Integration

Section 5: Integrating other functions

Notes and Examples

These notes contain subsections on
- Integrating exponential functions
- Logarithmic integrals
- Using the methods of substitution and inspection
- Integrating trigonometric functions

Integrating exponential functions

Remember that the derivative of e^x is e^x. Therefore

$$\int e^x \mathrm{d}x = e^x + c.$$

Similarly, since the derivative of e^{kx} is ke^{kx}:

$$\int e^{kx} \mathrm{d}x = \frac{1}{k} e^{kx} + c.$$

Example 1

Evaluate $\int_0^2 \sqrt{e^x}\,\mathrm{d}x$, giving your answer in terms of e.

Solution

$$\begin{aligned}\int_0^2 \sqrt{e^x}\,\mathrm{d}x &= \int_0^2 e^{\frac{1}{2}x}\,\mathrm{d}x \\ &= \left[2e^{\frac{1}{2}x}\right]_0^2 \\ &= \left[2e^{\frac{1}{2}x}\right]_0^2 \\ &= 2e^1 - 2e^0 \\ &= 2e - 2\end{aligned}$$

Logarithmic integrals

The derivative of $\ln x$ is $\frac{1}{x}$. It follows that $\int \frac{1}{x}\mathrm{d}x = \ln x + c$, provided $x > 0$. Also, if $x < 0$, the derivative of $\ln(-x)$ is $\frac{1}{(-x)}.(-1) = \frac{1}{x}$, so $\int \frac{1}{x}\mathrm{d}x = \ln(-x) + c$. These two results are sometimes combined using a modulus sign:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$$

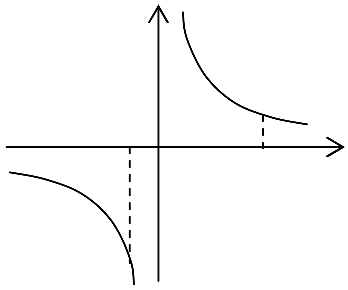
Care needs to be taken when using this result. The domain of x must be either $x > 0$ or $x < 0$ – you cannot integrate across zero, as the next example illustrates.

Example 2

Find $\int_{-1}^{3} \frac{1}{x}\,\mathrm{d}x$.

Solution

The integral is the area between the curve $y = \frac{1}{x}$ and the lines $x = -1$ and $x = 3$.



From the graph, you can see that this area is not defined, as it includes the value $x = 0$ for which the function $y = \frac{1}{x}$ is not defined!

Example 3

Integrate $\int \frac{1}{3x}\,\mathrm{d}x$

Solution

$$\int \frac{1}{3x}\,\mathrm{d}x = \tfrac{1}{3}\int \ln x \,\mathrm{d}x$$
$$= \tfrac{1}{3}\ln|x| + c$$

The example above illustrates an important point. In the same way that the integral of e^{kx} is $\frac{1}{k}\mathrm{e}^{kx} + c$, it is logical that the integral of $\frac{1}{kx}$ is $\frac{1}{k}\ln|kx| + c$, and in fact this is quite true. But in the example above, the integral of $\frac{1}{3x}$ is given as $\tfrac{1}{3}\ln|x| + c$ rather than as $\tfrac{1}{3}\ln|3x| + c$.

The answer to this problem is that in fact these two expressions are the same. Remember that by the laws of logarithms, $\ln|3x| = \ln 3 + \ln|x|$. So $\tfrac{1}{3}\ln|3x| + c$ may be written as $\tfrac{1}{3}\ln|x| + \tfrac{1}{3}\ln 3 + c$. But ln 3 is just a constant, and so it can be considered as part of the arbitrary constant.

In general, it is easier to take any constant outside the integral, as in Example 3, since this gives you a simpler expression to work with.

Using the methods of substitution and inspection

Example 4

Find $\int \frac{x}{1+x^2}\,\mathrm{d}x$.

Solution

Use the substitution $u = 1 + x^2$

$$u = 1 + x^2 \;\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$\Rightarrow \mathrm{d}x = \frac{1}{2x}\,\mathrm{d}u$$

$$\int \frac{x}{1+x^2}\,\mathrm{d}x = \int \frac{x}{u} \times \frac{1}{2x}\,\mathrm{d}x$$

$$= \int \frac{1}{2u}\,\mathrm{d}x$$

$$= \tfrac{1}{2}\ln u + c$$

$$= \tfrac{1}{2}\ln\left(1+x^2\right) + c$$

Notice that modulus signs are not needed here, since $1 + x^2$ is always positive.

Notice that this integral can be done quicker by inspection, by noting that the derivative of $\ln(1 + x^2)$ is $\frac{1}{1+x^2} \times 2x = \frac{2x}{1+x^2}$. This is twice the integrand, so it follows that $\int \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\ln\left(1+x^2\right) + c$.

It is easy to spot integrals which can be done by substitution to give a logarithm. If the integrand is a fraction, the numerator of which is the derivative of the denominator, then the integral is the natural logarithm of the denominator.

This can be generalised:

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + c$$

As in Example 4, in some cases the numerator is a multiple of the derivative of the denominator, so you need to adjust things a little. In Examples 5 and 6 this approach is used.

Example 5

Find $\int \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\,\mathrm{d}x$

The derivative of the denominator is $2\mathrm{e}^{2x}$, which is twice the numerator, so you can integrate by inspection.

Solution

$$\int \frac{e^{2x}}{1+e^{2x}}\,dx = \tfrac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx$$
$$= \tfrac{1}{2}\ln\left(1+e^{2x}\right)+c$$

Again, since $1+e^{2x}$ is always positive, no modulus signs are needed.

Example 6

Find $\int_1^2 \frac{1+x^2}{3x+x^3}\,dx$, expressing the answer as a single logarithm.

Solution

The derivative of $3x + x^3$ is $3 + 3x^2 = 3(1 + x^2)$. This is three times the numerator of the integrand. So

$$\int_1^2 \frac{1+x^2}{3x+x^3}\,dx = \tfrac{1}{3}\int_1^2 \frac{3+3x^2}{3x+x^3}\,dx$$
$$= \tfrac{1}{3}\Big[\ln\left|3x+x^3\right|\Big]_1^2$$
$$= \tfrac{1}{3}(\ln 14 - \ln 4)$$
$$= \tfrac{1}{3}\ln\left(\tfrac{14}{4}\right)$$
$$= \tfrac{1}{3}\ln\left(\tfrac{7}{2}\right)$$

You may also like to look at the **Integration that leads to log functions video**.

Integrals of trigonometric functions

The derivative of $\sin x$ is $\cos x$; the derivative of $\cos x$ is $-\sin x$. It follows that

$$\int \sin x\,dx = -\cos x + c \qquad \int \cos x\,dx = \sin x + c$$

Another useful result to bear in mind is that the derivative of $\tan x$ is $\sec^2 x$, so

$$\int \sec^2 x\,dx = \tan x + c$$

Similarly, by looking at the derivatives of $\sin kx$, $\cos kx$ and $\tan kx$, you can see that

$$\int \sin kx\,dx = -\tfrac{1}{k}\cos kx + c \qquad \int \cos kx\,dx = \tfrac{1}{k}\sin kx + c \qquad \int \sec^2 kx\,dx = \tfrac{1}{k}\tan kx + c$$

Example 7

Find (i) $\int \sin 3x \, \mathrm{d}x$

(ii) $\int_0^{\pi/6} \sec^2 2x \, \mathrm{d}x$

(iii) $\int \sin^2 x \cos x \, \mathrm{d}x$

Solution

(i) $\int \sin 3x \, \mathrm{d}x = -\tfrac{1}{3}\cos 3x + c$

(ii) $\int_0^{\pi/6} \sec^2 2x \, \mathrm{d}x = \left[\tfrac{1}{2}\tan 2x\right]_0^{\pi/6}$

$= \tfrac{1}{2}\left(\tan \tfrac{\pi}{3} - \tan 0\right)$

$= \tfrac{1}{2}\sqrt{3}$

remember that $\tan 60^\circ = \sqrt{3}$

(iii) $\int \sin^2 x \cos x \, \mathrm{d}x$ can be done using the substitution $u = \sin x$

$$u = \sin x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \Rightarrow \mathrm{d}x = \frac{1}{\cos x}\mathrm{d}u$$

$$\int \sin^2 x \cos x \, \mathrm{d}x = \int u^2 \cos x \times \frac{1}{\cos x}\mathrm{d}u$$

$$= \int u^2 \mathrm{d}u$$

$$= \tfrac{1}{3}u^3 + c$$

$$= \tfrac{1}{3}\sin^3 x + c$$

You could also do this by inspection: notice that the integral is a product of $\sin^2 x$ (a function of $\sin x$) and $\cos x$ (the derivative of $\sin x$)