the centres of gravity of insects.

9.12 The centre of gravity of composite bodies



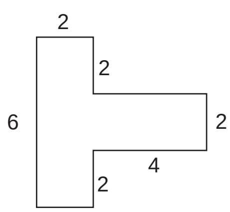
A composite body is one made up of two or more simple bodies joined together, for example a framework or a crane. The weight of the body is the sum of the weights of the bodies that make it up.

For the purposes of calculating the centre of gravity of the composite body each of the simpler bodies can be regarded as equivalent to a particle of the same weight situated at its centre of gravity. When the composite body is made up of two bodies of weights W_1 and W_2 and centres of gravity G_1 and G_2 its centre of gravity coincides with the centre of gravity of two particles of weights W_1 and W_2 at G_1 and G_2 respectively.

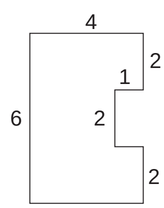
Example

Find the positions of the centres of gravity of the T- and U- shapes shown. Lengths are in metres.

(a)

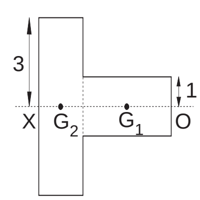


(b)



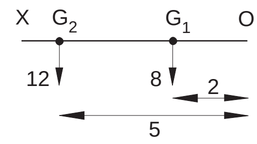
Solution

(a) Since the T-shape is symmetric about the line OX its centre of gravity lies on OX.



The shape can be divided into two rectangles as shown and these rectangles replaced by particles at their centres of gravity G_1 and G_2, the weights of the particles being proportional to the areas of these rectangles.

The resultant of these weights is proportional to the sum of these areas and acts at the centre of gravity, G, of the shape.

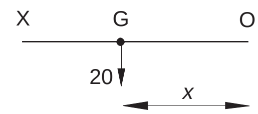


Equating moments about O for these two systems gives

$$20x = 8 \times 2 + 12 \times 5 = 76,$$

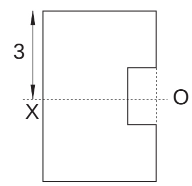
so that

$$x = \frac{76}{20} = 3.8.$$



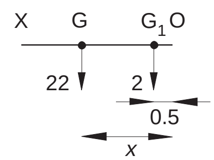
The centre of gravity of the shape is on OX, distance 3.8 m from O.

(b) The centre of gravity of the shape lies on its line of symmetry OX.



The U-shape can be regarded as a large rectangle with a small rectangle cut out as shown.

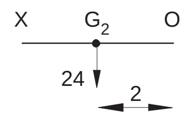
The U-shape and the small rectangle can be replaced by particles at the centres of gravity G_1 and G_2, where weights are proportional to the areas.



The resultant of these weights is proportional to the area of the large rectangle and acts at its centre of gravity G_2.

Since the areas of the large and small rectangles are 24 m^2 and 2 m^2, the area of the U-shape is 22 m^2.

Taking moments about O for the two systems gives



$$24 \times 2 = 2 \times \frac{1}{2} + 22 \times x,$$

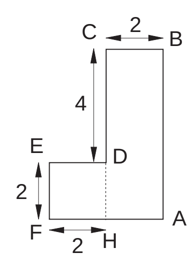
so that $\quad 22x = 48 - 1 = 47$

and $\quad x = \frac{47}{22} = 2.14.$

The centre of gravity of the shape is on OX, distance 2.14 m from O.

Example

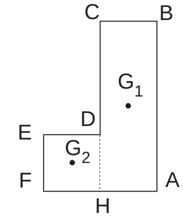
Find the distance from AB of the centre of gravity of the L-shape shown. Lengths are in metres.



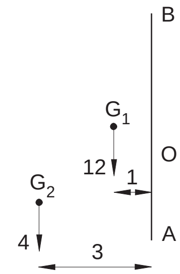
Solution

The area of the L-shape is 16 m^2.

The L-shape can be regarded as made up of two rectangles ABCH and DEFH of areas 12 m^2 and 4 m^2 respectively, with centres of gravity G_1 and G_2. The centre of gravity, G, of the shape lies on $G_1 G_2$.



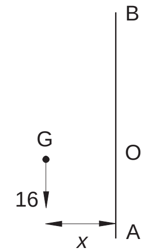
Imagine the shape to be in a vertical plane. Then the weight of the shape acts at its centre of gravity, G, and is parallel to AB. This is equivalent to the weights of the two rectangles acting at G_1 and G_2 and also parallel to AB.



The moments of the two systems about any point O on AB are equal. If x is the distance of the line of the weight of the shape from AB, then

$$16x = 12 \times 1 + 4 \times 3 = 24$$

so $$x = \frac{24}{16} = \frac{3}{2}.$$



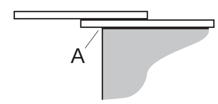
The centre of gravity of the shape is distant 1.5 m from AB.

Activity 12 Overhanging rules

You need a set of metre rules (as near equal in weight as possible) and a table. Lay one rule on the table at right angles to its edge and find out what is the most overhang you can get.

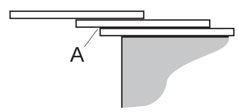
How far is the centre of gravity of the rule from the table edge when it is just about to topple?

Find out the most overhang you can get with two rules.



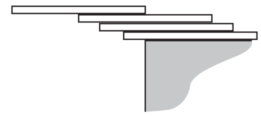
How far is the centre of gravity of the two rules from the table edge when they are just about to topple?

Imagine the two rules put on the table and then the first two rules put as they are on top of this third rule with the point A of the bottom rule, originally at the edge of the table, now at the overhanging end of the third rule. Assuming all three rules are equal in length and weight, calculate by how much the third rule overhangs the table when the rules are just on the point of toppling.



Check your result experimentally.

Now imagine the three rules lifted off as they are and then placed on top of the fourth rule on the table, again with the point of the lowest of the three rules originally at the table edge now at the end of the fourth rule. Calculate the amount the fourth rule overhangs the table when the rules are just on the point of toppling.

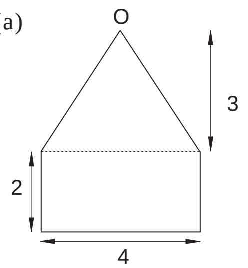


Check your result experimentally. Can you get the top rule to completely overhang the table?

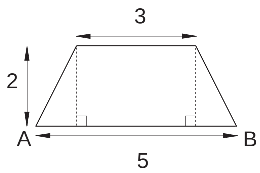
Exercise 9H

1. Find the centres of gravity of the shapes shown. Lengths are in metres.

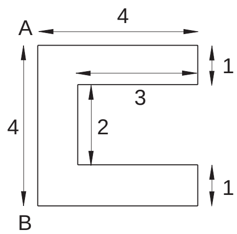
(a)



(b)

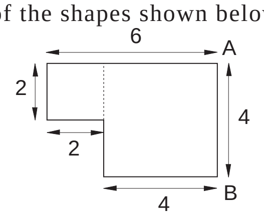


(c)

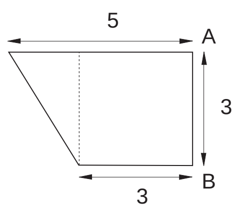


2. Find the distances from AB of the centres of gravity of the shapes shown below.

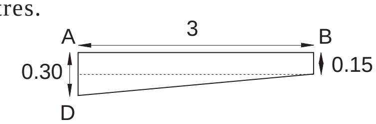
(a)



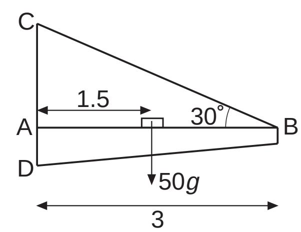
(b)



3. The jib of a wall crane has the shape and dimensions shown in the diagram, lengths being in metres.



It is made of 20 mm thick steel plate of density 800 kg m^{-3}. Find its mass and the distance of its centre of gravity from AD.

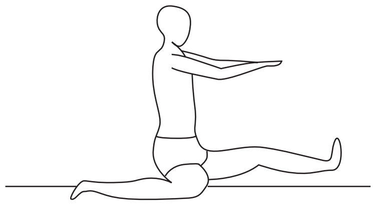


The jib is hinged to the wall at A by a pin-joint. A trolley of weight $50g$ N is on the jib 1.5 m from A. Find the tension in the rod BC and the horizontal and vertical components of the contact force of the wall in the joint at A. (This model of the wall crane takes account of the weight of the jib.)

9.13 Some applications of centres of gravity

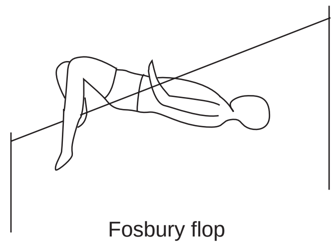
High jumping

The diagrams below show three high jump techniques - the scissors, the straddle and the Fosbury flop.



Scissors

Straddle



Fosbury flop

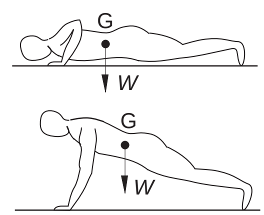
Where are the centres of gravity of the jumpers likely to be, relevant to the bars?

Which is likely to be the most and which the least effective technique?

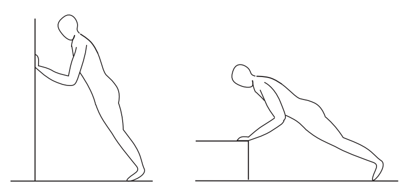
If you have access to a slow motion video of high jumping, you might like to look at it.

Keeping fit

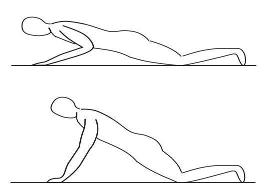
Exercises help you to keep fit and improve your performance at sport. Several exercises involve raising the whole or part of your body. During a press-up from the floor the body rotates about the feet due to the action of the arms. Muscles in the arms must overcome the moment about the feet of the body's weight at the centre of gravity, G; the smaller this moment, the easier the exercise is in terms of the strain on the arms.



It is easier to do press-ups against a wall or a bar than against the floor.

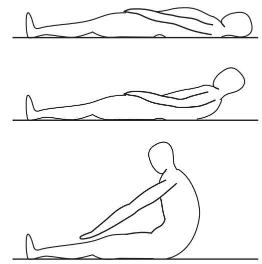


Why? Which is easier, the wall or the bar?



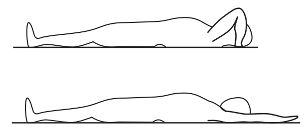
Why is it easier to do press-ups using the knees rather than the feet?

Activity 13 The trunk curl - sit up



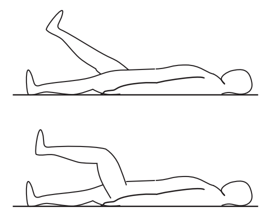
Try the trunk curl - sit up exercise shown in the three diagrams opposite.

The first stage of this exercise involves a trunk curl as shown in the second diagram, before the sit up is completed. What is the advantage of this?



Now try a trunk curl - sit up with your arms in the positions shown.

Why are these more difficult to do? Which is most difficult?



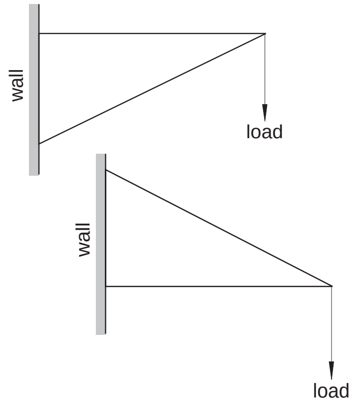
Now try an easier exercise, just raising one leg at a time.

Why is this easy?

Why is it even easier when the leg is bent?

Climbing trees

When a squirrel climbs a tree, it needs to overcome gravity. Its weight is balanced by an equal and upward force on its hind feet which it sets up either by digging in its claws or by friction. Since the squirrel's centre of gravity is not directly above its claws, this upward force and the weight exert a clockwise moment on the squirrel. To balance this, the squirrel must pull on the tree trunk with its forefeet and push with its hind feet.

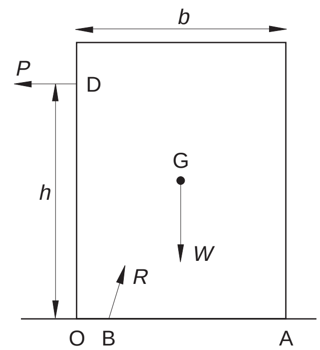


How does a woodpecker stay in equilibrium on a tree? Why does it rest its tail on the tree trunk?

Compare the contact forces exerted on the wall by the two structures on the right with those exerted by the squirrel and the woodpecker. Which best models the squirrel and the woodpecker?

9.14 Sliding and toppling of a block

In Activity 10 you found experimentally when a block slides and when it topples. To obtain mathematical criteria for sliding and toppling it is necessary to know the forces acting on the block.



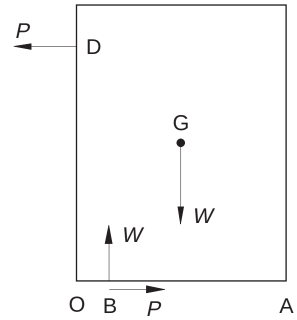
These are

the weight, W, acting vertically downwards through the centre of gravity, G, of the block;

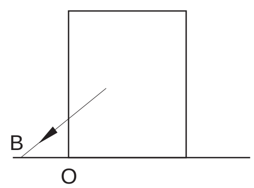
the longitudinal pull, P, on the block acting at the point D;

the force, R, the ground exerts on the block acting at a point B in the base OA of the block.

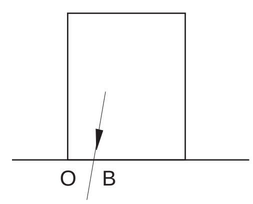
When the block is in equilibrium, the resultant of W and P must be equal and opposite to R and act along a line passing through B. The force due to the ground therefore has a normal component $N = W$ upwards to balance the weight and a friction component $F = P$ horizontally to balance the pull, this component being due to friction between the ground and the block.



For the block not to topple, B must lie between O and A. Otherwise, the resultant of W and P has an anticlockwise moment about O and the block topples about O.



topples



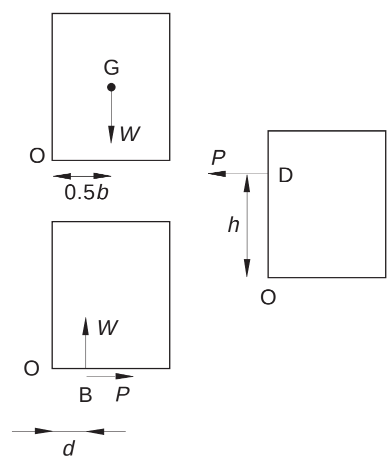
does not topple

The position of B is found by taking moments about O:

the moment of the weight W about O is clockwise and is $\dfrac{-Wb}{2}$, b the width of the block;

the moment of the pull P about O is anticlockwise and is Ph;

the moment of the force due to the ground about O is anticlockwise and is Wd, where $\mathrm{OB} = d$.



Provided the block is in equilibrium, the sum of these moments is zero. Hence

$$\frac{-Wb}{2} + Ph + Wd = 0,$$

which gives

$$Wd = \frac{Wb}{2} - Ph$$

or $$d = \frac{b}{2} - \frac{Ph}{W}.$$

Since B is between O and A, $\mathrm{OB} = d \geq 0$, so

$$\frac{b}{2} \geq \frac{Ph}{W}$$

or $$\frac{Wb}{2} \geq Ph.$$

This says that the block does not topple provided the moment of the weight about O is greater than or equal to the moment of the pull about O.

The block is just on the point of toppling when

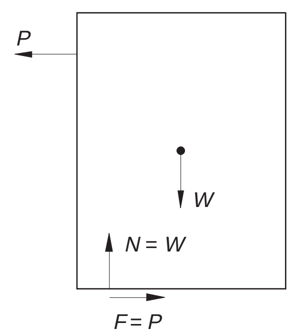
$d = 0$ or $\frac{Wb}{2} = Ph$. This gives the value of P for toppling as

$$P = \frac{Wb}{2h},$$

in other words, P is inversely proportional to h.

Is this the relation you found in Activity 10?

When does the block slide?



The block slides when there is not enough friction to hold it in position.

The force on the block due to the ground has a vertical component $N = W$, the normal contact force, and a horizontal component $F = P$, due to friction.

The law of friction says that

$$F \le \mu N,$$

where μ is the coefficient of friction. If the block has not already toppled, it is on the point of sliding when the friction becomes limiting, that is, $F = \mu N$.

It does not slide, however, provided $F \le \mu N$, that is,

$$P \le \mu W$$

or $$\frac{P}{W} \le \mu.$$

It does not topple provided

$$\frac{P}{W} \le \frac{b}{2h}.$$

If $\frac{b}{2h} > \mu$, then, as P increases from zero, $\frac{P}{W}$ reaches the value μ before the value $\frac{b}{2h}$. The block slides. Similarly, if $\frac{b}{2h} < \mu$, $\frac{P}{W}$ reaches the value $\frac{b}{2h}$ before the value μ. The block topples.

> As P increases from zero, the block slides first if $\frac{b}{2h} > \mu$ and topples first if $\frac{b}{2h} < \mu$.

Activity 14 Sliding and toppling of a block

You need your data from Activity 10.

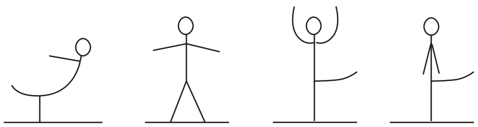
Choose values of P and h for which the block slides before it topples.

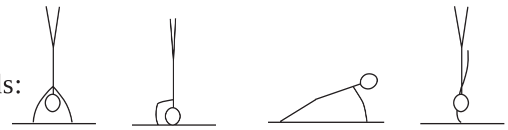
Use these values to find the coefficient of friction between the block and the table.

Investigate how well the remaining data you obtained in Activity 10 fit the criteria for sliding and toppling of a block.

Activity 15 Gymnasts

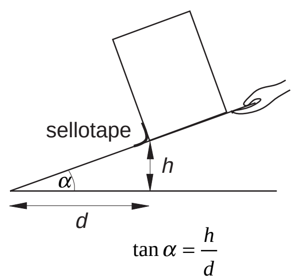
Arrange each group of gymnasts in the order in which they are most in danger of toppling over, explaining why you choose this order.

Balancing on the feet: 

Balancing on the hands: 

Activity 16 Toppling packets

You need two pieces of wood or rules, Sellotape, rectangular packets of soap flakes, cereal, sugar, jelly, etc. (alternatively you can use multilink).



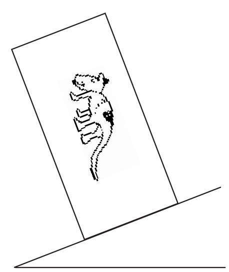
Sellotape one piece of wood to the other (or to the table) so as to make an inclined plane when you hold it.

Investigate practically the angles α at which packets of different heights topple.

Use a piece of Sellotape at the front of the packet to prevent it sliding.

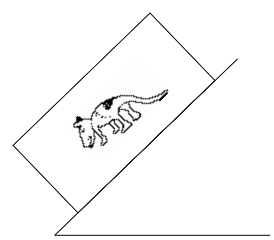
The angle α can be found by measuring h and d as in the diagram.

Obtain a theoretical criterion for the toppling of a rectangular packet on an inclined plane.



Check how well your theoretical criterion agrees with your actual results. Use your criterion to predict the angle at which a packet should topple and see what actually happens.

Place a packet with its largest side at right angles to the plane and then along the plane. In each case measure the angle at which the packet topples.



Can you find a relation between these angles?

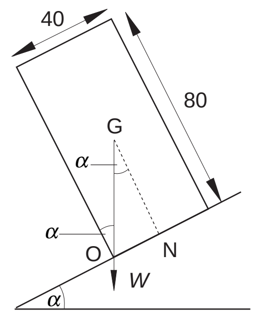
Can you confirm this relation theoretically?

Example

A uniform rectangular block of width 40 mm and height 80 mm is placed on a plane which is then gently raised until the block topples. What angle does the plane make with the horizontal when this occurs?

Solution

The block is about to topple when the moment about its lowest point, O, is zero. This is when the line along which the weight acts, the vertical through G, passes through O.



In triangle ONG the side $\text{ON} = 20$, $\text{NG} = 40$

and so

$$\tan\alpha = \frac{\text{ON}}{\text{NG}} = 0.5,$$

which gives $\alpha = 26.6°$. From the diagram the angle the plane makes with the horizontal is also $\alpha = 26.6°$.

Exercise 9I

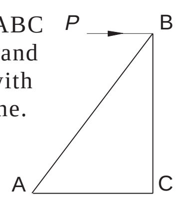
1. A block of mass 1 kg, width 40 cm and height 100 cm, rests on a table. Find the horizontal force, P, which causes it to topple when:

   (a) P acts at the top of the block;

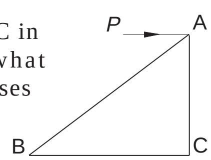
   (b) P acts halfway up.

   What is the least value of the coefficient of friction for which the block topples rather than slides whichever of the forces (a) or (b) is applied?

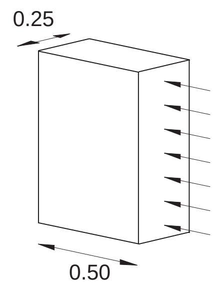
2. A right-angled triangular sheet ABC has sides $\text{AC} = 30$ cm, $\text{BC} = 40$ cm and mass 1 kg. It is placed vertically with the side AC along a horizontal plane. What horizontal force P at B causes the sheet just to topple?



   When the sheet has the side BC in contact with the plane, what horizontal force P at A then causes the sheet to topple?



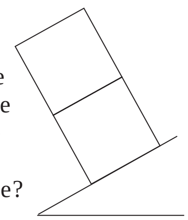
3. A brick column is 0.5 m wide and 0.25 m deep and weighs 18 000 N per cubic metre. There is a uniform wind pressure of 750 N per square metre on one side as shown in the diagram opposite. The column rests on a block but is not attached to it.



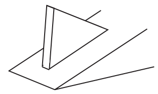
   What is the greatest height the column can be if it is not to topple?

   Assume the wind pressure produces a horizontal force on the column whose magnitude is the wind pressure times the area over which it acts, the force acting at the point of intersection of the diagonals of the force.

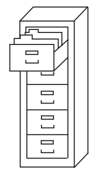
4. A tower is made of multilink cubes, each of side 20 mm, and is placed on a plane. The plane is gently raised until the tower topples. At what angle of the plane to the horizontal does a tower of 2 cubes topple? At what angles do towers of 3 and 10 cubes topple?



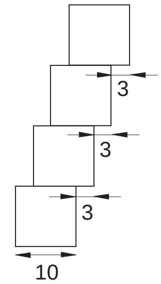
5. An equilateral triangular sheet is placed vertically with one side in contact with a horizontal plane. If the plane is raised gently, at what angle to the horizontal does the sheet topple?



6. A filing cabinet has four drawers, each of which has mass W kg when empty, the mass of the rest of the cabinet being $3W$ kg. When the bottom three drawers are empty and the papers in the top drawer weigh six times as much as the drawer itself, how far can this drawer be opened without the cabinet toppling? Treat the drawers as rectangles.



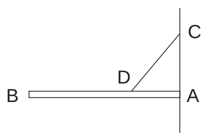
7. A pile of equal cubical blocks, each of edge 10 cm, is made by placing the blocks one on top of the other, with each displaced a distance 3 cm relative to the block below. Show that a pile of four blocks does not topple but one of five does.



9.15 Miscellaneous Exercises

1. A uniform shelf of length 46 cm is hinged to a vertical wall as shown. The shelf is supported by a rod and the tension in the rod is 100 N.

   If $\mathrm{AD} = 15$ cm and $\mathrm{A\hat{D}C} = 50°$, find the mass of the shelf.

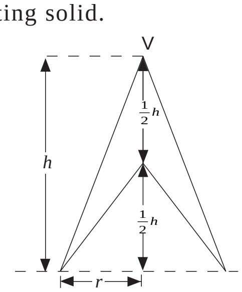


2. A circular hole of radius 2 m is made in a circular disc of radius 8 m. If the centre of the hole is 4 cm from the centre of the disc, find the position of the centre of gravity of the disc with its hole.

3. A hollow cylinder of diameter 5 cm is placed on a rough plane which is inclined at $30°$ to the horizontal. What is the maximum height of the cylinder if it is not to topple over?

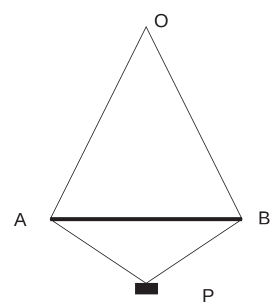
4. The diagram shows a solid uniform right circular cone of height h, base radius r and vertex V from which has been removed a solid coaxial cone of height $\frac{1}{2}h$, base radius r.

   Find the distance from V of the centre of mass of the resulting solid.



(AEB)

5. [In this question you should assume $g = 9.8\text{ms}^{-2}$.]



The diagram shows a body, P, of mass 13 kg, which is attached to a continuous inextensible string of length 15.4 m. The string passes over a small smooth peg O and the hanging portions of the string are separated by a heavy uniform horizontal rigid rod AB, which is 4 m long, the string fitting into small grooves at the ends of the rod. Given that in the equilibrium position $\mathrm{AP} = \mathrm{BP} = 2.5$ m, show that

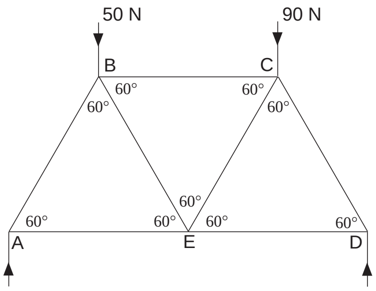
(a) $\sin \mathrm{P\hat{A}B} = \frac{3}{5}$ and $\sin \mathrm{O\hat{A}B} = \frac{12}{13}$;

(b) the tension in the string has magnitude $\frac{637}{6}$ N.

(c) Hence find the mass of the bar AB.

(AEB)

6. The diagram shows a framework consisting of seven equal smoothly jointed light rods AB, BC, DC, DE, AE, EB and EC. The framework is in a vertical plane with AE, ED and BC horizontal and is simply supported at A and D. It carries vertical loads of 50 N and 90 N at B and C respectively.



Find

(a) the reactions at A and D;

(b) the magnitudes of the forces in AB, AE and BC.

(AEB)

7. A uniform solid right circular cylinder of radius a and height $3a$ is fixed with its axis vertical, and has a uniform solid sphere of radius r attached to its upper face. The sphere and the cylinder have the same density and the centre of the sphere is vertically above the centre of the cylinder. Given that the centre of mass of the resulting composite body is at the point of contact of the sphere with the cylinder, find r in terms of a.

(AEB)

8. An equilateral triangular frame ABC is made up of three light rods, AB, BC and CA which are smoothly jointed together at their end points. The frame rests on smooth supports at A and B, with AB horizontal and with C vertically above the rod AB. A load of 50 N is suspended from C. Find the reactions at the supports and the tensions or thrusts in the rods.

(AEB)

9. A composite body B is formed by joining, at the rims of their circular bases, a uniform solid right circular cylinder of radius a and height $2a$ and a uniform right circular cone of radius a and height $2a$. Given that the cylinder and the cone have masses M and λM respectively, find

(a) the distance of the centre of mass of B from the common plane face when $\lambda = 1$;

(b) the value of λ such that the centre of mass of B lies in the common plane face.

(AEB)

10. The base of a uniform solid hemisphere has radius $2a$ and its centre is at O. A uniform solid S is formed by removing, from the hemisphere, a solid hemisphere of radius a and centre O. Determine the position of the centre of mass of S. (The relevant result for a solid hemisphere may be assumed without proof.) (AEB)

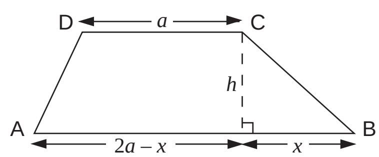
11. A light square lamina ABCD of side $2a$ is on a smooth horizontal table and is free to turn about a vertical axis through its centre O. Forces of magnitude P, $2P$, $3P$ and $7P$ act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{AD}$ respectively. Find the magnitude of the couple required to maintain equilibrium and also the magnitude of the reaction at O.

(AEB)

12. A uniform rod AB of weight 40 N and length1.2 m rests horizontally in equilibrium on two smooth pegs P and Q, where $AP = 0.2$ m and $BQ = 0.4$ m. Find the reactions at P and Q. Find also the magnitude of the greatest vertical load that can be applied at B without disturbing the equilibrium.

(AEB)

13. The diagram shows a uniform plane trapezium ABCD, in which AB and DC are parallel and of lengths $2a$ and a respectively. The foot of the perpendicular from C onto AB is E, $CE = h$ and $BE = x$.



Prove that the distance of the centre of mass of the trapezium from AB is $\frac{4}{9}h$. (AEB)

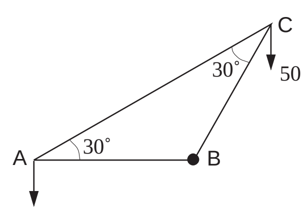
14. ABCD is a square of side 2 m. Forces of magnitude 2 N, 1 N, 3 N, 4 N and $2\sqrt{2}$ N act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ and $\overrightarrow{BD}$ respectively. In order to maintain equilibrium a force F, whose line of action cuts AD produced at E, has to be applied. Find

(a) the magnitude of F;

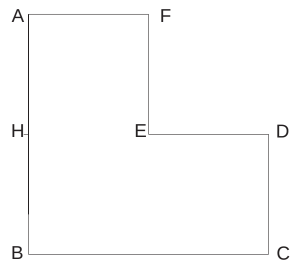
(b) the angle F makes with AD;

(c) the length AE. (AEB)

15. The diagram shows three light rods which are smoothly jointed together to form a triangular framework ABC in which the angles BAC and BCA are both 30°. The framework can turn in a vertical plane about a horizontal axis through B. When a load of 50 N is suspended from C the framework is kept in equilibrium with AB horizontal by means of a vertical force of magnitude P N applied at A. Determine P and the thrust in BC.



16. The diagram shows a uniform L-shaped lamina ABCDEF, of mass $3M$, where $AB = BC = 2a$ and $AF = FE = ED = DC = AH = a$.



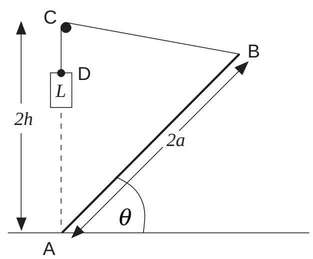
Find the perpendicular distances of the centre of mass of the lamina from the sides AB and BC. The lamina is suspended freely from H, the mid-point of AB, and hangs in equilibrium.

(a) Show that the tangent of the angle which the side AB makes with the horizontal is $\frac{1}{5}$.

(b) When a particle of mass m is attached to F the lamina hangs in equilibrium with AB horizontal. Find m in tems of M.

(c) The particle is now removed and the lamina hangs in equilibrium with BE horizontal when a vertical force of magnitude P is applied at B. Find P in terms of M and g.

17. The diagram shows a uniform rod AB of weight W and length $2a$ with the end A resting on a rough horizontal plane such that AB is inclined at an angle θ to the horizontal. The rod is maintained in equilibrium in a vertical plane in this position by means of a light inextensible string BD which passes over a small peg at C and which carries a load L at D. The peg is at height $2h$ vertically above A.



(a) Find, by taking moments about C and B, the horizontal and vertical components of the reaction on the rod at A in terms of W, a, h and θ. Given that the coefficient of friction at A is μ, show that

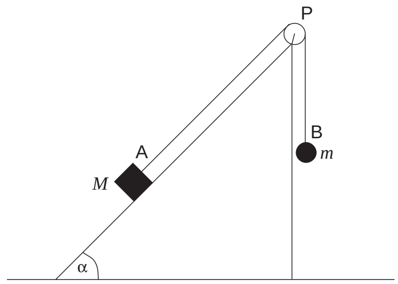
$$\frac{a\cos\theta}{h + a\sin\theta} \le \mu.$$

Hence find, in terms of h and a, the least value of μ so that equilibrium is possible for any angle θ satisfying $0 \le \theta \le 60°$.

(b) For the case when $a = \frac{1}{2}h$ and angle ABC is $90°$, find in terms of W, the load L and the vertical component of the force acting on the peg C.

(AEB)

18. The diagram shows a fixed rough inclined plane whose lines of greatest slope make an angle α with the horizontal. The plane has a smooth pulley P fixed at its highest point. Two particles, A and B, of mass M and m respectively, are attached one to each end of a light inextensible string which passes over the pulley. The masses are at rest with A in contact with the plane, B hanging freely vertically below P and the portion of string AP parallel to a line of greatest slope of the plane.



Find, in terms of M, m, g and α, the components, normal and parallel to the plane, of the force exerted on A by the plane. Deduce that

$$\sin\alpha - \mu\cos\alpha \le \frac{m}{M} \le \sin\alpha + \mu\cos\alpha,$$

where μ is the coefficient of friction.

It is found that when $\alpha = 60°$, A is on the point of moving down the plane but when $\alpha = 30°$, A is on the point of moving up the plane. Find μ.

(AEB)