# 2 ONE-DIMENSIONAL MOTION

## Objectives

After studying this chapter you should

- be able to derive and use formulae involving constant acceleration;
- be able to understand the concept of force;
- be able to use Newton's Laws of Motion in various contexts;
- know how to formulate and solve equations of motion;
- be able to use the principles of conservation of momentum.

## 2.0 Introduction

The physical world is full of moving objects. **Kinematics** is the study of motion; **dynamics** is the study of forces that produce motion. In this chapter the mathematics for describing motion is developed and then the links between the forces acting and the change in motion are described.

To describe the motion of real objects you usually need to make simplifying assumptions. Perhaps the most important simplification in applied mathematics is ignoring the size and shape of an object. In Chapter 1 the notion of replacing a real object by a point was introduced. For example, in defining Newton's Law of Gravitation for the force acting on an object near to the earth's surface, the object and the earth were considered as points. Now the normal terminology is to consider objects as **particles**, and this is then called the **particle model**.

This simplification provides a starting point for many problems, but it does mean that some features of the motion of objects have to be ignored. For example, consider the description of the motion of a tennis ball. At first sight the ball may appear to follow the typical parabolic path of any object thrown near the earth's surface (such motions are studied in detail in Chapter 5). However, a closer study of the motion shows that the ball will be spinning, causing it to swing and dip. The particle model will be good enough to describe the overall parabolic motion but the effects of spin will have to be neglected. A less simple model which includes the features of size and shape would be required to describe the effects of spin.

Consider the motion of a snooker ball . It is possible to make the ball slide or roll on the table or to move with a combination of both types of motion. For which type of motion will the particle model be most appropriate? What features of the motion of a rolling snooker ball will be neglected with the particle model?

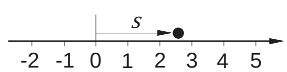
This chapter will concentrate on the description of objects which move along a **straight line**. The objects will be modelled as particles, and represented on diagrams by 'thick' points.

# 2.1 How to represent motion

## Displacement

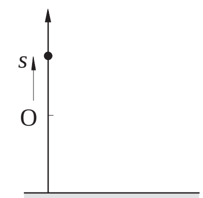
Typical of the questions to be answered are: if a ball is thrown vertically upwards, how long does it take to fall to the floor? What is its velocity as it hits the floor?

To answer questions like these you need to find the **position** and **velocity** of the ball as functions of time. The first step is to represent the ball as a particle. The position of a particle moving in a straight line at any given instant of time is represented on a straight line by a single point. In order to describe the exact position you choose a directed axis with a fixed origin 0 and a scale as shown opposite.



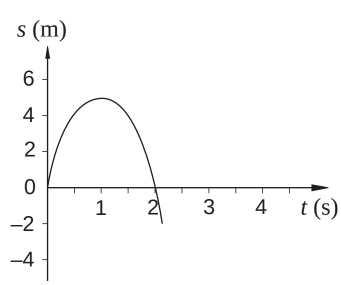

The position of the particle relative to the origin is called the **displacement** and is often denoted by the letter *s* measured in metres (m).

The choice of the origin in a problem will depend on what motion is being modelled. For example, in the problem of throwing a ball in the air an origin at the point of release would be a sensible choice.




## Displacement - time graphs

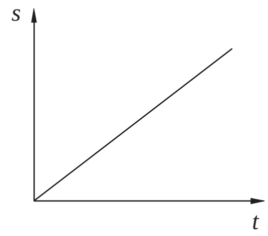
As a particle moves the displacement changes so that *s* is a function of time, *t*. A graphical method of showing motion is the **displacement-time graph** which is a plot of *s* against *t*. As an example, the figure shows the displacement-time graph for the motion of a ball thrown in the air and falling to the floor.

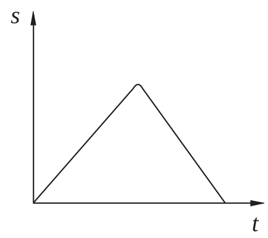


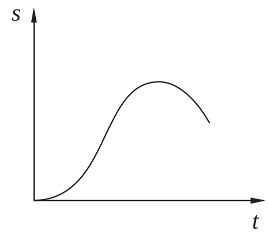

From the graph a qualitative description of how the position of the particle varies in time can be given. The ball starts at the origin and begins to move in the positive *s* direction (upwards) to a maximum height of 5 m above point of release. It then falls to the floor which is 2 m below the point of release. It takes just over 2 seconds to hit the floor but this part of the motion has not been shown.

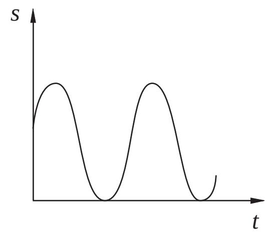
### Activity 1 Interpreting displacement - time graphs

Discuss the motion represented by each of the displacement - time graphs shown here.




## Velocity

Once the position of a particle has been specified its motion can be described. But other quantities, such as its speed and acceleration, are often of interest. For example, when travelling along a road in a car it is not the position that is of interest to the police but the **speed** of the car!

The statement that the speed of a car on the M1 is 60 mph means that if the speed remains unchanged then the car travels for 60 miles in one hour. However the statement gives no information about the direction of motion. The statement that the **velocity** of a car on the M1 is 60 mph due north tells us two things about the car. First its speed is 60 mph (the **magnitude**) and the car is heading due north (the **direction**). Quantities which have magnitude and direction are called **vectors** and these are discussed more fully in Chapters 3 and 4.

The average velocity of a particle over a given time period, $T$ say, is probably familiar to you and is defined by

$$\text{average velocity} = \frac{\text{displacement}}{\text{time taken}}.$$

Such a definition does not describe the many changes in velocity that may occur during the motion of the particle.

In the time interval $T$ from $t = t_0$ to $t = t_0 + T$ the distance travelled is $s(t_0 + T) - s(t_0)$ so the average velocity is

$$\frac{s(t_0 + T) - s(t_0)}{T}$$

Now the quantity that is more interesting is not the average velocity of the particle but the **instantaneous** velocity. You will have seen from the definition of differentiation in the Foundation Core that the link between average changes and instantaneous changes is the derivative. As the time interval $T$ tends to zero the ratio

$$\frac{s(t_0 + T) - s(t_0)}{T}$$

tends towards the derivative of $s(t)$.

Thus velocity is defined in the following way.

> If $s(t)$ is the displacement of a particle then its velocity is defined by
>
> $$v = \frac{ds}{dt}$$
>
> In the SI system of units velocity is measured in metres per second written as $\text{ms}^{-1}$.

## Activity 2 Limits of average velocity

The displacement of a particle is given by

$$s = t^2 + 2t.$$

Calculate the average velocity of the particle during each of the time intervals

from $t = 1$ to $t = 2$

from $t = 1$ to $t = 1.1$

from $t = 1$ to $t = 1.01$

from $t = 1$ to $t = 1.001$.

Estimate the value that the average velocity is tending towards as $t \to 1$ .

Does this value agree with $\frac{ds}{dt}$ when $t = 1$?

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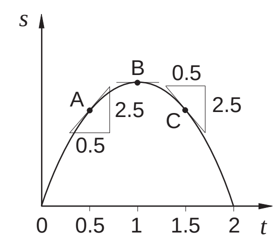
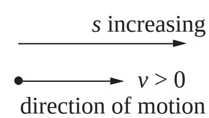
The definition of the velocity as a derivative can be interpreted geometrically as the slope of the tangent to the displacement - time graph. For example, consider the displacement - time graph opposite for the ball thrown into the air:



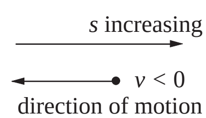

The slopes of the tangents to the graph at each of the points $t = 0.5$, $t = 1$ and $t = 1.5$ are equal to the velocities at these points.

When $t = 0.5$ (point A) the slope of the tangent is given by $\frac{2.5}{0.5} = 5$ ms$^{-1}$. At $t = 1$ (point B) the slope is zero and at $t = 1.5$ (point C) the slope is $-5$ ms$^{-1}$. You can now say more about the motion of the ball. At point A, the ball is going upwards with speed 5 ms$^{-1}$. At point B the ball is instantaneously at rest and it is at its highest point. At point C the ball is falling to the floor with speed 5 ms$^{-1}$.

For one-dimensional motion the sign of the velocity indicates the direction of motion of the particle.



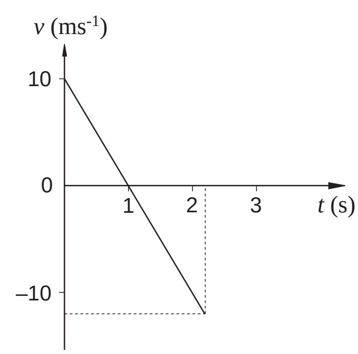

If $v > 0$ then $s$ is increasing with time since $\frac{ds}{dt} > 0$.




If $v < 0$ then $s$ is decreasing with time since $\frac{ds}{dt} < 0$.

The magnitude of the velocity of a particle is called its **speed**. For example, if $v = -3$ ms$^{-1}$ then you can say that the particle moves with a speed 3 ms$^{-1}$ in the direction of $s$ decreasing.

## Velocity - time graphs

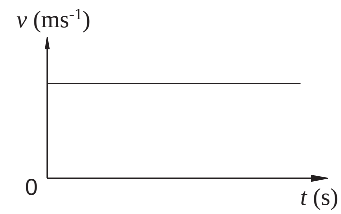



If you find the velocity, $v$, at several times, $t$, then a graph of the velocity against time is called a **velocity - time graph**. The figure shows the velocity - time graph for the motion of a ball thrown into the air and falling to the floor.

The ball begins its motion with a speed of 10 ms$^{-1}$ and this speed falls to zero during the first second of the motion. The speed then increases to approximately 12 ms$^{-1}$ when the ball hits the

## Activity 3    Interpreting velocity - time graphs

The diagram shows the velocity -time graph for Graph 1 in Activity 1.




Describe the motion of the particle from this graph.

Sketch the velocity - time graphs for the other displacement - time graphs in Activity 1.

Describe the motion of the particle in each case.

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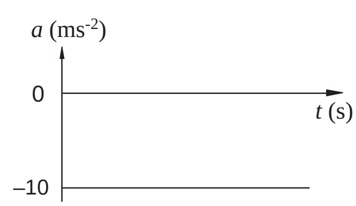
# Acceleration

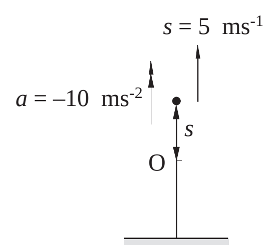
Chapter 1 identified the *change* in motion as an important quantity in the link between force and motion.  Hence in many situations it is not the velocity that is important but the change in velocity.  This is described by the **acceleration**.

You have seen that the velocity is defined as the rate of change of position; the acceleration is defined in a similar way to the velocity.

> If $v(t)$ is the velocity of a particle at time $t$ then the acceleration of the particle is defined by
>
> $$a = \frac{dv}{dt}.$$
>
> In the SI system of units, acceleration is measured in metres per second per second,  written as $\text{ms}^{-2}$.




The acceleration can be obtained from the slope of the velocity - time graph.  For example, for the motion of the ball thrown into the air, the diagram shows the acceleration - time graph.  The acceleration is constant with magnitude 10 $\text{ms}^{-2}$. The negative sign implies that the velocity decreases continuously with time.




On a diagram velocities are shown with single arrows and accelerations with double arrows.

### Activity 4 Interpreting acceleration - time graphs

Sketch the acceleration - time graphs from your velocity - time graphs in Activity 3.

Describe the motion of the particle in each case.

## Relationships between displacement, velocity and acceleration

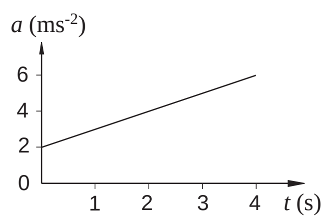
In this section you have seen that the motion of a particle along a straight line can be described by a displacement $s(t)$. The velocity and acceleration of the particle are then given by

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

respectively. Graphical descriptions of motion are given by displacement - time, velocity - time and acceleration - time graphs.

Now in mechanics problems the acceleration is often known and the velocity and displacement have to be found. This is achieved by integration. You will know from your knowledge of pure mathematics that integration is equivalent to finding the area under a graph.

### Activity 5 Finding a velocity and displacement




The graph opposite shows the acceleration of a particle over 4 seconds of its motion. The particle starts from rest, at $t = 0$.

Estimate the velocity of the particle at the end of 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4 seconds.

Use your results to sketch a velocity - time graph for the particle.

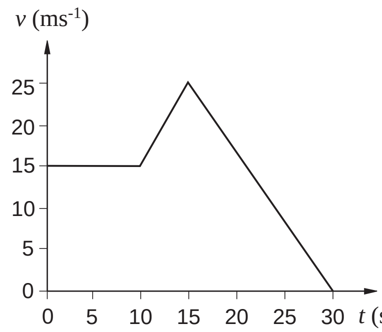
From your velocity - time graph estimate the distance travelled during the 4 seconds of motion between $t = 0$ and $t = 4$.

The equation for the graph above is $a = 2 + t$. By integrating $a$ to find $v$ and then integrating $v$ to find $s$, check your answer.

# Exercise 2A

1. Sketch displacement - time and velocity - time graphs for the following:
   (a) A car starts from rest and increases its velocity steadily to 10 $\text{ms}^{-1}$ in 5 seconds. The car holds this velocity for another 10 seconds and then slows steadily to rest in a further 10 seconds.
   (b) A ball is dropped on a horizontal floor from a height of 3 m. The ball bounces several times before coming to rest.
   (c) A person jumps out of an aircraft and falls until the parachute opens. The person glides steadily to the ground.

2.




The diagram represents the motion of an object for 30 seconds. Calculate the acceleration for each of the following intervals:

(a) $0 < t < 10$ (b) $10 < t < 15$ (c) $15 < t < 30$

Calculate the displacement of the object over the 30 seconds.

3.

| $t$(s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $v(\text{ms}^{-1})$ | 0 | 8 | 16 | 24 | 32 | 30 | 24 | 14 | 0 |

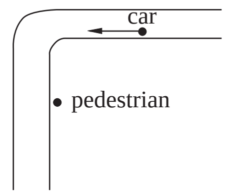
   (a) Plot these figures on a velocity - time graph.
   (b) Verify that for $0 \le t \le 4$, the figures are consistent with constant acceleration.
   (c) Calculate an estimate for the acceleration at $t = 6$.
   (d) Calculate an estimate for the displacement after 8 seconds.

4. During the launch of a rocket the velocity was noted every second for 10 seconds and the following table of values obtained.

| $t$(s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$(kph) | 0 | 32 | 80 | 128 | 176 | 224 | 272 | 320 | 368 | 400 | 448 |

   Estimate the distance travelled by the rocket during the first 10 seconds of its motion.

5. A road has a sharp bend around which approaching traffic cannot be seen. The width of the road is 6 m and the speed limit is 30 mph. The bend is on the near side of the road. How far from the bend should a pedestrian cross the road to avoid an accident? Assume the man walks at 3 mph.




6. A particle is set in motion at time $t = 0$ and its position is subsequently given by $s = 4 + 5t - 2t^2$.
   (a) Calculate the velocity of the particle after 1 second and after 2 seconds. What is the speed and direction of motion at each of these times?
   (b) Find the time at which the particle is instantaneously at rest.
   (c) Calculate the acceleration of the particle at $t = 1$ s and $t = 2$ s.
   (d) Describe the motion of the particle.

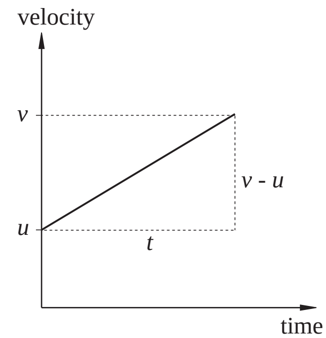
7. Repeat Problem 6 for
   (a) $s = t^2 - 4t + 1$
   (b) $s = 3 + 18t - 7.5t^2 + t^3$.

8. The acceleration of a particle is given by $a = -10\ \text{ms}^{-2}$. At the instant $t = 1$ s, the particle is at position $s = 2$ m and has velocity 3 $\text{ms}^{-1}$.
   (a) Find the velocity and displacement of the particle as functions of time.
   (b) Calculate the position and velocity of the particle when $t = 2$ s.

## 2.2 Modelling motion under constant acceleration

There are several simple formulae which can be used when dealing with motion under **constant** (also known as uniform) **acceleration**.

**Discuss common types of motion where you think that constant acceleration is likely to occur.**




The diagram represents the motion of an object with initial velocity $u$ and final velocity $v$ after $t$ seconds has elapsed. The gradient of the line is calculated from the expression

$$\frac{v-u}{t}.$$

Since the gradient is equal to the value of the acceleration, $a$, then

$$a=\frac{v-u}{t}.$$

This can be rewritten as

$$\boxed{v=u+at} \qquad (1)$$

The area under the velocity - time graph is equal to the displacement of the object. Using the rule for the area of a trapezium gives

$$\boxed{s=\frac{(u+v)t}{2}} \qquad (2)$$

Note that $\frac{(u+v)}{2}$ is the average velocity of the object, so (2) is the algebraic form of the result that the displacement is equal to the average velocity multiplied by the time.

### Example

A motorbike accelerates at a constant rate of $3\ \mathrm{ms}^{-2}$. Calculate

(a) the time taken to accelerate from 20 mph to 40 mph.

(b) the distance in metres covered during this time.

#### Solution

You can use equation (1) to find the time and then equation (2) to find the distance travelled. But first you must convert mph to $\mathrm{ms}^{-1}$.

Using the conversion rule 5 miles $\approx 8$ kilometres, gives

$$10 \text{ mph} = 16 \text{ kph} = \frac{16 \times 1000}{60 \times 60} \text{ ms}^{-1}$$

$$= 4.445 \text{ ms}^{-1}.$$

Hence 20 mph $= 8.89 \text{ ms}^{-1}$ and 40 mph $= 17.78 \text{ ms}^{-1}$.

(a) Using equation (1) the time taken to accelerate from $8.89 \text{ ms}^{-1}$ to $17.78 \text{ ms}^{-1}$ at $3 \text{ ms}^{-2}$ is given by

$$t = \frac{17.78 - 8.89}{3} = 2.96 \text{ s (to 3 sig. fig.)}.$$

(b) Using equation (2) the distance travelled in this time is

$$s = \left(\frac{8.89 + 17.78}{2}\right) \times 2.96 = 39.5 \text{ m}.$$

There are two further useful formulae which can be obtained from (1) and (2). Writing (1) in the form

$$t = \frac{v - u}{a}$$

and substituting into (2) gives

$$s = \left(\frac{u + v}{2}\right)\left(\frac{v - u}{a}\right),$$

so $$s = \left(\frac{v^2 - u^2}{2a}\right).$$ Remember

$$v^2 - u^2 = (v - u)(v + u)$$

This expression can be rearranged to give

$$\boxed{v^2 = u^2 + 2as} \qquad (3)$$

Similarly, substituting for $v$ from (1) into (2) gives

$$s = \left(\frac{u + u + at}{2}\right) t$$

or $$\boxed{s = ut + \tfrac{1}{2} at^2} \qquad (4)$$

Formula (3) is useful when the time $t$ has not been given or is not required, while formula (4) is useful when the final velocity $v$ has not been given nor is required.

## Example

A car accelerates from a velocity of 16 $\text{ms}^{-1}$ to a velocity of 40 $\text{ms}^{-1}$ in a distance of 500 m. Calculate the acceleration of the car.

### Solution

Using (3) $40^2 = 16^2 + 2 \times a \times 500$

$$a = \frac{1600 - 256}{1000} = 1.344 \text{ ms}^{-2}.$$

## Example

A car decelerates from a velocity of 36 $\text{ms}^{-1}$. The magnitude of the deceleration is 3 $\text{ms}^{-2}$. Calculate the time required to cover a distance of 162 m.

### Solution

When an object is slowing down it is said to be **decelerating**. You then use equations (1) - (4) but with a negative value for *a*. In this problem set $a = -3$.

Let *t* seconds be the time required to cover 162 m.

Using (4), $162 = 36t + \frac{1}{2} \times (-3) \times t^2$.

Rearranging this gives

$$1.5t^2 - 36t + 162 = 0.$$

Dividing by 1.5 gives

$$t^2 - 24t + 108 = 0.$$

Factorizing gives

$$(t - 6)(t - 18) = 0,$$

so that $t = 6$ or18.

Discuss the significance of the two solutions to the quadratic equation. Which is the required time?

## Activity 6 Stopping distance on the road

The Highway Code gives the following data for overall stopping distances of vehicles at various speeds.

| Speed (mph) | Thinking Distance (metres) | Braking Distance (metres) | Stopping Distance (metres) |
|---|---|---|---|
| 30 | 9 | 14 | 23 |
| 50 | 15 | 38 | 53 |
| 70 | 21 | 75 | 96 |

What is meant by the thinking distance?

Show that the deceleration during braking is roughly the same at each of the three speeds.

Use a graph plotter to fit a suitable curve through the data for speed and stopping distance. Use your results to estimate the speed corresponding to a stopping distance of 150 metres.

The amber warning light on traffic signals is intended to give drivers time to slow before a red stop light. Time the duration of amber lights in your locality. Do they give sufficient warning at the speed limit in operation on the road?

## Exercise 2B

1. A car accelerates uniformly from a speed of 50 kph to a speed of 80 kph in 20 seconds.

   Calculate the acceleration in $ms^{-2}$.

2. For the car in Question 1, calculate the distance travelled during the 20 seconds.

3. A train signal is placed so that a train can decelerate uniformly from a speed of 96 kph to come to rest at the end of a platform. For passenger comfort the deceleration must be no greater than 0.4 $ms^{-2}$. Calculate

   (a) the shortest distance the signal can be from the platform;

   (b) the shortest time for the train to decelerate.

4. A rocket is travelling with a velocity of 80 $ms^{-1}$. The engines are switched on for 6 seconds and the rocket accelerates uniformly at 40 $ms^{-2}$. Calculate the distance travelled over the 6 seconds.

5. In 1987 the world record for the men's 60 m race was 6.41 seconds.

   (a) Assuming that the race was carried out under constant acceleration, calculate the acceleration of the runner and his speed at the end of the race.

   (b) Now assume that in a 100 m race the runner accelerates for the first 60 m and completes the race by running the next 40 m at the speed you calculated in (a).

   Calculate the time for the athlete to complete the race.

6. The world record for the men's 100 m was 9.83 s in 1987. Assume that the last 40 m was run at constant speed and that the acceleration during the first 60 m was constant.

   (a) Calculate this speed.

   (b) Calculate the acceleration of the athlete.

7. Telegraph poles, 40 m apart stand alongside a railway line. The times taken for a locomotive to pass the two gaps between three consecutive poles are 2.5 seconds and 2.3 seconds respectively. Calculate the acceleration of the train and the speed past the first post.

8. The world record for the women's 60 m and 100 m are respectively 7.00 seconds and 10.49 seconds. Analyse this information using the method given in Question 6.

9. A set of traffic lights covers road repairs on one side of a road in a 30 mph speed limit area. The traffic lights are 80 m apart so time must be allowed to delay the light changing from green to red. Assuming that a car accelerates at $2\,\mathrm{ms}^{-2}$ what is the least this time delay should be?

10. A van travelling at 40 mph skids to a halt in a distance of 15 m. Find the acceleration of the van and the time taken to stop, assuming that the deceleration is uniform.

# 2.3 How do bodies move under gravity?

For many centuries it was believed that:

(a) heavier bodies fell faster than light ones

and

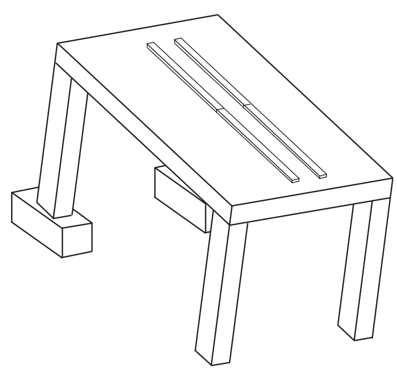
(b) the speed of a falling body was constant.

**Discuss why such views were held and suggest ways in which they could be refuted.**

*Galileo Galilei* (1564 - 1642) was the first person to state clearly that all objects fall with the same constant acceleration. Since there were no accurate timers available in the seventeenth century, he demonstrated this principle by timing balls which were allowed to roll down inclined planes.

## Activity 7 *Galileo's rolling ball experiment*

You will need balls of different masses, a table, some blocks, four metre rulers and a stop watch.



(a) Use the blocks to set up the table as shown in the diagram so that it takes about 4 seconds for the ball to roll down. Fix the rulers to make a channel down which the ball can roll. Measure the time it takes the ball to roll distances of 0.5, 0.75, 1, 1.25, 1.5, 1.75 and 2.0 m. Repeat the experiment and find the average time in each case.

(b) Find a relationship between the distance travelled, *s* metres, and time taken, *t* seconds.

(c) Do your results support Galileo's statement?

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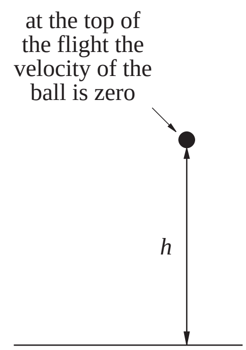
Modern determination of the acceleration of falling bodies gives values in the region of $9.81\ \mathrm{ms}^{-2}$ although the value varies slightly over the surface of the Earth. The magnitude of this acceleration is denoted by $g$. A common approximation is to take $g = 10\ \mathrm{ms}^{-2}$ and this value will be used in this text for solving problems.

Since the acceleration acts towards the Earth's surface, its sign must be opposite to that of any velocities which are upwards.

When dealing with problems involving motion under gravity, you can use the formulae for constant acceleration developed in the previous section.




## Example

A ball is thrown vertically upwards with an initial speed of $30\ \mathrm{ms}^{-1}$. Calculate the height reached.

### Solution

Since the ball slows down, take $a = -g = -10\ \mathrm{ms}^{-2}$.

Using (3) $\quad 0 = 30^2 - 2 \times 10 \times h$

where $h$ is the maximum height reached.

Thus $\quad h = \dfrac{900}{20} = 45$ m (to 2 significant figures).

## Example

A ball is thrown vertically upwards with a speed of $40\ \mathrm{ms}^{-1}$. Calculate the time interval between the instants that the ball is 20 m above the point of release.

### Solution

Using (4) with $a = -g = -10\ \mathrm{ms}^{-2}$

$$20 = 40t - 0.5 \times 10 \times t^2$$

where $t$ seconds is the time passed since the ball was thrown up.

$$5t^2 - 40t + 20 = 0$$

$$t^2 - 8t + 4 = 0.$$

Using the quadratic formula

$$t = \frac{8 \pm \sqrt{48}}{2} = 7.62 \text{ or } 0.54.$$

The ball is 20 m above the point of release twice, at $t = 0.54$ s (on way up) and $t = 7.62$ s (on way down).

The required time interval is $7.62 - 0.54 = 7.08$ seconds.

Two useful formulae which can be used on a body falling from rest through a height $h$ metres can be found by putting $u = 0$, $a = 10$ and $s = h$ in equation (4) to give

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{h}{5}} \qquad (5)$$

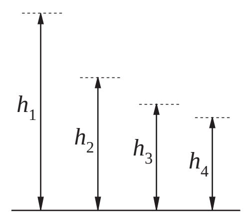
and by putting $u = 0$, $a = 10$ and $s = h$ in equation (3) to give

$$v = \sqrt{2gh} = \sqrt{20h} \qquad (6)$$

where $t$ is the time of fall in seconds, $v$ $\text{ms}^{-1}$ is the final velocity and $g = 10\ \text{ms}^{-2}$.

## Activity 8 Estimating the value of $g$

You will need a bouncy ball and a metre rule for this activity.




Drop the ball onto a hard floor from a height of 2 m. Measure the height of the rebound. Repeat this several times and average your results. Now drop the ball from the rebound height and measure the new height of rebound. Repeat this procedure several times and average your results. Keep measuring new rebound heights for two further cases.

Now drop the ball from 2 m and measure the time elapsed until the fourth bounce with the floor. Repeat several times and average your results.

Show that the total time, $t$, up to the fourth bounce is

$$t = \sqrt{\frac{2h_1}{g}} + 2\sqrt{\frac{2h_2}{g}} + 2\sqrt{\frac{2h_3}{g}} + 2\sqrt{\frac{2h_4}{g}}.$$

Use this, together with your measurements, to calculate a value for $g$.

# Exercise 2C

(Take $g = 10\ \text{ms}^{-2}$)

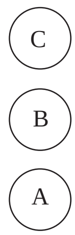
1. A ball is dropped on to level ground from a height of 20 m.

   (a) Calculate the time taken to reach the ground.

   The ball rebounds with half the speed it strikes the ground.

   (b) Calculate the time taken to reach the ground a second time.

2. A stone is thrown down from a high building with an initial velocity of 4 $\text{ms}^{-1}$. Calculate the time required for the stone to drop 30 m and its velocity at this time.
3. A ball is thrown vertically upwards from the top of a cliff which is 50 m high. The initial velocity of the ball is 25 $\text{ms}^{-1}$. Calculate the time taken to reach the bottom of the cliff and the velocity of the ball at that instant.
4. The diagram show three positions of a ball which has been thrown upwards with a velocity of $u$ $\text{ms}^{-1}$




Position A is the initial position.

Position B is halfway up.

Position C is at the top of the motion.

Copy the diagram and for each position put on arrows where appropriate to show the direction of the velocity.

On the same diagram put on arrows to show the direction of the acceleration.

5. An aircraft is flying at a height of 4 km when it suddenly loses power and begins a vertical dive. The pilot can withstand a deceleration of 5 g before becoming unconscious. What is the lowest height that the pilot can pull out of the dive ?
6. If the Earth is assumed to be a perfect sphere then the acceleration due to gravity at a height $h$ m above the surface of the Earth is given by

$$\frac{k}{(R+h)^2}$$

where $R$ is the radius of the Earth in metres and $k$ is a constant.

(a) Given that $R$ = 6400 km and at the Earth's surface $g = 9.8$ $\text{ms}^{-2}$ , estimate the value of $k$.

(b) Use your calculator to find the height at which the acceleration differs by 1% from its value at the Earth's surface.

(c) Use a graphic calculator to draw the variation of acceleration with height.

7. When a ball hits the ground it rebounds with half of the speed that it had when it hit the ground. If the ball is dropped from a height $h$, investigate the total distance travelled by the ball.
8. One stone is thrown upwards with a speed of 2 $\text{ms}^{-1}$ and another is thrown downwards with a speed of 2 $\text{ms}^{-1}$. Both are thrown at the same time from a window 5 m above ground level.

(a) Which hits the ground first?

(b) Which is travelling fastest when it hits the ground?

(c) What is the total distance travelled by each stone?

## 2.4 What causes changes in motion?

Think about the following types of motion:

- an athlete running around a bend in a 200 m race;
- a ball being thrown over a fence;
- a car braking to a halt;
- a rocket accelerating in space;
- a snowball picking up snow as it rolls on level ground.

In each case there is a change in motion.

**Discuss what these changes are in each case.**

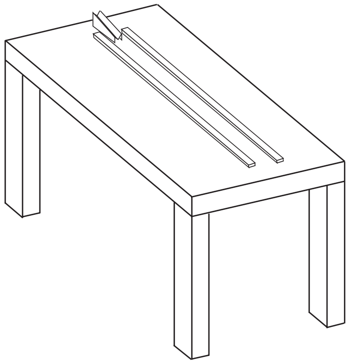
A change in motion is caused by **a force**. In medieval times it was thought that a force was required to keep a body in motion and that the only state which corresponded to an absence of forces was a state of rest.

The true relationship between forces and motion was stated by Newton using ideas of Galileo. In the absence of any forces there must be no change in the motion of the body, that is, the body must be at rest or moving with uniform velocity. Although first stated by Galileo, this is now generally known as Newton's First Law of Motion.

## Newton's First Law

A body remains in a state of rest or moves with uniform motion, unless acted on by a force.

## Activity 9 The chute experiment.



You will need a level table, 2 metre rules, a billiard ball, a chute and a stopwatch. You can make a chute out of a piece of folded cardboard.

Use the metre rules to make a channel on the table.

Allow the ball to roll down the chute so that it takes about 3 seconds to travel 1 m. Mark the point on the chute from which you release the ball.

Releasing the ball from this mark for each run, time the ball to roll 0.25, 0.5, 0.75 and 1 m.

Draw a displacement - time graph.

**What forces act on the marble as it rolls on the table ?**
**Are your results consistent with Newton's First Law?**

If the total force acting on a body is **zero** (for example, if equal and opposite forces act on the same body), then the body remains at rest or moves in a state of uniform motion.

Newton's First Law defines what happens to the motion of a body if no force is present. When a force acts on a body, there is a change in motion. Firstly, you need a clear definition of how to measure the motion of a body. Newton did a great deal of experimental work on this and came to the conclusion that the motion of a body was measured by the product of its mass and its velocity. This quantity is known as the **momentum** of the body.

$$\boxed{\text{Momentum} = m \times v}$$

A force produces a change in the momentum of a body, through a combination of changes in mass and/ or velocity.

**Discuss the change in momentum in each of the cases given at the beginning of the section.**

The physical law relating change in motion and the force acting on a body is given by Newton's Second Law.

## Newton's Second Law

The rate of change of momentum of a body is proportional to the applied force.

In the case where the mass of the body is constant, this leads to the result that the force is proportional to the product of the mass and the acceleration of the body. By choosing appropriate units to measure mass, acceleration and force, the constant of proportionality can be made equal to one so that

$$\boxed{F = ma}$$

where $m$ is in kilograms ( kg)

$a$ is in metres per second per second $(\text{ms}^{-2})$

$F$ is in newtons.

One newton is the force sufficient to produce an acceleration of one $\text{ms}^{-2}$ in a body of mass one kg. The abbreviation for a newton is N.

The approximate magnitude of some typical forces are

- force exerted by an adult arm $\approx 250$ N
- force exerted by Earth's gravity on an adult $\approx 700$ N
- force exerted by a car engine $\approx 2000$ N

### Example

A car of mass 1100 kg can accelerate from rest to a speed of 30 mph in 12 seconds. Calculate the force required.

#### Solution

Using the conversion $10 \text{ mph} = 4.445 \text{ ms}^{-1}$ gives

$$30 \text{ mph} = 13.3 \text{ ms}^{-1}$$

Assuming that the car accelerates at a constant rate, then using $v = u + at$ with $v = 13.3, u = 0$ and $t = 12$ gives

$$a = \frac{13.3}{12} = 1.11 \text{ ms}^{-2}.$$

Applying Newton's Second Law gives

$$F = ma$$

$$= 1100 \times 1.11$$

$$= 1220 \text{ N (to 3 significant places).}$$

### Example

A force of magnitude 20 N is applied to a particle of mass 4 kg for 6 seconds. Given that the initial velocity of the body is 15 $\text{ms}^{-1}$,

(a) calculate the acceleration, $a$, of the body;

(b) calculate its velocity, $v$, after 6 seconds.

After 6 seconds a force, $F$, is applied to bring the body to rest in a further 125 m.

(c) Calculate the magnitude of the force.

#### Solution

(a) Using Newton's Second Law,

$$20 = 4a$$

$$a = 5 \text{ ms}^{-2}$$

(b) Using (1), $v = 15 + 5 \times 6 = 45 \text{ ms}^{-1}$

(c) Since the particle is slowing down, the acceleration in equation (3) will be negative so put $a = -A$. Then using (3)

$$0 = 45^2 - 2 \times A \times 125.$$

Solving for $A$,

$$A = \frac{45^2}{250} = 8.1 \text{ ms}^{-2}$$

Using Newton's Second Law,

$$F = 4 \times 8.1 = 32.4 \text{ N.}$$

### Activity 10 Forces produced by car engines

(a) Collect data on typical accelerations and masses of cars. Calculate the average force produced during the acceleration.

(b) Use the data for braking distances shown in Activity 6 to calculate the average force applied during braking.

Since all objects fall with an acceleration of $g$ (neglecting air resistance) near the Earth's surface, the force acting on an object of mass $m$ is given by

$$F = mg.$$

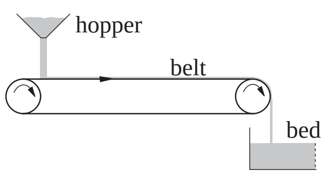
This force is known as the weight of the object.

## Exercise 2D

1. A force of 50 N is applied to a particle of mass 4 kg for 5 seconds only.
   (a) Calculate the acceleration for the first 5 seconds.
   (b) Write down the acceleration for the next 5 seconds.
   (c) Calculate the distance travelled during the first 10 seconds given that the particle was at rest initially.

2. A world class sprinter can accelerate from rest to $10\ \mathrm{ms}^{-1}$ in about 2 seconds. Estimate the magnitude of the force required to produce this acceleration.

3. The brakes of a train are required to bring it to rest from a speed of 80 kph in a distance of 500 m. The mass of the train is 200 tonnes.
   (a) Calculate the average deceleration.
   (b) Calculate the average force required to be exerted by the brakes.

4. 


   The diagram shows a conveyor belt which is designed to convey coal dust from a hopper to a bed.

   Initially, there is no dust on the belt. The force required to drive the belt in this case is 200 N and the velocity of the belt is $2\ \mathrm{ms}^{-1}$. The length of the belt is 40 m.

   Coal is now allowed to fall on the belt at a rate of 8 kg per second. The force driving the belt is adjusted so that the velocity stays at $2\ \mathrm{ms}^{-1}$.

   Draw a diagram of force against time for the first 30 seconds of operation.

5. A toy car of mass 0.04 kg is propelled from rest by an engine which provides a pulling force of $2.7\times10^{-3}$ N lasting for 8 seconds.
   (a) Calculate the acceleration.
   (b) Calculate the velocity after 8 seconds.

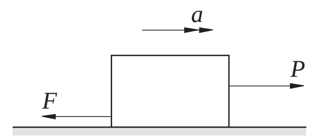
   If the speed of the toy car decreases uniformly to zero during the next 32 seconds, find the total distance travelled by the car.

6. A rocket has a mass of 40 tonnes of which 80% is fuel. The rocket motor develops a thrust of 1200 kN at all times.
   (a) Calculate the acceleration when the rocket is full of fuel.
   (b) Calculate the acceleration just before the fuel is exhausted.
   (c) What is the acceleration after the fuel is exhausted ?

7. A parachute reduces the speed of a parachutist of mass 70 kg from $40\ \mathrm{ms}^{-1}$ to $10\ \mathrm{ms}^{-1}$ in 3 seconds. Calculate the average force exerted by the parachute.

8. A car travelling at 30 mph is brought to rest in 3 seconds during a collision. Calculate the average force exerted on the car during the collision. Assume mass of driver is 70 kg and that of the car is 1100 kg.

## 2.5 How to deal with more than one force




The diagram shows a block of mass $M$ kg being pulled by a force $P$ N over a horizontal floor. The floor is rough so that there is a force, $F$ N, due to frictional resistance acting in the opposite direction to $P$.

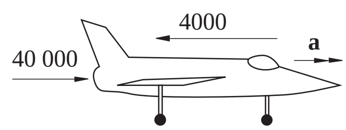
Since $P$ and $F$ act in opposite directions, their effect is a net force $P - F$ in the direction of $P$. Newton's Second Law is then applied using the net force to give

$$P - F = Ma \qquad (7)$$

where $a$ is the acceleration of the block.

The net force is often called the **resultant** force.

### Example




A jet aircraft of mass 8 tonnes has a single engine which generates a force of 40 000 N. Resistance to motion amounts to a constant value of 4000 N. [Reminder: 1 tonne $= 1000$ kg]

(a) Calculate the acceleration of the aircraft.

(b) The aircraft starts from rest. Calculate the speed after 12 seconds.

After 12 seconds the engine is switched off.

(c) Calculate the distance travelled by the aircraft before coming to a halt.

### Solution

(a) The net force acting on the aircraft is $40\,000 - 4000$ N.
So using equation (7), $40\,000 - 4000 = 8000a$.

Solving gives $a = 4.5 \text{ ms}^{-2}$.

(b) Equation (1) with $u = 0$, $a = 4.5$ and $t = 12$ gives
$$v = 0 + 4.5 \times 12 = 54 \text{ ms}^{-1}.$$

(c) The only horizontal force acting on the aircraft is resistance.

Using (7), $8000a = 0 - 4000$

$$\Rightarrow \quad a = -0.5 \text{ ms}^{-2}$$

where $a$ is the acceleration of the aircraft when the engine is switched off. Since $a < 0$ the aircraft is slowing down.

Using (3), $0 = 54^2 + 2 \times (-0.5) \times s$

$$s = 2916 \text{ m}$$

where $s$ is the distance travelled by the aircraft.

When equal but opposite forces act on a body, their net effect is zero, so there is no acceleration and the body remains at rest or in a state of constant velocity. Similarly if a body is at rest then the net force acting on the body must be zero. If the forces are not equal then there must be an acceleration.

In the diagram the sitter is not accelerating. Yet there is at least one force acting - his weight. Since people sitting on chairs do not accelerate downwards there must be an equal but opposite force acting. This force is provided by the chair acting on the sitter and is known as the normal contact force or reaction.

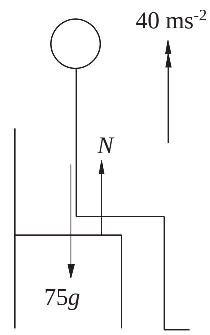
This leads to the following observation for two bodies in contact.

## Newton's Third Law

Whatever the nature of the forces, for two bodies in contact, **equal** but **opposite** forces must act from one to the other.

### Example

A manned rocket takes off with an acceleration of 40 $\text{ms}^{-2}$. An astronaut has a mass of 75 kg. Calculate the magnitude of the normal contact force acting from the chair on the astronaut.




#### Solution

The normal contact force is $N$. Using (7)

$$75 \times 40 = N - 75 \times g.$$

Using $g = 10$ , the value of $N$ is 3750 N.

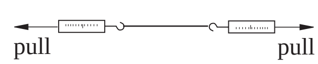
(This means that the astronaut will experience a force of 5 times her/his weight .)

## Connected bodies

Bodies are generally pushed or pulled by strings, bars or chains. In a tug of war match, the opposing teams have no direct contact. If they are not accelerating then the pulling forces must be equal and opposite. Also each member is pulling so there must be an equal and opposite force in the rope. Forces in ropes and strings are known as tensions. Tensions act inwards along the ropes and act in equal but opposite pairs.

### Activity 11 Tension and pulleys

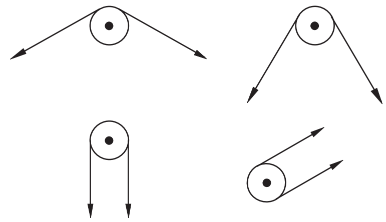
You will need a pulley, string and 2 newton-meters for this activity.




Tie each end of the string to a newton-meter as shown.

Pull to tension the string and note the readings.

What is the reading on one newton-meter if the other reads 2 N, 5 N, 9 N? Is it possible to get different readings?



Now note the readings on the newton-meters if you tension the string over the pulley in each of the arrangements opposite.

Is it possible to get different readings on each newton-meter?

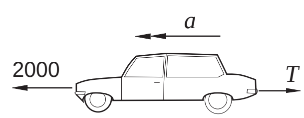
What can you say about the tension in the string on either side of a pulley?

---

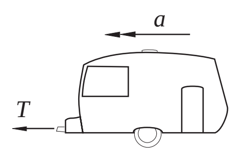



Bars and rods can also support thrusts as well as tensions. Thrusts are denoted by outward facing arrows and they keep objects apart rather than bring them together. By using equal but opposite forces in connectors, it is possible to solve problems in which more than one body is involved.

Each body can be treated separately by considering the forces acting on it in isolation together with any tensions or thrusts from the connectors.




In the diagram a car of mass 1300 kg is connected to a caravan of mass 700 kg via a coupling. The car engine develops a pulling force of 2 kN. (1 kN = 1000 N)




The motion of the car and caravan can be considered separately. For the caravan, the pulling force is supplied by the tension in the coupling. The car is driven forward by the pulling force of the engine and held back by the tension, $T$, in the coupling.

For the car, using equation (7)

$$2000 - T = 1300a$$

where $a$ is the acceleration of the car.

For the caravan

$$T = 700a$$

since the car and caravan must have the same acceleration.

Adding the two equations gives

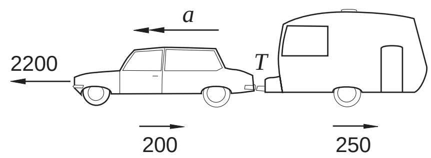
$$2000 = 2000a$$

$$\Rightarrow \quad a = 1 \text{ ms}^{-2}.$$

The tension in the coupling is $T = 700 \times 1 = 700$ N.

## Example

A car of mass 1100 kg pulls a caravan of mass 800 kg. The force exerted by the engine is 2200 N. In addition, friction and air resistance amount to 200 N on the car and 250 N on the caravan.




Calculate the acceleration of the car and the tension, $T$, in the coupling between the car and the caravan.

## Solution

Using (7) for the car

$$2200 - 200 - T = 1100a$$

where $a$ is the acceleration of the car.

Using (7) for the caravan

$$T - 250 = 800a\,.$$

Adding gives

$$1750 = 1900a$$

$$\Rightarrow \; a = 0.921 \text{ ms}^{-2}\,.$$

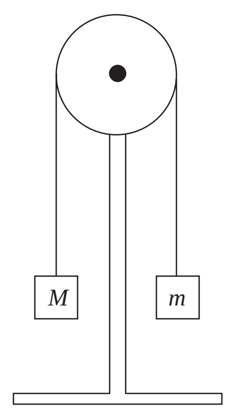
Now from $\quad T - 250 = 800a$

$$T = 800 \times 0.921 + 250$$

$$= 987 \text{ N}\,.$$

## Activity 12 Investigating the motion of connected bodies.

You will need a pulley, string, masses and a stop watch for this activity.




Set up the pulley and masses as shown in the diagram with the length of string slightly greater that the distance of the pulley from the floor. This distance should be about 1.5 m. Take $M = 120\,\text{g}$ and $m = 90\,\text{g}$.

Pull the 90 g mass down to the floor and measure the distance of the 120 g mass above it.

Release the system from rest and measure the time taken for the 120 g mass to reach the floor.

Repeat the measurements and average your results.

Use your results to calculate a value for the acceleration of the system assuming constant acceleration.

Derive an expression for the acceleration of the system in terms of $M$, $m$ and $g$.

How well do your results agree? You might like to repeat the experiment for other sets of masses.

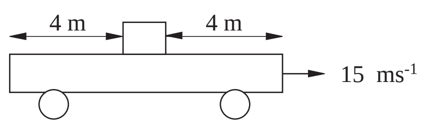
---

An assumption generally made in the motion of connected bodies is that the mass of the connector can be neglected compared to the masses of the bodies. Such connectors are called ‘light’. Another assumption is that the length of the connector remains constant, that is, the connector does not stretch.

**Discuss cases of motion where you think that one or both of the above assumptions would not be valid.**

## Exercise 2E

1. A block of wood of mass 500 gram is pulled by a force of 4 N. The resistance between the wood and the floor is 2 N. Calculate the acceleration.
2. Explain what you feel when you get into a lift which then ascends with an initial acceleration 2 $ms^{-2}$ followed by a steady velocity and finally has a deceleration of 11 $ms^{-2}$.
3. During a car collision, the car’s speed decreases from 50 mph to zero in 4 seconds. The driver is restrained by a car seat belt. Calculate the average force on the chest of an 80 kg driver.
4.




The diagram shows a box on the top of a trolley. The whole system is travelling on level ground at 15 $ms^{-1}$. The trolley reduces speed from 15 $ms^{-1}$ to 5 $ms^{-1}$ in 5 seconds. The mass of the box is 20 kg. As the trolley decelerates, the frictional force between the box and the trolley is 25 N.

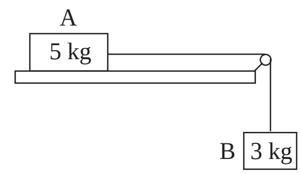
Describe, in detail, the motion of the box.

5. A shunting train pushes trucks along a level line. The mass of the locomotive is 20 tonnes and the mass of each truck is 4 tonnes. The locomotive can develop a force of 2000 N. Given that the locomotive pushes one truck and ignoring any resistances,

(a) calculate the acceleration of the locomotive.

(b) calculate the thrust on the truck.

6.




The diagram shows a mass A of 5 kg initially at rest on a horizontal table. A resistance force of 10 N acts against the motion of A which is connected to mass B of 3 kg by a light, inextensible string. The system is released from rest.

(a) Calculate the acceleration of A.

(b) Calculate the tension in the string.

After a short time, B reaches the floor.

(c) Calculate the acceleration of A now.

7. A breakdown truck tows a car of mass 1200 kg. Calculate the tension in the tow rope if the car is

(a) accelerating at 0.5 $ms^{-2}$ and experiencing a resistance force of 500 N;

(b) travelling at constant speed but experiencing a resistance force of 400 N.

## 2.6 Forming differential equations

In Section 2.1 differentiation is used to describe the rates of change of displacement and velocity. The resulting equations, which include derivatives such as $\frac{ds}{dt}$ and $\frac{dv}{dt}$, are called **differential equations**.

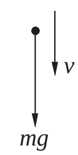
A differential equation is a relation between a **function** and its **derivatives** with respect to some **variable**, often time or distance.

If a differential equation is used to model a situation, then a general method is needed to solve the differential equation.

For a ball of mass $m$ falling vertically downwards, neglecting air resistance, the equation of motion is given by Newton's Second Law

$$mg = m\frac{dv}{dt}$$

giving $$\boxed{\frac{dv}{dt} = g} \qquad (8)$$

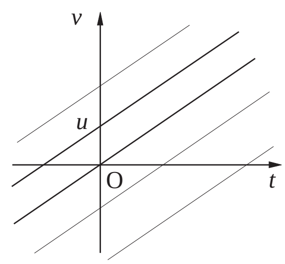



which is a first order differential equation with a constant right hand side. A **first order** differential equation is a relation between a function $v$ and its first derivative $\frac{dv}{dt}$.

Integrating equation (8) with respect to $t$,

$$v = gt + C, \qquad (9)$$

where $C$ is some constant. For different values of $C$ the graphs of $v$ against $t$ are parallel straight lines with gradient $g$. To pick out a particular line it is necessary to specify $v$ at some time $t$. If the ball is released from rest when $t = 0$, then




$$v = 0 \text{ when } t = 0.$$

Putting $t = 0$ in equation (9) gives

$$0 = 0 + C$$

so that $C = 0$ and equation (9) becomes

$$v = gt, \qquad (10)$$

which is the straight line passing through the origin O.

For the differential equation (8), expression (9) is called the **general solution**. The general solution of a first order differential equation has **one** arbitrary constant. The condition $v = 0$ when $t = 0$ is called an **initial condition** and expression (10) is a **particular solution**. A particular solution satisfies the differential equation **and** the initial condition.

If the ball had been given a velocity $u$ at $t = 0$ then $v = u$ at $t = 0$ and, from equation (9),

$$u = 0 + C \quad \Rightarrow \quad C = u,$$

so that the particular solution is now

$$v = u + gt. \qquad (11)$$

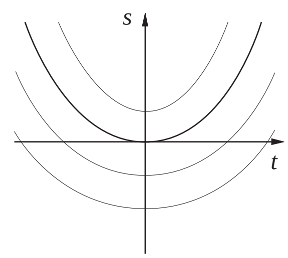
This straight line has intercept $u$ on the $v$ axis.

Since $v = \frac{ds}{dt}$, equation (11) gives

$$\frac{ds}{dt} = u + gt, \qquad (12)$$

which is another first order differential equation.

Integrating with respect to $t$ gives




$$s = ut + \frac{1}{2}gt^2 + B, \qquad (13)$$

where $B$ is some constant. Expression (13) is the general solution of equation (12). When $u = 0$, the graphs of $s$ against $t$ are parabolas with $t = 0$ as axis of symmetry. This family of parabolas have equal gradients

$$\frac{ds}{dt} = gt$$

at the same time $t$. If $s = 0$ when $t = 0$, then $B = 0$ and $s = \frac{1}{2}gt^2$.

## Exercise 2F

1. Taking $g = 10\text{ms}^{-2}$ sketch the graphs of $s$ against $t$ from Equation (4),
   (a) if $s = 0$ when $t = 0$ and
   (i) $u = 5\text{ ms}^{-1}$ (ii) $u = -10\text{ ms}^{-1}$
   (b) if $s = 6$ m when $t = 0$ and $u = 10\text{ ms}^{-1}$.

2. Integrate to find the general solutions of
   (a) $\frac{dy}{dt} = t^2$ (b) $\frac{dy}{dt} = \cos t$
   (c) $\frac{dy}{dt} = e^t$ (d) $\frac{dy}{dt} = -\frac{1}{t^2}$
   and sketch their graphs.

# 2.7 Direct integration

All the differential equations in the previous section are of the form

$$\frac{dy}{dt} = f(t), \tag{14}$$

where $f(t)$ is a function of $t$ only. They can be solved by direct integration, giving

$$y = \int f(t)dt + C, \tag{15}$$

where $C$ is some constant. Equation (15) is the general solution of equation (14).

So for example, if

$$\frac{dy}{dt} = t^3$$

then $$y = \int t^3 dt + C \quad \Rightarrow \quad y = \frac{1}{4}t^4 + C.$$

### Example

Solve the differential equation

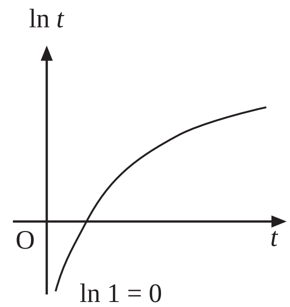
$$\frac{dy}{dt} = \frac{t+1}{t}$$

given that $y = 2$ when $t = 1$ for $t > 0$.

**Solution**

The general solution is

$$
\begin{aligned}
y &= \int \frac{t+1}{t}\,dt + C \\
&= \int \left(1+\frac{1}{t}\right) dt + C \\
&= \int dt + \int \frac{1}{t}\,dt + C \\
&= t + \ln t + C.
\end{aligned}
$$




When $t = 1, y = 2$, so that

$$
\begin{aligned}
2 &= 1 + \ln 1 + C \qquad (\ln 1 = 0) \\
&= 1 + C
\end{aligned}
$$

giving $C = 1$ and the required solution is

$$y = t + \ln t + 1.$$

## Exercise 2G

1. Find the general solutions of the following:

(a) $\dfrac{dy}{dt} = 2t^3 + 3$ (b) $\dfrac{dy}{dt} = \sin 2t$

(c) $\dfrac{dy}{dt} = \dfrac{1}{(1+3t)^{\frac{1}{2}}}$ (d) $\dfrac{dy}{dt} = (1+2t)^{\frac{1}{2}}$

(e) $\dfrac{dy}{dt} = \dfrac{1}{1+t}$ (f) $\dfrac{dy}{dt} = \dfrac{t}{1+2t^2}$.

2. Find the particular solutions of

(a) $\dfrac{dy}{dt} = 3t + 4, \quad y = 0$ when $t = 0$,

(b) $\dfrac{dy}{dt} = t^4 + \dfrac{1}{t^4}, \ y = 1$ when $t = 1$,

(c) $e^{2t}\dfrac{dy}{dt} + 1 = 0, \ y \to 2$ when $t \to \infty$,

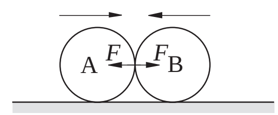
(d) $\dfrac{dy}{dt} = 2\sin 2t, \quad y = 1$ when $t = \dfrac{\pi}{4}$,

and sketch their graphs.

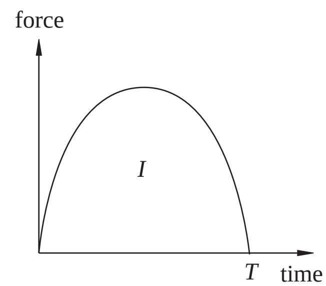
# 2.8 What happens during collisions?

When two particles collide, short term forces act during contact. The forces act to separate the particles although it is possible for them to stick together.

In the diagram opposite, A and B approach each other and have a head-on collision. During the collision, forces act to prevent the particles passing through each other. The force $F$ acts from B to A for the time that they are in contact and an equal opposite one from A to B.




The area under the $F-t$ curve is known as the impulse of B on A. You can replace $F$ by an average force $F_{av}$ such that the area remains the same. In that case the impulse, $I$, is given by




$$I = F_{av}T$$

where $T$ is the total time that the force acts.

**Discuss whether it is true that being struck by a hard ball is more painful than being struck by a soft ball of the same mass travelling with the same speed.**

**What happens when you bring your fist down on a table - fleshy side first and then knuckles first?**

Since there is a force acting from B to A, A will undergo a change of motion. Letting $a$ be the average acceleration of A during the collision,

$$F = ma.$$

Now $$a = \frac{v-u}{T}$$

where $u$ and $v$ are the velocities of A before and after the collision respectively.

Combining the two equations gives

$$F = \frac{m(v-u)}{T} \quad \text{or}$$

$$FT = mv - mu. \qquad (16)$$

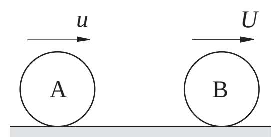
This is known as the **impulse equation** and is interpreted in the form

impulse on A = change in momentum of A.

Similar reasoning leads to

impulse on B = change in momentum of B.

When the two bodies are in contact, Newton's Third Law states that the impulses are equal and opposite.



Before impact

**Discuss why the impulses are equal and opposite.**

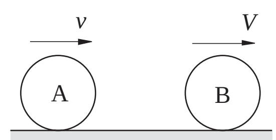
Adding the two equations above gives

change in momentum of A + change in momentum of B = 0.

This can be written algebraically to give

$$(mv - mu) + (MV - MU) = 0$$

or $$\boxed{MV + mv = MU + mu} \qquad (17)$$



After impact

In words, this can be expressed as

initial total momentum of the colliding bodies

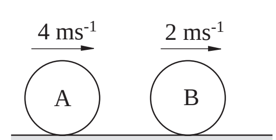
= final total momentum of the colliding bodies.

This is known as the principle of **conservation of momentum**.

In the diagrams showing the motion before and after impact, the two bodies are shown as if they are always moving in the same direction, i.e. to the right. If, in fact, the bodies are moving towards each other as shown by the first diagram on the previous page, then the value of $U$ must be negative when used in equation (17).

## Example

In the diagram A and B have masses 2 kg and 3 kg and move with initial velocities as shown. The collision reduces the velocity of A to 1 $ms^{-1}$. Find the velocity of B after the collision.




### Solution

Impulse on A $= 2\times 1 - 2\times 4 = -6$ kg $ms^{-1}$.

Impulse on B $= +6$ kg $ms^{-1}$.

Using the impulse equation

$$+6 = 3v - 3\times 2$$

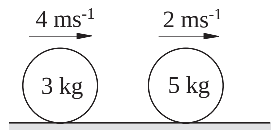
where $v$ is the velocity of B after the collision, giving

$$v = 4 \text{ ms}^{-1}.$$

In general, if the masses and original velocities are known, then the algebraic form of the conservation of momentum is a single equation in the two unknowns $V$ and $v$. So one of $V$ and $v$ must be measured in order to predict the other. There is one exception to this that is easy to deal with. This occurs when the bodies stick together or **coalesce**. In that case $V = v$ .

## Example

The two bodies in the diagram have masses 3 kg and 5 kg respectively. They are travelling with speeds 4 $ms^{-1}$ and 2 $ms^{-1}$. The bodies coalesce on impact. Find the speed of this body.




### Solution

Letting the final common velocity of the coalesced body be $v$, the principle of conservation of momentum gives

$$\text{initial total momentum} = \text{final total momentum}$$

$$\Rightarrow \quad 3\times4+5\times2=(3+5)\times v$$

$$\Rightarrow \quad v=\frac{22}{8} = 2.75 \text{ ms}^{-1}.$$

# Case study

Police forces have a well-established procedure for analysing car crashes. One of the simplest cases to consider is a head-on collision with a stationary vehicle.

**Discuss the stages in the incident starting from the instant the car driver sees the stationary vehicle and the data which the police would have to collect to analyse the incident.**

## Example

A car with total mass, including passengers, of 1300 kg collided with a stationary car of mass 1100 kg parked without lights on a dark road. The police measured the length $L$ metres of the skid marks of the moving car before impact and the length $l$ metres of the skid tracks after impact. They also took note of the state of the road. In order to estimate the decelerations of the cars, the police drove a car under the same road conditions at 50 mph and found the length of the skid to be 30 m. $L$ was found to be 14 and $l$ to be 3. It is required to find the speed of the moving car before impact.

### Solution

First of all, define symbols

$U$ = speed of car before braking in $ms^{-1}$

$V$ = speed of car just before impact in $ms^{-1}$

$v$ = speed of the two cars just after impact in $ms^{-1}$

$a$ = acceleration in $ms^{-2}$ on the surface where the accident took place.

The acceleration is found from the trial skid.

Using (3)

$$0 = 22.2^2 + 2 \times a \times 30$$

$$a = -8.23 \text{ ms}^{-2}$$

The negative acceleration indicates deceleration, as expected.

Next, consider the motion after the collision

$$0 = v^2 - 2 \times 8.23 \times 3.$$

So $$v = 7.03 \text{ ms}^{-1}.$$

Conservation of momentum now yields

$$1300V = (1300 + 1100) \times 7.03$$

$$V = 13.0$$

The velocity of the car before braking can be found from (3).

$$13^2 = U^2 - 2 \times 8.23 \times 14$$

$$U = 20.0$$

So the initial speed of the car was 44.9 mph.

## Exercise 2H

1. A particle A of mass 250 g collides with a particle B of mass 150 g. Initially A has velocity 7 $\text{ms}^{-1}$ and B is at rest. After the collision, the velocity of B is 5 $\text{ms}^{-1}$.
   (a) Calculate the impulse of A on B.
   (b) Calculate the velocity of A after the impact.
2. Two railway trucks, each of mass 8 tonnes, are travelling in the same direction and along the same tracks with velocities 3 $\text{ms}^{-1}$ and 1 $\text{ms}^{-1}$ respectively. When the trucks collide they couple together. Calculate the velocity of the coupled trucks.
3. A service in tennis can result in a speed of 90 mph for the ball. The return of service is typically 60 mph. Given that the mass of a tennis ball is 60 g, calculate the impulse on the racket of the return of service.
4. A ball is dropped from a height of 2 m and rebounds to a height of 75 cm. Given that the mass of the ball is 70 g, calculate the magnitude and direction of the impulse of the floor on the ball.
5. The world record for the men's high jump is approximately 2 m 45 cm. Estimate the magnitude of the impulse needed for a 70 kg athlete to clear this height. [Hint: For a simple model assume that the athlete jumps vertically].
6. A child of mass 40 kg runs and jumps onto a skateboard of mass 4 kg. If the child was moving forward at 0.68 $\text{ms}^{-1}$ when he jumped onto the skateboard, find the speed at which they move.
7. A tow truck of mass 3 tonnes is attached to a car of mass 1.2 tonnes by a rope. The truck is moving at a constant 3 $\text{ms}^{-1}$ when the tow rope becomes taut and the car begins to move. Assume that both vehicles move at the same speed once the rope is taut, and find this speed.

## 2.9 Miscellaneous Exercises

1. Bill is going to take a penalty in a hockey game. He can hit the ball at a speed of $25\,\text{ms}^{-1}$. Initially the ball is placed 2.5 m from the goal line. The width of the goal is 1.5 m. John, the goalie, can move himself at a speed of $6\ \text{ms}^{-1}$. John stands in the middle of the goal.

   (a) What range of directions should Bill hit the ball to score? (Assume he hits the ball along the surface.)

   (b) What difference would it make to the answer to (a) if John's reaction time of 0.25 seconds were taken into consideration ?

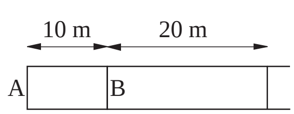
   (c) Generalise to arbitrary velocities for John and Bill.

2. Two cars A and B are initially at rest side by side. A starts off on a straight track with an acceleration of $2\ \text{ms}^{-2}$. Five seconds later B starts off on a parallel track to A, with acceleration 3.125 $\text{ms}^{-2}$.

   (a) Calculate the distance travelled by A after 5 seconds.

   (b) Calculate the time taken for B to catch up A.

   (c) Find the speeds of A and B at that time.

3. 


   The diagram shows part of an athletics track laid out for the changeover in a 4 by 100 m relay race. Assume the incoming runner A is moving at 10 $\text{ms}^{-1}$ and the receiving runner B starts from rest with an acceleration of 4.5 $\text{ms}^{-2}$. The receiver starts running as soon as the incoming runner enters the box. Where does the changeover occur?

4. A motorist approaches a set of traffic lights at 45 mph. Her reaction time is 0.7 seconds and the maximum safe deceleration of the car is 6.5 $\text{ms}^{-2}$.

   (a) The motorist sees the lights turn to amber. What is the minimum distance from the lights that she can safely bring the car to a stop?

   (b) The lights remain on amber for 2 seconds, before turning red. As the motorist approaches the lights she sees them turn amber and decides to try to get past the lights before they turn red. What is the maximum distance from the lights that the motorist can do this at a constant speed of 45 mph?

   (c) Suggest an improvement to the model in (b), where constant speed was assumed. What difference does this make to the maximum distance ?

   (d) Generalise your answers to (a), (b) and (c) for an arbitrary approach velocity $v$ $\text{ms}^{-1}$.

5. A college campus has a road passing through it. The speed of vehicles along the road is to be reduced by placing speed bumps at intervals along the road. The purpose of the bumps is to force vehicles to go very slowly over them.

   Suppose the maximum speed of a vehicle is 30 mph and the bumps are placed every $D$ metres. Suppose the vehicle drives over a bump at 5 mph and its maximum acceleration is 2 $\text{ms}^{-2}$.

   (a) Sketch a velocity time diagram. Given that the vehicle just achieves its maximum speed, calculate the value of $D$, making simplifying assumptions where necessary.

   (b) Consider a range of speeds at which the vehicle crosses the bumps. Find the dependence of $D$ on the speed, $V$, that the vehicle crosses the bumps. Assume a maximum speed of 20 mph.

6. A ball is thrown vertically upwards with an initial velocity of 30 $\text{ms}^{-1}$. One second later, another ball is thrown upwards with an initial velocity of $u$ $\text{ms}^{-1}$. The particles collide after a further 2 seconds. Find the value of $u$.

7. The velocity of a car every 10 seconds is given in the following table.

| $t$ (sec) | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $v$(kph) | 0 | 34 | 54 | 66 | 74 | 78 | 80 |

   Draw a velocity time graph for the motion of the car. Use your graph to estimate

   (a) the acceleration of the car after 25 seconds.

   (b) the displacement of the car after 60 seconds.

8. A force $F$ N acts on a particle of mass 2 kg initially at rest. After 4 seconds the displacement of the particle is 20 m. Find the value of $F$.

9. A locomotive has a pulling force of 125 kN and a mass of 120 tonnes. On a level track it travels at a steady speed of 72 kph.

   (a) What is the resistance to motion? Assuming that the resistance is proportional to the square of the velocity, find the constant of proportionality.

   (b) Calculate the acceleration at 54 kph.

10. A particle of mass 4 kg is released from rest and falls under gravity against a resistance to motion of $2v$ N where $v$ is its velocity in $ms^{-1}$.

    Determine its terminal velocity. How long does it take to reach a velocity of 15 $ms^{-1}$?

11. A particle of mass 1 kg is released from rest and falls under gravity against resistance of $\frac{v^2}{2}$N where $v$ is its velocity in $ms^{-1}$. Determine its terminal velocity. How far does the body fall as its velocity increases from 2 $ms^{-1}$ to 4 $ms^{-1}$?

12. Crowd control in some countries can involve the use of high pressure hoses. These spray water at people and can knock them over. The velocity of the water is 20 $ms^{-1}$ and the jet has a diameter of 10 cm. Assuming that the momentum of the jet is destroyed on hitting the person, calculate the force on the person.

13. The gravitational attraction of the Earth on a particle of mass $m$ is $\frac{km}{r^2}$ where $r$ is the distance of the particle from the Earth's centre and $k$ is a constant.

    Calculate the acceleration of a particle of mass $m$ at a distance 120 km above the Earth's surface. [The radius of the Earth is 6400 km and at the Earth's surface $g = 9.8$ $ms^{-2}$.]

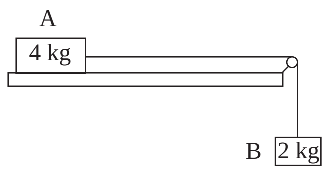
14. A transport system is to be designed around a rail running over a set of pulse generators spaced 10 m apart and giving a force of 10 N acting 1 m either side of each generator to boost the velocity of the vehicle.

    (a) Sketch a velocity-time diagram.

    (b) Find the velocity increase after 55 m if the mass of the vehicle is 50 kg.

15. A locomotive of mass 80 tonnes pulls two trucks each with mass 9 tonnes. The pulling force of the locomotive is 14000 N. Resistance to motion can be ignored. Calculate the acceleration of the system and the tension in each of the couplings.

16.




Two bodies A and B of mass 4 kg and 2 kg respectively are attached by a light inextensible string passing over a smooth pulley. A rests on a table and B hangs over the side. Resistance forces on A amount to 8 N.

The system is released from rest. Calculate the acceleration of the system and the tension in the string. Find the speed of B when it has fallen 2m.

17. A locomotive of mass $M$ kg pulls a train of trucks. The mass of each truck is $\frac{M}{10}$ kg.

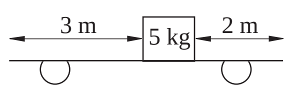
    The pulling force of the engine is $F$. How many trucks can the engine pull so that the tension in the coupling between the locomotive and the first truck is greater than $\frac{F}{2}$?

18. Rockets are made in sections known as stages. This is to enable a stage to be jettisoned once its fuel is used up. In a 3 stage rocket the mass of the lower stage is 1800 tonnes, the mass of the middle stage is 800 tonnes and the mass of the upper stage is 180 tonnes. The stages are coupled together by rings which can withstand a thrust of $1.65 \times 10^7$ N. Calculate the maximum safe acceleration of the rocket

    (a) when all 3 stages are present

    (b) when the upper 2 stages are present.

19.




The diagram shows a case of mass 5 kg on the floor of a trolley. Resistance force amounts to 5 N. The trolley is subject to a deceleration of 8 $ms^{-2}$ until it is brought to rest from a speed of 40 $ms^{-1}$.

Describe in detail the motion of the case.

20. A simple device for measuring acceleration (an accelerometer) consists of a mass $m$ attached by a light inextensible string to the roof of a vehicle.

    Sketch diagrams to show what happens to this arrangement when :

    (a) the vehicle accelerates;

    (b) the vehicle decelerates;

    (c) the vehicle moves with uniform velocity.

21. A train of total mass 110 tonnes and velocity 80 kph crashes into a stationary locomotive of mass 70 tonnes.

    (a) Calculate the velocity of the combined system immediately after impact.

    The trains plough on for a further 40 m.

    (b) Calculate the average deceleration and the resistance to motion.

22. A pile-driver consists of a pile of mass 200 kg and a driver of mass 40 kg. The driver drops on the pile with velocity 6 $\text{ms}^{-1}$ and sticks to the top of the pile.

    (a) Calculate the velocity of the pile immediately after impact.

    Resistances to motion of the pile amount to 1400 N.

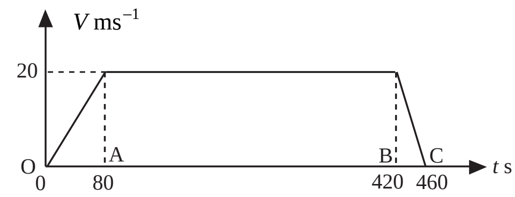
    (b) Calculate the distance penetrated by the pile.

23. Two uniform smooth spheres, A of mass 0.03 kg and B of mass 0.1 kg, have equal radii and are moving directly towards each other with speeds of 7 $\text{ms}^{-1}$ and 4 $\text{ms}^{-1}$ respectively. The spheres collide directly and B is reduced to rest by the impact. State the magnitude of the impulse experienced by B and find the speed of A after impact. (AEB)

24. Two particles A and B of masses $m$ and $2m$ respectively, are attached to the ends of a light inextensible string which passes over a fixed smooth pulley. The particles are released from rest with the parts of the string on each side of the pulley hanging vertically. When particle B has moved a distance $h$ it receives an impulse which brings it momentarily to rest. Find, in terms of $m$, $g$ and $h$, the magnitude of this impulse. (AEB)

25. A vehicle travelling on a straight horizontal track joining two points A and B accelerates at a constant rate of 0.25 $\text{ms}^{-2}$ and decelerates at a constant rate of 1 $\text{ms}^{-2}$. It covers a distance of 2.0 km from A to B by accelerating from rest to a speed of $v$ $\text{ms}^{-1}$ and travelling at that speed until it starts to decelerate to rest. Express in terms of $v$ the times taken for acceleration and deceleration.

    Given that the total time for the journey is 2.5 minutes find a quadratic equation for $v$ and determine $v$, explaining clearly the reason for your choice of the value of $v$. (AEB)

26. 


    The diagram shows the speed-time graph for a train which travels from rest in one station to rest at the next station. For each of the time intervals OA, AB and BC, state the value of the train's acceleration.

    Calculate the distance between the stations. (AEB)

27. When a train accelerates its acceleration is always $f$ $\text{km h}^{-2}$ and when it decelerates its retardation is always $3f$ $\text{km h}^{-2}$. The acceleration is such that the train can accelerate from rest to 60 $\text{km h}^{-1}$ in a distance of 1.5 km. Find

    (a) $f$,

    (b) the time taken to reach a speed of 60 $\text{km h}^{-1}$ from rest,

    (c) the distance travelled in decelerating from 60 $\text{km h}^{-1}$ to rest.

    On a journey of over 40 km the train is accelerated from rest to a speed of 60 $\text{km h}^{-1}$ and kept at that speed until retarded to rest at the end of the journey. On one such journey the train is required, roughly half way through the journey, to slow down to rest, stay at rest for 3 minutes and then accelerate back to a speed of 60 $\text{km h}^{-1}$.

    (d) Determine how late the train is on arrival at rest at its destination. (AEB)

28. Show that, in the usual notation, $v\dfrac{dv}{dx} = \dfrac{d^2x}{dt^2}$.

    A particle P moves along the positive $x$-axis such that when its displacement from the origin O is $x$ m, its acceleration in the positive $x$ direction is $(10x - 2x^3)$ $\text{ms}^{-2}$. The speed of P is $\sqrt{15}$ $\text{ms}^{-1}$ when $x = 2$. Find an expression for the speed of P for any value of $x$.

    Determine the values of $x$ for which P comes instantaneously to rest. (AEB)

29. A particle moving in a straight line with speed $u$ ms$^{-1}$ is retarded uniformly for 16 seconds so that its speed is reduced to $\frac{1}{4}u$ ms$^{-1}$. It travels at this reduced constant speed for a further 16 seconds. The particle is then brought to rest by applying a constant retardation for a further 8 seconds. Draw a time-speed graph and hence, or otherwise,

   (a) express both retardations in terms of $u$,

   (b) show that the total distance travelled over the two periods of retardation is $11u$ m,

   (c) find $u$ given that the total distance travelled in the 40 seconds in which the speed is reduced from $u$ ms$^{-1}$ to zero is 45 m.

   (AEB)

30. A tram travelling along a straight track starts from rest and accelerates uniformly for 15 seconds. During this time it travels 135 metres. The tram now maintains a constant speed for a further one minute. It is finally brought to rest decelerating uniformly over a distance of 90 metres. Calculate the tram's acceleration and deceleration during the first and last stages of the journey. Also find the time taken and the distance travelled for the whole journey.

   (AEB)

31. A train travelling at 50 ms$^{-1}$ applies its brakes on passing a yellow signal at a point A and decelerates uniformly, with a deceleration of 1 ms$^{-2}$, until it reaches a speed of 10 ms$^{-1}$. The train then travels for 2 km at the uniform speed of 10 ms$^{-1}$ before passing a green signal. On passing the green signal the train accelerates uniformly, with acceleration 0.2 ms$^{-2}$, until it finally reaches a speed of 50 ms$^{-1}$ at a point B. Find the distance AB and the time taken to travel that distance.

   (AEB)