3 VECTORS 1

Objectives

After studying this chapter you should

- understand that a vector has both magnitude and direction and be able to distinguish between vector and scalar quantities;
- understand and use the basic properties of vectors in the context of position, velocity and acceleration;
- be able to manipulate vectors in component form;
- recognise that vectors can be used in one, two and three dimensions;
- understand the significance of differentiation of vectors;
- be able to differentiate simple vector functions of time.

3.0 Introduction

"Set course for Zeeton Mr Sulu, warp factor 5"

"Bandits at 3 o'clock, 1000 yards and closing"

There are many situations in which simply to give the size of a quantity without its direction, or direction without size would be hopelessly inadequate. In the first statement above, both direction and speed are specified, in the second, both direction and distance. Another example in a different context is a snooker shot. Both strength and direction are vital to the success of the shot.

Activity 1 Size and direction

Suggest some more situations where both the size and direction of a quantity are important. For two of the situations write down why they are important.

Quantities which require size (often called magnitude) and direction to be specified are called **vector quantities**. They are very different from **scalar quantities** such as time or area, which are completely specified by their magnitude, a number.

Activity 2 Vectors or scalars?

Classify the following as either vector or scalar quantities: temperature, velocity, mass, length, displacement, force, speed, acceleration, volume.

'And after re-entry to earth's atmosphere, Challenger's velocity has been reduced to 800 mph.'

Discuss the statement above and whether the correct meaning is given to the terms speed, velocity and acceleration in everyday language.

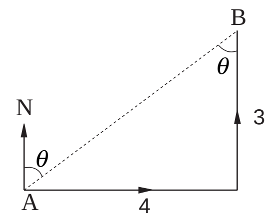
Displacement

One of the most common vector quantities is **displacement**, that is distance and direction of an object from a fixed point.

Example

An aircraft takes off from an airport, A. After flying 4 miles east, it swings round to fly north. When it has flown 3 miles north to B, what is its displacement (distance and direction) from A?

Solution



The distance of from A is AB in the diagram shown opposite. Using Pythagoras' theorem

$$AB^2 = 4^2 + 3^2$$

$$\Rightarrow \quad AB = 5 \text{ miles.}$$

The direction of B from A is the bearing $\theta°$ where

$$\tan\theta = \frac{4}{3}$$

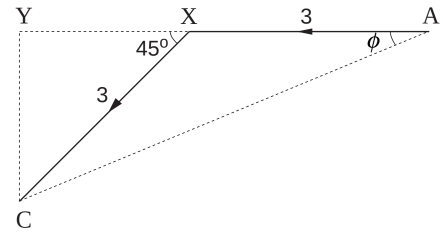
$$\Rightarrow \quad \theta = 53°.$$

The displacement of B from A is 5 miles on a bearing of $053°$.

Example

An aircraft takes off from A facing west and flies for 3 miles before swinging round to fly in a south-westerly direction to C. After it has flown for a further 3 miles, what is its displacement from A?

Solution

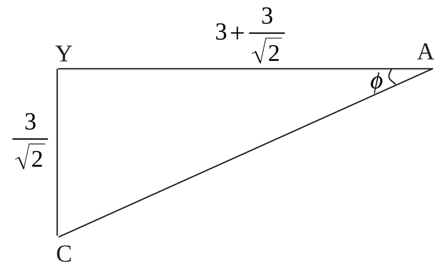


From Δ XYC

$$XY = 3\cos 45 = 2.12$$

$$YC = 3\cos 45 = 2.12.$$

Using Pythagoras' theorem on triangle AYC



$$AC^2 = (3+2.12)^2 + (2.12)^2$$

$$\Rightarrow \quad AC^2 = 26.23 + 4.50 = 30.73$$

$$\Rightarrow \quad AC = 5.54.$$

The direction is the bearing $(270-\phi)°$

where $$\tan\phi = \frac{YC}{AY} = \frac{2.12}{5.12} = 0.414$$

and so

$$\phi = 22.5°$$

The displacement of C from A is 5.54 miles on a bearing of 247.5°.

Column vectors

Distance and bearing is only one method of describing the displacement of an aircraft from A. An alternative would be to use a **column vector**. In the first example above, the displacement of aircraft B from A would be $\begin{pmatrix}4\\3\end{pmatrix}$. This means the aircraft is 4 miles east and 3 miles north of A.

In the second example, the displacement of C from A is

$$\begin{pmatrix}-3-XY\\-YC\end{pmatrix}$$
$$=\begin{pmatrix}-3-2.12\\-1.12\end{pmatrix}$$
$$=\begin{pmatrix}-5.1\\-2.1\end{pmatrix}.$$

Exercise 3A

Find the displacement of each of the following aircraft from A after flying the 2 legs given for a journey. For each, write the displacement using

(a) distance and bearing;

(b) column vector.

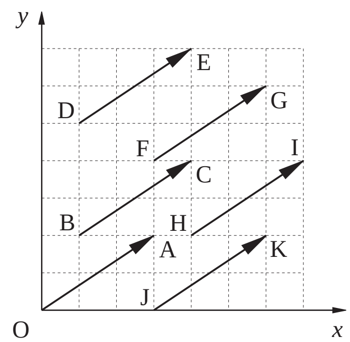
1. The aircraft flies 5 miles north then 12 miles east.
2. The aircraft flies 3 miles west then 6 miles north.
3. The aircraft flies 2 miles east then 5 miles south-east.

An aircraft is at a point X whose displacement from A is $\begin{pmatrix} -5 \\ 6 \end{pmatrix}$.

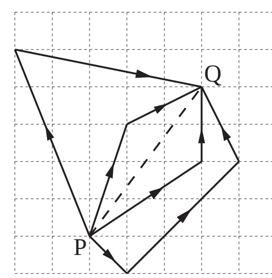
If there is no restriction on the direction it can fly, is there any way of knowing the route it took from A to X?

How is the column vector giving the displacement of A from X related to the column vector giving the displacement of X from A?

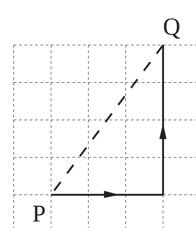
3.1 Vector notation and properties



In the diagram opposite, the displacement of A $(3,\ 2)$ from O $(0,\ 0)$ is described by the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. This time, the entries in the vector give distances in the positive x and positive y directions. The displacement of C $(4,\ 4)$ from B $(1,\ 2)$ is also $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. In fact each of the line segments $\overrightarrow{OA}$, $\overrightarrow{BC}$, $\overrightarrow{DE}$, ..., $\overrightarrow{JK}$ is represented by the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. In general, any displacement of '3 along, 2 up' has vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. However $\overrightarrow{OA}$ is special. It is the only vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which starts at the origin. The **position vector** of the point A $(3,\ 2)$ relative to the origin 0 is said to be $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



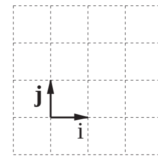
Any displacement such as $\overrightarrow{PQ}$ in the diagram can be thought of as the result of two or more separate displacements. Some possibilities are shown in the diagram, each starting at P and ending at Q.



Of the two stage displacements which are equivalent to the vector $\overrightarrow{PQ}$, only one has its first segment parallel to the positive x direction, and second parallel to the positive y direction. Because this is unique and is extremely useful, it has its own representation.

Components of a vector

A displacement of one unit in the positive x direction is labelled **i** and a displacement of one unit in the positive y direction is labelled **j**. Because each has length one unit, they are called **unit vectors**.



So $\qquad \overrightarrow{PQ} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

can also be written as

$$\overrightarrow{PQ} = 3\mathbf{i} + 4\mathbf{j}$$

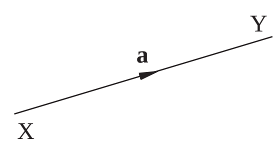
$3\mathbf{i}$ and $4\mathbf{j}$ are known as the **components** of the vector $\overrightarrow{PQ}$.

When working in three dimensions, a third unit vector **k** is introduced (see Section 3.4).

Notation

Besides using the end points of the line segment with an arrow above to denote a vector, you may see a single letter with a line underneath in handwritten text (e.g. $\underline{a}$) or the letter in bold type (e.g. **a)** without the line underneath in printed text.

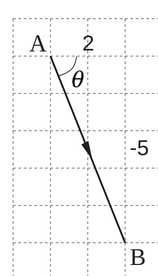
In the diagram opposite, $\overrightarrow{XY}$ and $\mathbf{a}$ are two ways of referring to the vector shown. It should be noted also that $\overrightarrow{YX} = -\mathbf{a}$ is a vector of equal length but in the opposite direction to $\overrightarrow{XY}$ or $\mathbf{a}$.



The reason why underlining letters has become the standard method of denoting a vector is because this is the instruction for a printer to print it in bold type. It is essential that you always remember to underline vectors, otherwise whoever reads your work will not know when you are using vectors or scalars.

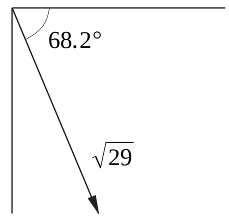
Magnitude and direction of a vector

Consider the vector $\overrightarrow{AB} = 2\mathbf{i} - 5\mathbf{j}$ The magnitude or modulus of vector $\overrightarrow{AB}$, written $|\overrightarrow{AB}|$, is represented geometrically by the length of the line AB.



Using Pythagoras' theorem

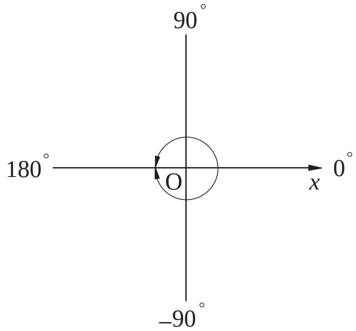
$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{2^2 + (-5)^2}\\ &= \sqrt{4+25}\\ &= \sqrt{29}\\ &= 5.39 \text{ units.}\end{aligned}$$



Its direction is defined by the angle AB makes with the positive x direction. This angle is $-\theta$, where

$$\tan\theta = \frac{5}{2}$$
$$\theta = 68.2°$$

$\overrightarrow{AB}$ has magnitude 5.39 units and its direction makes an angle $-68.2°$ with the x-axis.

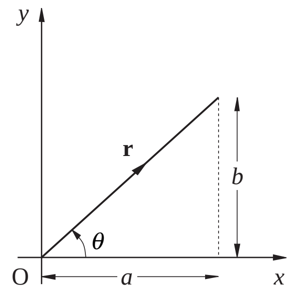


The convention used to define direction in the example above is that angles are measured **positive anticlockwise** from $0x$ up to and including $180°$ and **negative clockwise** from $0x$ up to, but not including, $-180°$.

How many other ways can you find of uniquely defining the direction of a vector?

In general for a vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ its magnitude is given by

$$|\mathbf{r}| = \sqrt{a^2 + b^2}$$



and its direction is given by θ where

$$\cos\theta = \frac{a}{|\mathbf{r}|} \quad \sin\theta = \frac{b}{|\mathbf{r}|} \quad \text{and} \quad \tan\theta = \frac{b}{a}.$$

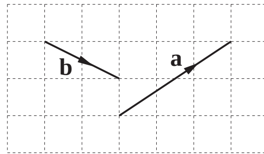
The angle θ can be found from one of these together with a sketch. The vector $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$ has magnitude zero and is called the zero vector.

Adding vectors

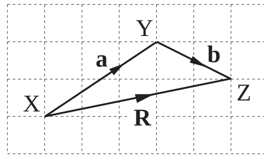
Vectors have magnitude and direction. To complete the definition of a vector, it is necessary to know how to add two vectors.

When vectors are added, it is equivalent to one displacement followed by another.

When $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ is added to $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ it is the same as displacement $\overrightarrow{XY}$ followed by displacement $\overrightarrow{YZ}$. From the diagrams opposite,



$$\begin{aligned} \mathbf{R} &= \mathbf{a} + \mathbf{b} \\ &= (3\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \\ &= (3+2)\mathbf{i} + (2-1)\mathbf{j} \\ &= 5\mathbf{i} + \mathbf{j} \end{aligned}$$



The single vector **R** is equivalent to $\mathbf{a} + \mathbf{b}$, with **R**, **a** and **b** forming a triangle XYZ.

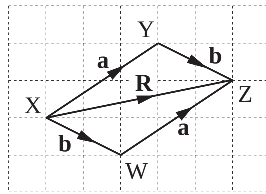
The components are added independently of each other. **R** is called the **resultant** of **a** and **b** and this property of vectors is called the **triangle law of addition**.

In general, in component form, if $\mathbf{p} = d\mathbf{i} + e\mathbf{j}$ and $\mathbf{q} = f\mathbf{i} + g\mathbf{j}$ then $\mathbf{p} + \mathbf{q} = (d+f)\mathbf{i} + (e+g)\mathbf{j}$

Adding vectors can be considered in terms of a parallelogram law as well as a triangle law. In fact, the parallelogram law includes the triangle law

$$\overrightarrow{XZ} = \overrightarrow{XY} + \overrightarrow{YZ}$$

or $\qquad \mathbf{R} = \mathbf{a} + \mathbf{b}$.



From the lower triangle XWZ the result is obviously just as valid. This shows that the resultant vector **R** of **a** and **b** is either **a** followed by **b** or **b** followed by **a**.

> A vector is any quantity possessing the properties of magnitude and direction, which obeys the triangle law of addition

Activity 3 Resultant vectors

Find the resultant vector $\mathbf{a} + \mathbf{b}$

(a) by drawing;

(b) by adding components;

for the following values of **a** and **b**:

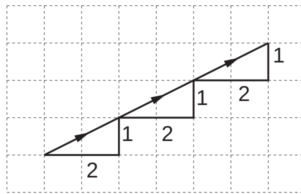
(i) $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ $\quad \mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$

(ii) $\mathbf{a} = -2\mathbf{i} + \mathbf{j}$ $\quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

(iii) $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ $\quad \mathbf{b} = \mathbf{i} + \mathbf{j}$

(iv) $\mathbf{a} = -3\mathbf{i} + \mathbf{j}$ $\quad \mathbf{b} = -2\mathbf{i} - 4\mathbf{j}$

Multiplication by a scalar



Suppose a displacement $2\mathbf{i}+\mathbf{j}$ is repeated three times. This is equivalent to adding three equal vectors:

$$(2\mathbf{i}+\mathbf{j})+(2\mathbf{i}+\mathbf{j})+(2\mathbf{i}+\mathbf{j}).$$

The result is $6\mathbf{i}+3\mathbf{j}$ or $3(2\mathbf{i}+\mathbf{j})$, and the scale factor of 3 scales each component separately.

The vector $2\mathbf{i}+\mathbf{j}$ is said to have been multiplied by the scalar 3. In general, for 2 non-zero vectors **a** and **b**, if $\mathbf{a}=s\mathbf{b}$ where *s* is a scalar, then **a** is **parallel** to **b**.

If $s>0$, **a** and **b** are in the same direction, but if $s<0$ then **a** and **b** are in opposite directions.

In general, in component form, if vector **a** is given as

$$\mathbf{a}=x\mathbf{i}+y\mathbf{j}$$

and *s* is a scalar, then the vector *s***a** is

$$\begin{aligned} s\mathbf{a} &= s(x\mathbf{i}+y\mathbf{j}) \\ &= sx\mathbf{i}+sy\mathbf{j} \end{aligned}$$

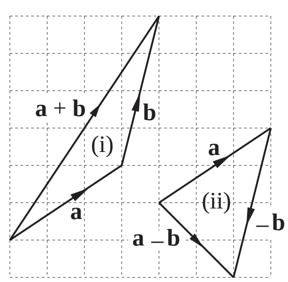
Subtracting vectors

You have seen so far that vectors can be added and multiplied by scalars. What then of subtracting vectors?

Take $\mathbf{a}=3\mathbf{i}+2\mathbf{j}$ and $\mathbf{b}=\mathbf{i}+4\mathbf{j}$

Operating as with addition,

$$\begin{aligned} \mathbf{a}-\mathbf{b} &= \mathbf{a}+(-\mathbf{b}) \\ &= 3\mathbf{i}+2\mathbf{j}+(-\mathbf{i}-4\mathbf{j}) \\ &= (3-1)\mathbf{i}+(2-4)\mathbf{j} \\ &= 2\mathbf{i}-2\mathbf{j} \end{aligned}$$



Geometrically, $-\mathbf{b}$ is equal in magnitude but opposite in direction to **b**, so while $\mathbf{a}+\mathbf{b}$ is shown in (i) on the right, $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$ is shown in (ii).

You can see that both addition and subtraction of vectors use the triangle law of addition.

Exercise 3B

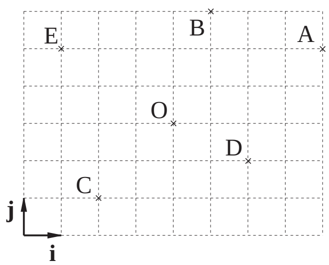
1. Write in the form $a\mathbf{i}+b\mathbf{j}$ the vectors:

   (a) $\overrightarrow{OA}$ (b) $\overrightarrow{OB}$ (c) $\overrightarrow{AB}$

   (d) $\overrightarrow{BA}$ (e) $\overrightarrow{BC}$ (f) $\overrightarrow{CD}$

   (g) $\overrightarrow{BD}$ (h) $\overrightarrow{CE}$ (i) $\overrightarrow{DA}$

   (j) $\overrightarrow{EA}$ (k) $\frac{1}{2}\overrightarrow{EC}$ (l) $5\overrightarrow{CA}$.



2. In the diagram for question 1:

   (a) the point Q has position vector $3\mathbf{i}+\mathbf{j}$ Find the vectors $\overrightarrow{QO}$, $\overrightarrow{QC}$, $\overrightarrow{DQ}$;

   (b) the point R has position vector $p\mathbf{i}+q\mathbf{j}$ Find in terms of p and q, $\overrightarrow{RO}$, $\overrightarrow{RC}$ and $\overrightarrow{AR}$.

3. The triangle law of addition for $\overrightarrow{AB}$ can be verified using triangle OAB from the diagram of question 1:-

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$$

L.H.S. $=\overrightarrow{AB}$
$= -3\mathbf{i}+\mathbf{j}$

R.H.S. $=\overrightarrow{AO}+\overrightarrow{OB}$
$=(-4\mathbf{i}-2\mathbf{j})+(\mathbf{i}+3\mathbf{j})$
$=(-4\mathbf{i}+\mathbf{i})+(-2\mathbf{j}+3\mathbf{j})$
$=-3\mathbf{i}+\mathbf{j}$

Hence L.H.S. = R.H.S. showing $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$.

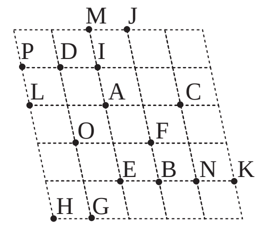
Verify the triangle law of addition for $\overrightarrow{CD}$ using triangle OCD.

4. Using the grid shown, write down as many vectors as you can that are equal to:

   (a) $\overrightarrow{OA}$ (b) $\overrightarrow{EB}$

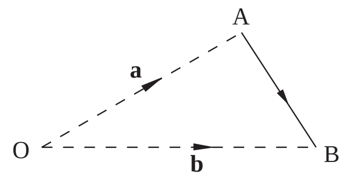
   (c) $\overrightarrow{BO}$ (d) $\overrightarrow{EI}$

   (e) $\overrightarrow{FL}$ (f) $\overrightarrow{GP}$.



3.2 Relative position vectors

The position vector of B relative to A is simply $\overrightarrow{AB}$. Then, using the triangle law of addition,



$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$$
$$=-\mathbf{a}+\mathbf{b}$$
$$=\mathbf{b}-\mathbf{a}.$$

where **a** and **b** are the position vectors of A and B, respectively, relative to the origin O.

Example

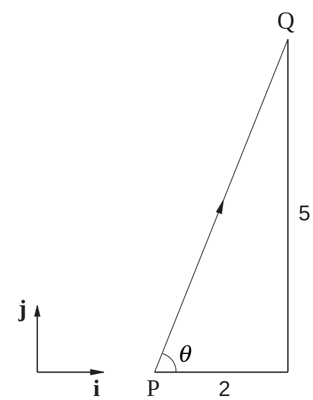
The point P has position vector $-\mathbf{i}+\mathbf{j}$ the point Q, $\mathbf{i}+6\mathbf{j}$ and the point R, $-\mathbf{j}$ Find the magnitudes and directions of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

Solution

$$\vec{PQ} = \vec{PO} + \vec{OQ}$$
$$= -\vec{OP} + \vec{OQ}$$
$$= \vec{OQ} - \vec{OP}$$
$$= (\mathbf{i} + 6\mathbf{j}) - (-\mathbf{i} + \mathbf{j})$$
$$= 2\mathbf{i} + 5\mathbf{j}$$

The magnitude of $\vec{PQ}$ is given by

$$|\vec{PQ}| = \sqrt{2^2 + 5^2}$$
$$= \sqrt{29} = 5.39.$$



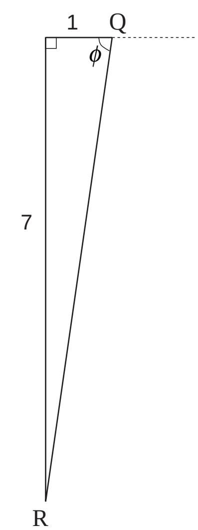
The direction is given by

$$\tan\theta = \frac{5}{2}$$

so that $\theta = 68.2°$.

Similarly

$$\vec{QR} = \vec{OR} - \vec{OQ}$$
$$= -\mathbf{j} - (\mathbf{i} + 6\mathbf{j})$$
$$= -\mathbf{i} - 7\mathbf{j}$$



The magnitude of $\vec{QR}$ is given by

$$|\vec{QR}| = \sqrt{(-1)^2 + (-7)^2}$$
$$= \sqrt{50} = 7.07.$$

The direction of $\vec{QR}$ is $-180° + \phi$, where

$$\tan\phi = \frac{7}{1}$$

giving $\phi = 81.9°$.

So the direction of $\vec{QR}$ is $-180° + 81.9° = -98.1°$.

Exercise 3C

1. Find the magnitudes and directions of:

   $\mathbf{a} = 3\mathbf{i} - 9\mathbf{j}$ $\quad \mathbf{b} = -2\mathbf{i} - \mathbf{j}$

   $\mathbf{c} = \sqrt{3}\mathbf{i} + 3\mathbf{j}$ $\quad \mathbf{d} = -5\mathbf{i} + 4\mathbf{j}$

2. Points P, Q, R and S have position vectors

   $\mathbf{p} = 2\mathbf{i} - \mathbf{j}$ $\mathbf{q} = -\mathbf{i} + \mathbf{j}$ $\mathbf{r} = 2\mathbf{j}$ and $\mathbf{s} = a\mathbf{i} + 6\mathbf{j}$

   Find the magnitudes of $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{QR}$ and $\overrightarrow{PS}$.

   Given that $\left|\overrightarrow{PS}\right| = \sqrt{50}$, find the possible values of a.

3. Points A to F have position vectors **a** to **f** respectively, defined in terms of **p**, **q** and **r** as follows:

   $\mathbf{a} = \mathbf{p} + \mathbf{q} + \mathbf{r}$, $\quad \mathbf{b} = \mathbf{p} + \mathbf{q} - \mathbf{r}$, $\quad \mathbf{c} = 2\mathbf{p}$,

   $\mathbf{d} = 3\mathbf{q} - \mathbf{r}$, $\quad \mathbf{e} = -\mathbf{p} + 4\mathbf{q}$, $\quad \mathbf{f} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

   Find in terms of **p**, **q** and **r**, the vectors

   (a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AC}$

   (d) $\overrightarrow{EC}$ (e) $\overrightarrow{BD}$ (f) $\overrightarrow{FA}$

   (g) $\overrightarrow{DF}$ (h) $\overrightarrow{CE}$ (i) $\overrightarrow{ED}$

   (j) $\overrightarrow{BF}$.

3.3 Unit vectors

You should be getting used to using **i** and **j** which are unit vectors in the directions Ox and Oy. A **unit vector** is simply a vector having **magnitude one**, and can be in **any direction**. To find a unit vector in the direction of $\mathbf{c} = 3\mathbf{i} - 4\mathbf{j}$ you multiply by a scalar, so that its direction is unchanged but its magnitude is altered to one.

The magnitude of **c** is

$$|\mathbf{c}| = \sqrt{3^2 + (-4)^2}$$
$$= 5$$

which is five times as big as the magnitude of a unit vector. **c** must be multiplied by $\frac{1}{5}$ or divided by 5 to make a unit vector in its direction.

A unit vector in the direction of **c** is

$$\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$$
$$= \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

In general, if a vector **a** has magnitude $|\mathbf{a}|$, then a unit vector in the direction of **a** is denoted $\hat{\mathbf{a}}$ and

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Equal vectors

If the vectors $c\mathbf{i} + d\mathbf{j}$ and $e\mathbf{i} + f\mathbf{j}$ are equal, then

$$c\mathbf{i} + d\mathbf{j} = e\mathbf{i} + f\mathbf{j}$$

and it follows that

$$c = e \text{ and } d = f.$$

These are the only possible conclusions if the vectors are equal. Note that $c = e$ comes from equating the **i** components of the equal vectors and $d = f$ comes from equating the **j** components.

You will see in later chapters that the technique of 'equating components' is very useful in the solution of problems.

Example

Vectors **p** and **q** are defined in terms of x and y as

$$\mathbf{p} = 3\mathbf{i} + (y-2)\mathbf{j} \text{ and } \mathbf{q} = 2x\mathbf{i} - 7\mathbf{j}$$

If $\mathbf{p} = 2\mathbf{q}$, find the values of x and y.

Solution

Since $\mathbf{p} = 2\mathbf{q}$,

$$3\mathbf{i} + (y-2)\mathbf{j} = 2(2x\mathbf{i} - 7\mathbf{j})$$

giving $3\mathbf{i} + (y-2)\mathbf{j} = 4x\mathbf{i} - 14\mathbf{j}$

Equating **i** components gives

$$3 = 4x$$

and so

$$x = \frac{3}{4}.$$

Equating **j** components gives

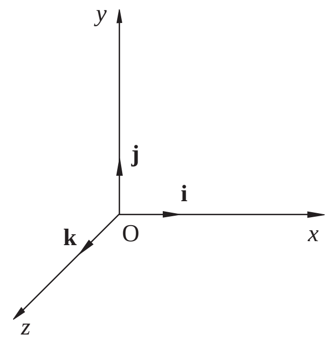
$$y - 2 = -14$$

and so

$$y = -12.$$

3.4 Vectors in three dimensions

The results obtained so far have all been applied to vectors in one or two dimensions. However, the power of vectors is that they can be applied in one, two or three dimensions. Although the applications of mechanics in this book will be restricted to one or two dimensions, by taking a vector approach, the extensions to three dimensional applications will be easier.



In order to work in three dimensions, it is necessary to define a third axis Oz, so that Ox, Oy and Oz form a right-handed set as in the diagram opposite. An ordered trio of numbers such as (2, 3, 4) is necessary to define the coordinates of a point and a vector must have 3 components. For example, the position vector of the point (2, 3, 4) is $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ where **k** is a unit vector in the direction Oz.

The properties of vectors considered so far are all defined in three dimensions as the following examples show.

Example

If $\mathbf{p}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{q}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, find $\mathbf{p}+\mathbf{q}$ and $4\mathbf{q}$.

Solution

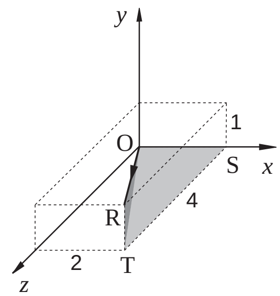
$$\begin{aligned}\mathbf{p}+\mathbf{q} &= (3\mathbf{i}-2\mathbf{j}+\mathbf{k})+(\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= (3+1)\mathbf{i}+(-2+3)\mathbf{j}+(1-2)\mathbf{k}\\ &= 4\mathbf{i}+\mathbf{j}-\mathbf{k}.\\ 4\mathbf{q} &= 4(\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= 4\mathbf{i}+12\mathbf{j}-8\mathbf{k}.\end{aligned}$$

Example

The point R has position vector $\mathbf{r}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$. Find OR.

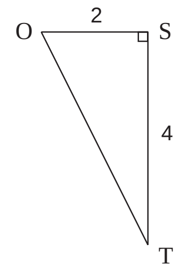
Solution

From the diagram, it can be seen that OR is the diagonal of a cuboid with dimensions 2, 1, 4 units.



Using Pythagoras' theorem, OR can be found from the right angled triangles OST, OTR.

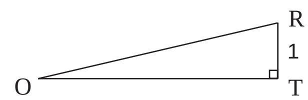
From triangle OST,



$$\begin{aligned}OT^2 &= OS^2 + ST^2\\ &= 2^2 + 4^2\\ &= 4 + 16\end{aligned}$$

giving $\quad OT = \sqrt{20}.$

From triangle OTR,



$$\begin{aligned}OR^2 &= OT^2 + RT^2\\ &= 20 + 1^2\end{aligned}$$

giving $\quad OR = \sqrt{21} = 4.58.$

Note that $OR = |\mathbf{r}|$ and that extending the two dimensional result for modulus of a vector gives

$$\begin{aligned}|\mathbf{r}| &= \sqrt{2^2 + 1^2 + 4^2}\\ &= \sqrt{4 + 1 + 16}\\ &= \sqrt{21} = 4.58.\end{aligned}$$

In general, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

then $\quad |\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}.$

Example

The points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Find $\overrightarrow{AB}$, $|\overrightarrow{BC}|$ and the unit vector in the direction of $\overrightarrow{BC}$.

Solution

The vector $\overrightarrow{AB}$ is given by

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a}\\ &= (-2\mathbf{i} + \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 3\mathbf{k})\\ &= -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}.\end{aligned}$$

$$\vec{BC} = \mathbf{c} - \mathbf{b}$$
$$= (3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (-2\mathbf{i} + \mathbf{k})$$
$$= 5\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}.$$
$$|\vec{BC}| = \sqrt{5^2 + 2^2 + (-5)^2}$$
$$= \sqrt{54} = 7.35$$

The unit vector in the direction of BC is $\frac{1}{\sqrt{54}}(5\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$

$$= \frac{5}{\sqrt{54}}\mathbf{i} + \frac{2}{\sqrt{54}}\mathbf{j} - \frac{5}{\sqrt{54}}\mathbf{k}.$$

Exercise 3D

1. If $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - 4\mathbf{k}$, find:
   (a) $\mathbf{a} + \mathbf{b} + \mathbf{c}$;
   (b) $5\mathbf{a}$;
   (c) $2\mathbf{b} + 3\mathbf{c}$.
2. If $\mathbf{p} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$, $\mathbf{q} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{r} = -3\mathbf{j} + 4\mathbf{k}$ are the position vectors of the points P, Q and R, find:
   (a) the position vector $\vec{PQ}$;
   (b) $\vec{QR}$;
   (c) $|\vec{PR}|$;
   (d) the unit vector in the direction $\vec{QP}$.
3. If the vectors $4\mathbf{i} - m\mathbf{j}$ and $(3n-2)\mathbf{i} + (3n+2)\mathbf{j}$ are equal in magnitude but opposite in direction, find the values of m and n.
4. Vectors **a** and **b** are defined in terms of x, y and z as
   $$\mathbf{a} = \mathbf{i} + (x+y)\mathbf{j} - (2x+y)\mathbf{k}$$
   $$\mathbf{b} = z\mathbf{i} + (3y+2)\mathbf{j} - (5x-3)\mathbf{k}.$$
   If $\mathbf{a} = \mathbf{b}$ then find the values of x, y and z and show that
   $$\left|\frac{5}{4}x\mathbf{i} + 5y\mathbf{j} + 6z\mathbf{k}\right| = \sqrt{46}.$$

3.5 Scalar products

So far, vectors have been added, subtracted and multiplied by a scalar. Just as the addition of two vectors is a different operation from the addition of two real numbers, the product of two vectors has its own definition.

The **scalar product** of two vectors, **a** and **b**, is defined as $ab\cos\theta$, where θ is the angle between the two vectors, and **a** and **b** are the moduli (or magnitude) of **a** and **b**. The scalar product is usually written as **a.b**, read as "a dot b", so

$$\mathbf{a}.\mathbf{b} = ab\cos\theta.$$

If the vectors **a** and **b** are perpendicular the scalar product **a.b** is zero since $\cos 90° = 0$.

So

$\mathbf{a}.\mathbf{b} = 0$ when **a** is perpendicular to **b**.

Also, if the vectors **a** and **b** are parallel, the scalar product **a.b** is given by *ab*, since $\cos 0 = 1$.

So

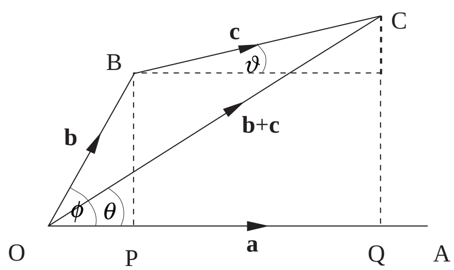
$\mathbf{a}.\mathbf{b} = ab$ when **a** is parallel to **b**.

The scalar product follows both the commutative and distributive laws.

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= ab\cos\theta \\ &= ba\cos\theta \\ &= \mathbf{b}.\mathbf{a}\end{aligned}$$

This shows that the scalar product is commutative ie $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

The scalar product **a** .(**b**+**c**) can be found by considering the diagram opposite.



Since $OQ = OP + PQ$

then $|\mathbf{b}+\mathbf{c}|\cos\theta = b\cos\phi + c\cos\vartheta$.

Multiplying by a $(= |\mathbf{a}|)$ gives

$$|\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\theta = ab\cos\phi + ac\cos\vartheta$$

So $\qquad \mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$

This result shows that the scalar product is distributive over addition.

Calculating the scalar product

To calculate the scalar product of two vectors you must find the magnitude of each vector and the angle between the vectors.
The magnitude of a vector $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$|p| = \sqrt{a^2 + b^2 + c^2}\ .$$

Since **i**, **j** and **k** are perpendicular unit vectors, the scalar product gives some valuable results.

Perpendicular vectors: $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = \mathbf{k}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$

Parallel vectors: $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

These results are used to find **p.q**

where $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and $\mathbf{q} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$

$$\begin{aligned} \mathbf{p}.\mathbf{q} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).(d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\ &= ad\,\mathbf{i}.\mathbf{i} + ae\,\mathbf{i}.\mathbf{j} + af\,\mathbf{i}.\mathbf{k} \\ &\quad + bd\,\mathbf{j}.\mathbf{i} + be\,\mathbf{j}.\mathbf{j} + bf\,\mathbf{j}.\mathbf{k} \\ &\quad + cd\,\mathbf{k}.\mathbf{i} + ce\,\mathbf{k}.\mathbf{j} + cf\,\mathbf{k}.\mathbf{k} \\ &= ad + 0 + 0 + 0 + be + 0 + 0 + 0 + cf \end{aligned}$$

$$\mathbf{p}.\mathbf{q} = ad + be + cf \qquad (1)$$

But $\mathbf{p}.\mathbf{q} = pq\cos\theta$

So $$\cos\theta = \frac{\mathbf{p}.\mathbf{q}}{pq} = \frac{ad + be + cf}{pq} \qquad (2)$$

Example

If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $b = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ find

a.b and the angle between **a** and **b.**

Solution

$$a = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$b = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

Using equation (1)

$$\begin{aligned} \mathbf{a}.\mathbf{b} &= 1 \times 2 + 1 \times 3 + 1 \times 4 \\ &= 9 \end{aligned}$$

and equation (2)

$$\cos\theta = \frac{9}{\sqrt{3}\sqrt{29}}$$
$$\theta = 15.2^\circ$$

Exercise 3E

1. Show that the vectors $2\mathbf{i}+5\mathbf{j}$ and $15\mathbf{i}-6\mathbf{j}$ are perpendicular.
2. Show that the vectors $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ are perpendicular.
3. If $\mathbf{a}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$, find **a.b** and the angle between **a** and **b**.
4. If two forces are described by the vectors $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ find the cosine of the angle between the forces.

3.6 Velocity as a vector

Velocity is the rate of change of displacement, a measure of how the position of an object is changing with time.

Contrast this with speed, which is the rate of change of distance travelled with respect to time.

To compare these quantities, consider a child on a roundabout in a playground, being pushed by her father. He pushes slowly but steadily so that it takes 4 seconds for one revolution. The circumference of the roundabout is 12 metres.

For one revolution, the child's average speed is

$$\frac{12 \text{ metres}}{4 \text{ seconds}} = 3 \text{ ms}^{-1}.$$

But after one revolution, the child's displacement from her initial position is **0**. (She is back to where she started.)for one revolution, her average velocity is

$$\frac{\mathbf{0}}{4} = \mathbf{0}.$$

The magnitude of her average velocity is 0 ms^{-1}.

The magnitude of the average velocity and the average speed are equal only when the motion is in a straight line, with no changes of direction.

If an object changes from position **r** to position **s** in a time *t*, then the average velocity is given by

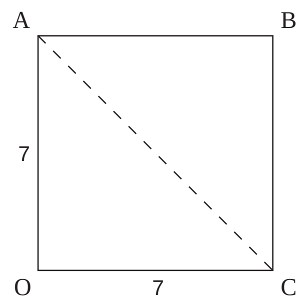
$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time}}$$

or

$$\mathbf{v} = \frac{\mathbf{s}-\mathbf{r}}{t} \text{ or } \frac{1}{t}\mathbf{s} - \frac{1}{t}\mathbf{r}.$$

Because this is the difference of two scaled displacement Because this is the difference of two scaled displacement vectors, each multiplied by a scalar, it is a vector quantity.

For example, Sally and Floella use $\mathbf{v} = \frac{\mathbf{s} - \mathbf{r}}{t}$ to calculate a known average velocity to verify that the result works. Their classroom is a square 7 metres by 7 metres and the intention is that one of the girls walks the diagonal at steady speed. They will then calculate her average velocity both directly and from $\mathbf{v} = \frac{\mathbf{s} - \mathbf{r}}{t}$ so that the results can be compared.



They use Pythagoras' theorem to calculate the length of the diagonal and find it is approximately 10 metres.

Sally walks the diagonal AC at steady speed. Floella times her at 10 seconds. So they know her average speed is $\frac{10}{10} = 1 \text{ ms}^{-1}$. Taking the origin as O, OC as x-axis and OA as y-axis, her direction is $-45°$ to the x-axis. So they take her average velocity to be 1 ms^{-1} at $-45°$ to Ox.

Sally and Floella take the magnitude of the average velocity to be the average speed. Why?

This is their verification:-

since the position vector of A is 7**j**

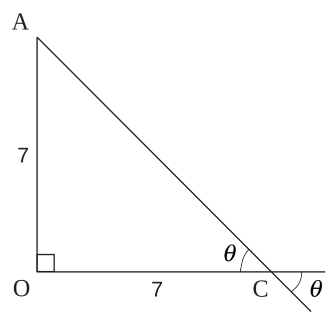
and the position vector of C is 7**i**,

the average velocity $\mathbf{v} = \frac{7\mathbf{i} - 7\mathbf{j}}{10}$ $\left(\text{using } \frac{\mathbf{s} - \mathbf{r}}{t}\right)$.

The magnitude of **v** is

$$|\mathbf{v}| = \frac{\sqrt{7^2 + 7^2}}{10} = \frac{\sqrt{98}}{10} \approx 1 \text{ ms}^{-1}$$

and its direction is given by $-\theta$ to Ox where



$$\tan\theta = \frac{7}{7}$$

giving $\theta = 45°$.

So using $\mathbf{v} = \frac{\mathbf{s} - \mathbf{r}}{t}$ gives them the same result as the known average velocity, i.e. 1 ms^{-1} at $-45°$ to Ox.

3.7 Acceleration as a vector

Acceleration is the rate of change of velocity with time. If an object changes from a velocity **v** to a velocity **w** in a time *t*, then average acceleration is

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time}}$$

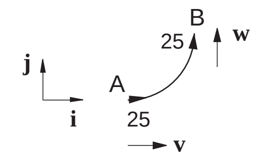
or

$$\mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t}.$$

This is also a vector quantity.

Example

A car travels round a bend which forms a quadrant of a circle at a constant speed of 25 ms^{-1}. If the bend takes 5 seconds to negotiate, find the average acceleration of the car during this period.

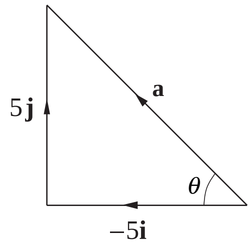


Solution

Define unit vectors **i**, **j** as shown.

The velocity of the car at A is $\mathbf{v} = 25\mathbf{i}$ and at B is $\mathbf{w} = 25\mathbf{j}$

so the average acceleration from A to B is



$$\mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t} = \frac{25\mathbf{j} - 25\mathbf{i}}{5}$$

$$= -5\mathbf{i} + 5\mathbf{j}$$

The magnitude of **a** is

$$|\mathbf{a}| = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07,$$

and its direction is $180 - \theta$ where

$$\tan\theta = 1, \quad \theta = 45°.$$

Note that the speed of the car is not changing as it negotiates the bend, but the velocity is changing.

Exercise 3F

1. Verify the result:

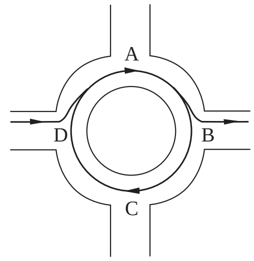
   $$\text{average velocity} = \frac{\mathbf{s} - \mathbf{r}}{t}$$

   based on walking the diagonal of a rectangular room 6 metres by 8 metres in 8 seconds.

2. The driver of a car is unsure of his route. He approaches a roundabout intending to turn right, but changes his mind, eventually doing $1\frac{1}{2}$ circuits of the roundabout and going straight on.

   If he travels at a constant 12 ms^{-1} and takes 16 seconds for one complete circuit ABCDA, then calculate his average acceleration for the $\frac{1}{4}$ circle AB. Hence write down the average acceleration for $\frac{1}{4}$ circles BC, CD, DA.

   Calculate the average acceleration for the $\frac{1}{2}$ circle CDA.



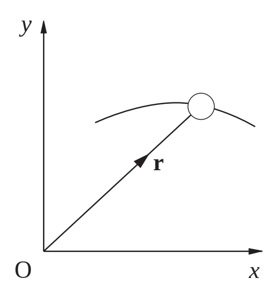
Discuss what you would understand to be the average acceleration for a complete circle of the roundabout in Question 2.

3.8 Instantaneous velocity and acceleration

You have seen that for a one dimensional motion, if the displacement x from a fixed point O is known as a function of time $x = f(t)$ then the velocity and acceleration at any time t can be found by successive differentiation:

$$\text{velocity} = \frac{dx}{dt} \qquad \text{acceleration} = \frac{d^2x}{dt^2}.$$

One dimensional motion is not very common however. The motion of a tennis ball during a game, or a child on a swing or a jumping frog takes place in two or three dimensions.



If the position vector **r** of a netball passed between 2 players is known, you can calculate the average velocity and average acceleration. But what about velocity and acceleration at an instant of time t?

If **r** is known as a function of t, can this be successively differentiated? If so, can the same meanings be attached to these derivatives as in the one dimensional case?

Two kinematics activities revisited

In your study of one dimensional kinematics, the motion of a rolling ball was investigated in two activities, 'Galileo's experiment' and the 'chute experiment'. The motion of a ball rolling down an incline was investigated in the first of these activities, and in the second, the motion of a ball along a level horizontal surface.

In 'Galileo's experiment', you will have obtained a relationship of the form

$$y = kt^2 \qquad (3)$$

where y is the distance in metres rolled by the ball in time t seconds, and k is a constant. This is a simple quadratic function model for a ball rolling down an incline and your k was probably about 0.1.

Choosing $k = 0.1$, equation (3) becomes

$$y = 0.1t^2.$$

Differentiating this gives the velocity, $\frac{dy}{dt}$, at time t,

$$\frac{dy}{dt} = 2(0.1)t$$

$$\frac{dy}{dt} = 0.2t.$$

Differentiating again gives the acceleration, $\frac{d^2y}{dt^2}$, at time t,

$$\frac{d^2y}{dt^2} = 0.2$$

This is constant, independent of time.

In the 'chute experiment', you will have obtained a relationship of the form

$$x = ct \qquad (4)$$

where x is the distance in metres rolled by the ball in time t seconds, and c is a constant. This is a simple linear function model for a ball rolling along a level horizontal surface and your value of c was probably about 0.5.

Choosing $c = 0.5$, equation (4) becomes

$$x = 0.5t.$$

Differentiating this gives the velocity $\frac{dx}{dt}$, at time t

$$\frac{dx}{dt} = 0.5 \qquad \text{(constant velocity).}$$

Differentiating again gives the acceleration, $\frac{d^2x}{dt^2}$, at time t

$$\frac{d^2x}{dt^2} = 0 \qquad \text{(zero acceleration).}$$

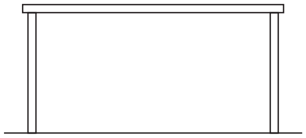
Activity 4 Combining 'Galileo's experiment' with the 'chute experiment'.

You will need the following for this activity:

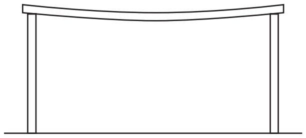
level table, a chute, billiard ball, stopwatch, two blocks to incline the table.

Stage 1

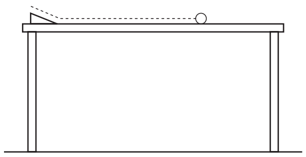
Set up the chute experiment. Mark a point on the chute from which the ball takes about 2 seconds to roll the length of the table. Make sure you **always** release the ball from this mark in stages 1 and 3.



Choose a level table



Not bowed in the middle



Take readings of time for **one** distance only, say 1 metre, for several releases of the ball. Average the results. Use this one pair of readings for x and t to substitute into the model $x = ct$ and find c, rounding your value to one decimal place. For example, if your values are

$$x = 1 \text{ metres}, \quad t = 1.45 \text{ seconds}$$

then substituting into $x = ct$ gives

$$1 = 1.45c$$

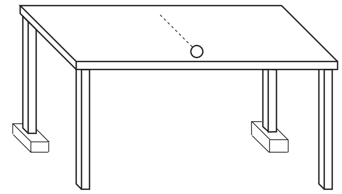
$$\Rightarrow \quad c = \frac{1}{1.45}$$

$$\Rightarrow \quad c = 0.7 \qquad \text{(1 decimal place).}$$

Stage 2

Incline the table so that the slope runs down its width and the ball takes about 2 seconds to roll down the width. Now do 'Galileo's experiment' as follows.

Take readings of time for **one** distance only for the ball to roll from rest down the width of the table. Average the results. Use this one pair of readings for y and t to substitute into the model $y = kt^2$ and find k, rounding your value to one decimal place. For example, if your values are



$$y = 0.6 \text{ m} \qquad t = 2.26 \text{ s}$$

then substituting into $y = kt^2$ gives

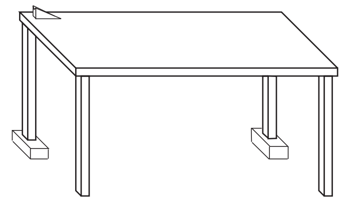
$$0.6 = k(2.26)^2$$

$$\Rightarrow \qquad k = \frac{0.6}{(2.26)^2}$$

$$\Rightarrow \qquad k = 0.1 \qquad \text{(1 decimal place).}$$

Stage 3

Leaving the table inclined exactly as in stage 2, set up the chute at the top left hand side of the table, but pointing **along** its length.



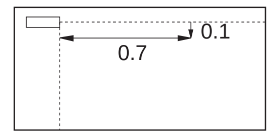
Before releasing the ball down the chute, predict its path using the values of c and k from stages 1 and 2.

If you have

$$x = 0.7t$$

and

$$y = 0.1t^2,$$



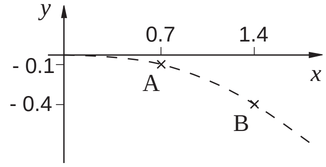
then, when $t = 1$ second

$$x = 0.7(1) = 0.7$$

and $$y = 0.1\left(1^2\right) = 0.1 \text{ m.}$$

This predicts the coordinates of the position of the ball after one second.

Make 4 or 5 predictions using your own values of c and k. Mark them lightly on the table in chalk and join them with a smooth curve.

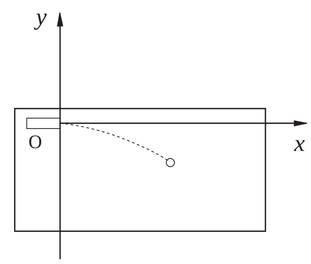


Having marked the predictions, release the ball from your mark on the chute and see how accurate your predictions are.

Release the ball a number of times; you should consistently obtain a curved path close to that predicted.

Position vector of the ball

Choose axes as shown in the diagram opposite. They are in the plane of the table. Ox is along the length of the table in line with the chute. Oy is perpendicular to Ox but pointing up the incline so that the origin O is at the foot of the chute.



For the data given as an example in Activity 4, the path of the ball is predicted by the set of coordinates $(x, y) = (0.7t, -0.1t^2)$, or by the position vector

$$\mathbf{r} = 0.7t\,\mathbf{i} - 0.1t^2\,\mathbf{j}. \quad (5)$$

This shows that the two one dimensional motions of Stages 1 and 2 have been combined to produce a single motion in two dimensions in Stage 3.

What is the path predicted by your data?

3.9 Investigating the velocity and speed of the ball

The average velocity of the ball in Activity 4 can be found between any two points on its path using the result of Section 3.6,

$$\text{average velocity } \mathbf{v} = \frac{\mathbf{s} - \mathbf{r}}{t}.$$

For the average velocity between $t = 1$ and $t = 2$ seconds,

when $t = 1$

$$\mathbf{r} = 0.7(1)\mathbf{i} - 0.1(1^2)\mathbf{j} \text{ from equation (5)}$$
$$= 0.7\mathbf{i} - 0.1\mathbf{j}$$

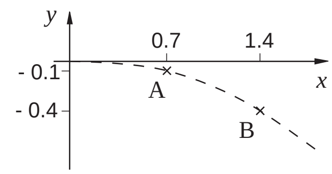
when $t = 2$

$$\mathbf{s} = 0.7(2)\mathbf{i} - 0.1(2^2)\mathbf{j}$$
$$= 1.4\mathbf{i} - 0.4\mathbf{j}$$

So the average velocity

$$= \frac{(1.4\mathbf{i} - 0.4\mathbf{j}) - (0.7\mathbf{i} - 0.1\mathbf{j})}{2 - 1}$$
$$= 0.7\mathbf{i} - 0.3\mathbf{j}$$

The diagram opposite shows the positions A and B of the ball when $t = 1$ second and when $t = 2$ seconds.



As well as the velocity of the ball between these two times, the approximate average speed can be calculated using

$$\text{approximate average speed} = \frac{\text{straight line distance AB}}{\text{time taken}}$$

$$= \frac{\sqrt{0.7^2 + 0.3^2}}{2 - 1}$$

$$= 0.76 \text{ ms}^{-1}.$$

What are the average velocity and approximate average speed between $t = 1$ and $t = 2$ seconds for your data from Activity 4?

Activity 5 Calculating average velocity

For the ball whose path is given by equation (5), find the average velocity and approximate average speed between the following time intervals.

time intervals	average velocity	approximate average speed
$t = 1$ to $t = 2$	$0.7\mathbf{i} - 0.3\mathbf{j}$	0.76
$t = 1$ to $t = 1.5$		
$t = 1$ to $t = 1.2$		
$t = 1$ to $t = 1.1$		
$t = 1$ to $t = 1.05$		

What do you notice about your results? If you were asked to predict the velocity and speed of the ball at the point A, when $t = 1$ second, what would you predict? Write down both of your predictions.

Why is the average speed used in Activity 5 approximate?

What could you have measured to make it exact?

How accurate do you consider the approximation to be?

Velocity and speed at a point on the path or at an instant of time *t*

Can a meaning now be given to differentiating the position vector

$$\mathbf{r} = 0.7t\,\mathbf{i} - 0.1t^2\,\mathbf{j}?$$

Since **i** and **j** are fixed unit vectors, they are constants and are treated as such when differentiating.

So $$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 2(0.1)t\,\mathbf{j}$$

giving $$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\,\mathbf{j} \qquad (6)$$

Since the position vector is a function of *t*, differentiating it is simply differentiating its components as functions of *t*.

Substituting $t = 1$ in equation (6) gives

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2\,\mathbf{j}$$

whose magnitude is

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{0.7^2 + 0.2^2} = 0.73.$$

Compare these values with your predictions of velocity and speed at $t = 1$ from Activity 5. You should find

$\frac{d\mathbf{r}}{dt}$ is the velocity of the ball at $t = 1$,

$\left|\frac{d\mathbf{r}}{dt}\right|$ is the speed of the ball at t = 1.

Example

A ball is rolled on a table and axes chosen so that its position vector at time *t* seconds is given in metres by

$$\mathbf{r} = \left(1 - 0.2t^2\right)\mathbf{i} + 0.6t\,\mathbf{j}$$

Find its velocity and speed after 2 s.

Solution

$$\mathbf{r} = \left(1 - 0.2t^2\right)\mathbf{i} + 0.6t\,\mathbf{j}.$$

Differentiating with respect to *t* gives

$$\frac{d\mathbf{r}}{dt} = -0.4t\mathbf{i} + 0.6\mathbf{j}$$

When $t = 2$

$$\frac{d\mathbf{r}}{dt} = -0.8\mathbf{i} + 0.6\mathbf{j}$$

and $$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(0.8)^2 + 0.6^2} = 1.$$

So velocity is $-0.8\mathbf{i} + 0.6\mathbf{j}\,\text{ms}^{-1}$, and the speed is 1 ms^{-1}.

Example

The position vector of a clay pigeon t seconds after release is given in metres by

$$\mathbf{r} = 3t\mathbf{i} + \left(30t - 5t^2\right)\mathbf{j} + 40t\mathbf{k}.$$

Calculate its velocity and speed after one second.

Solution

Differentiating **r** with respect to t gives

$$\frac{d\mathbf{r}}{dt} = 3\mathbf{i} + (30 - 10t)\mathbf{j} + 40\mathbf{k}.$$

When $t = 1$

$$\frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 20\mathbf{j} + 40\mathbf{k}$$

and

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3^3 + 20^2 + 40^2}$$

$$= 44.8\text{ ms}^{-1}.$$

After one second, its velocity is $3\mathbf{i} + 20\mathbf{j} + 40\mathbf{k}\text{ ms}^{-1}$ and its speed is 44.8 ms^{-1}.

Exercise 3G

1. The position vector at time t seconds of a football struck from a free kick is given in metres by:

   $\mathbf{r} = 24t\mathbf{i} + \left(7t - 5t^2\right)\mathbf{j}$

   Find its velocity at time t and determine the speed with which the ball was struck.

2. The position vector of an aircraft flying horizontally at time t seconds is given in metres by:

   $\mathbf{r} = 120t\mathbf{i} + 160t\mathbf{j}$

   where **i** is directed east and **j** north. What is the speed of the aircraft? On what bearing is it heading?

3. The position vector of a golf ball t seconds after it has been struck is given in metres by:

$$\mathbf{r} = 60t\,\mathbf{i} + \left(12t - 5t^2\right)\mathbf{j} - t\,\mathbf{k}.$$

Find its speed and its velocity after 2.4 seconds.

4. During the first three seconds of her fall, the position vector of a sky-diver is given in metres by:

$$\mathbf{r} = 100t\,\mathbf{i} + \left(1000 - 5t^2\right)\mathbf{j} - 30t\,\mathbf{k}.$$

Find her velocity after 2 seconds and speed after 2.5 seconds.

3.10 Average acceleration of the ball

For the combined Galileo/chute experiment the average acceleration of the ball over a second is the change in velocity in that second.

Activity 6 Calculating average acceleration

For the ball whose velocity is given in equation (6) by

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\,\mathbf{j}$$

assume that the table is large enough for the motion to take place over 5 s.

Complete these values of velocity:

time	velocity $\frac{d\mathbf{r}}{dt}$
$t = 0$	$0.7\mathbf{i} - 0\mathbf{j}$
$t = 1$	$0.7\mathbf{i} - 0.2\mathbf{j}$
$t = 2$	$0.7\mathbf{i} - 0.4\mathbf{j}$
$t = 3$	
$t = 4$	
$t = 5$	

Now use the result of Section 3.7,

$$\text{average acceleration } \mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t}$$

to complete the following values:

time interval	average acceleration
$t = 0$ to $t = 1$	$\dfrac{(0.7\mathbf{i} - 0.2\mathbf{j}) - (0.7\mathbf{i} - 0\mathbf{j})}{1 - 0} = 0\mathbf{i} - 0.2\mathbf{j}$
$t = 1$ to $t = 2$	$\dfrac{(0.7\mathbf{i} - 0.4\mathbf{j}) - (0.7\mathbf{i} - 0.2\mathbf{j})}{2 - 1} =$
$t = 2$ to $t = 3$	
$t = 3$ to $t = 4$	
$t = 4$ to $t = 5$	

What do these results suggest about the acceleration of the ball?

Acceleration at a point on the path or at an instant of time *t*

Differentiating the velocity vector $\dfrac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\mathbf{j}$ gives

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 0.2\mathbf{j}$$

which should agree with the values you obtained in Activity 6.

This suggests that the second derivative $\dfrac{d^2\mathbf{r}}{dt^2}$ of position vector **r** represents acceleration. The acceleration is constant in this case because $0\mathbf{i} - 0.2\mathbf{j}$ is independent of t.

The comparison with the one-dimensional case is now complete.

In two dimensions, starting with position vector

$$\mathbf{r} = 0.7t\mathbf{i} - 0.1t^2\mathbf{j}$$

differentiating once gives the velocity vector

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\mathbf{j}$$

and differentiating a second time gives the acceleration vector

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 0.2\mathbf{j}$$

Summary

If the position vector, **r**, of an object is known as a function of time, t, then the instantaneous velocity $\frac{d\mathbf{r}}{dt}$ and the instantaneous acceleration, $\frac{d^2\mathbf{r}}{dt^2}$ as functions of t can be calculated from **r** by differentiation:

if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$,

then $$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

and $$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

Example

The position vector of a golf ball t seconds after it has been struck is given in metres by:

$$\mathbf{r} = 50t\mathbf{i} + \left(14t - 5t^2\right)\mathbf{j} + 2t\mathbf{k}.$$

Find its speed when $t = 2.5$ and show that the acceleration is constant and in the negative y direction. What is its magnitude?

Solution

$$\mathbf{r} = 50t\mathbf{i} + \left(14t - 5t^2\right)\mathbf{j} + 2t\mathbf{k}$$

Differentiating **r** with respect to t gives

$$\frac{d\mathbf{r}}{dt} = 50\mathbf{i} + (14 - 10t)\mathbf{j} + 2\mathbf{k}.$$

When $t = 2.5$, the velocity is

$$\frac{d\mathbf{r}}{dt} = 50\mathbf{i} + (14 - 25)\mathbf{j} + 2\mathbf{k}$$
$$= 50\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}$$

and the speed is

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{50^2 + 11^2 + 2^2}$$
$$= 51.2 \text{ ms}^{-1}.$$

Differentiating again gives the acceleration

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 10\mathbf{j} + 0\mathbf{k}.$$

This is a constant acceleration of magnitude 10 ms^{-2} in the negative y direction.

Exercise 3H

1. The position vector of a netball at time t seconds is given in metres by

   $$\mathbf{r} = 12t\,\mathbf{i} + \left(16t - 5t^2\right)\mathbf{j}$$

   where **i** is horizontal and **j** is vertically upwards. Determine the velocity with which it was thrown and show that its acceleration is constant. What is the direction of this acceleration?

2. Determine the accelerations in each of Questions 3 and 4 from Exercise 3G. What conclusion can you draw about each of these motions?

3. The velocity of a particle is

   $$\frac{d\mathbf{r}}{dt} = (3t+1)\mathbf{i} - 4t\,\mathbf{j}$$

   What is its acceleration at time t?

3.11 Miscellaneous Exercises

1. (a) Find the magnitude and direction of the following vectors:

   (i) $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$ (ii) $\mathbf{b} = \mathbf{i} + \sqrt{3}\mathbf{j}$

   (iii) $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$

   (b) Find the magnitude of the following vectors:

   (i) $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (ii) $\mathbf{e} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

   (iii) $\mathbf{f} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$.

2. For the vectors in Question 1, find a unit vector in the direction of each vector.

3. The points A and B have position vectors

   $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$

   Find:

   (a) the vector $\overrightarrow{AB}$;

   (b) the magnitude of vector $\overrightarrow{OA}$;

   (c) the unit vector in the direction of $\overrightarrow{OB}$.

4. The vectors **p** and **q** are defined in terms of a, b and c as:

   $$\mathbf{p} = a\,\mathbf{i} + (a-b)\mathbf{j} + c\mathbf{k}$$

   and $\mathbf{q} = (5-b)\mathbf{i} + (2b+7)\mathbf{j} + (a+5b)\mathbf{k}$.

   If $\mathbf{p} = \mathbf{q}$, find the values of a, b and c. Determine the unit vector in the direction of $\mathbf{p} - \mathbf{s}$ where $\mathbf{s} = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

5. Given the vectors in Question 1, find :

   (a) $\mathbf{a} + 2\mathbf{b}$ (b) $2\mathbf{c} - 3\mathbf{d}$

   (c) $\mathbf{a} + \mathbf{d} - 2\mathbf{f}$ (d) the vector **x** if $2\mathbf{a} - \mathbf{x} = \mathbf{e}$.

6. Given the vectors

   $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$

   (a) describe the direction of $\mathbf{a} + \mathbf{b}$;

   (b) find the vectors $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ and $-\mathbf{a} + 2\mathbf{b} + \mathbf{c}$;

   (c) P is the end-point of the displacement vector $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ and $(1, -2, 3)$ is the starting point.

   What is the position vector of P?

7. Given the vectors

   $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$,

   (a) find the magnitude and describe the direction of $\mathbf{a} + \frac{1}{2}\mathbf{b}$;

   (b) find the vectors $3\mathbf{a} - \mathbf{b} + \mathbf{c}$ and $-2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$.

   (c) P is the end-point of the displacement vector $-2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$ and $(-1, 0, -2)$ is its starting point. What is the position vector of P?

8. Referred to the x and y axes, the coordinates of the points P and Q are (3, 1) and (4, –3) respectively.
   (a) Calculate the magnitude of the vector $\overrightarrow{OP} + 5\,\overrightarrow{OQ}$;
   (b) Calculate the magnitude and direction of the vector $\overrightarrow{PQ}$;
   (c) Calculate the coordinates of the point R if $\overrightarrow{OP} + \overrightarrow{OQ} = 2\overrightarrow{OR}$.

9. The velocity of a particle is given at time t seconds, as

$$\mathbf{v} = 6t\,\mathbf{i} - \mathbf{j} + t^2\mathbf{k}.$$

Find its acceleration at time t and its speed after 3s.

10. The position vector of a particle at time t seconds is

$$\mathbf{r} = t\,\mathbf{i} + \left(\frac{1}{t}\right)\mathbf{j}$$

What is its velocity at time t and speed after 2 s?

11. The coordinates of a moving point P at time t seconds are $(4t^2, 8t)$ metres.
   (a) Write down the position vector of P.
   (b) Find the velocity of P.
   (c) Show that the acceleration of P is always parallel to the x axis.

12. At time t, the position vectors of two points, P and Q, are given by:

$$\mathbf{p} = 2t\,\mathbf{i} + (3t^2 - 4t)\,\mathbf{j} + t^3\mathbf{k}$$

$$\mathbf{q} = t^3\mathbf{i} - 2t\,\mathbf{j} + (2t^2 - 1)\,\mathbf{k}.$$

Find the velocity and acceleration of Q relative to P when $t = 3$.

13. The position vector of smoke particles as they leave a chimney for the first 4 s of their motion is given by

$$\mathbf{r} = 4t\,\mathbf{i} + \left(\frac{3t^2}{2}\right)\mathbf{j} + 6t\,\mathbf{k},$$

where **i** and **k** are horizontal, directed north and east respectively, and **j** is vertically upward.
   (a) What is the magnitude and direction of the acceleration of the smoke?
   (b) What is (i) the velocity
   and (ii) the speed
   of the smoke particles after 2 s?
   (c) In what direction does the smoke go relative to the **i**, **k** plane (i.e. relative to the ground)?

14. A glider spirals upwards in a thermal (hot air current) so that its position vector with respect to a point on the ground is

$$\mathbf{r} = \left(100\cos\frac{t}{5}\right)\mathbf{i} + \left(200 + \frac{t}{3}\right)\mathbf{j} + \left(100\sin\frac{t}{5}\right)\mathbf{k}.$$

The directions of **i**, **j** and **k** are as defined in Question 13.
   (a) Determine the glider's speed at $t = 0$, 5π and 10π seconds.
   What do you notice?
   (b) Find **r** when $t = 0$ and 10π seconds and find the height risen in one complete turn of the spiral.

15. A particle P moves in such a way that its position vector **r** at time t is given by

$$\mathbf{r} = t^2\mathbf{i} + (t + 1)\,\mathbf{j} + t^3\mathbf{k}.$$

   (a) Find the velocity and acceleration vectors.
   (b) Find a unit vector along the direction of the tangent to the path of the motion.

16. A particle of mass 3 kg moves in a horizontal plane and its position vector at time t s relative to a fixed origin O is given by

$$\mathbf{r} = (2\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\text{ m}.$$

Find the values of t in the range $0 \le t \le \pi$ when the speed of the particle is a maximum.

(AEB)

17. Two particles P and Q have velocities $(3\mathbf{i} - 5\mathbf{j})\text{ ms}^{-1}$ and $(2\mathbf{i} - 3\mathbf{j})\text{ ms}^{-1}$ respectively. The line of motion of P passes through the point A with position vector $(5\mathbf{i} + 13\mathbf{j})$ m, relative to a fixed origin O, and the line of motion of Q passes through the point B with position vector $(7\mathbf{i} + 9\mathbf{j})$ m relative to O.

In the case when P and Q pass through the points A and B respectively at the same time, find the velocity of P relative to Q and deduce that the particles will collide two seconds after passing through these points. Find also the position vector, relative to O, of the point of collision. Given that the particles have equal mass and stick together upon collision, find the velocity of the combined mass after collision.

(AEB)

*18. Two particles P and Q are moving on a horizontal plane and at time t seconds

$$\mathbf{OP} = -a(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}), \qquad \mathbf{OQ} = (vt - a)\,\mathbf{i},$$

where a, v, ω are constants and O is a fixed point in the plane. Show that both P and Q are moving with constant speeds.

Denoting the speed of P by u show that the square of the speed of P relative to Q is,

$$v^2 + u^2 - 2vu\sin\omega t.$$

When $t = 0$ the speed of P relative to Q is $\sqrt{80}$ ms^{-1} and when $t = \dfrac{\pi}{6\omega}$, the speed of P relative to Q is $\sqrt{48}$ ms^{-1}. Given that P is moving faster than Q, find the speeds of P and Q.

(AEB)

19. A particle moves in a plane and at time t its position vector, **r**, is given by

$$\mathbf{r} = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

Find the values of t in the range $0 \le t \le \pi$ when

(a) the speed of the particle is a maximum,

(b) the force acting on the particle is perpendicular to the velocity.

(AEB)

20. A particle P of mass 0.25 kg moves on a smooth horizontal table with constant velocity $(17\mathbf{i} + 6\mathbf{j})$ ms^{-1}, where **i** and **j** are perpendicular constant unit vectors in the plane of the table. An impulse is then applied to the particle so that its velocity becomes $(29\mathbf{i} + 22\mathbf{j})$ ms^{-1}. Find this impulse in the form of $a\mathbf{i} + b\mathbf{j}$

Determine a unit vector **n** such that the component of the velocity of P along **n** is unchanged by the impulse. Obtain the magnitude of this component.

(AEB)