Binomial Theorem

If a and x are two real number and n is a positive integer then

$$(a+x)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}x^1+\binom{n}{2}a^{n-2}x^2+\ldots+\binom{n}{n-1}a\,x^{n-1}+\binom{n}{n}x^n$$

Proof

We will use mathematical induction to prove this so let $S(n)$ be the given statement.

Put $n=1$

$$S(1):(a+x)^1=\binom{1}{0}a^1+\binom{1}{1}a^{1-1}x^1=(1)a+(1)(1)\,x \quad\Rightarrow\quad a+x=a+x$$

$S(1)$ is true so condition I is satisfied.

Now suppose that $S(n)$ is true for $n=k$.

$$S(k):\ (a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k \quad \ldots..(i)$$

The statement for $n=k+1$

$$S(k+1):\ \ (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^{k+1-1}x^1+\binom{k+1}{2}a^{k+1-2}x^2+\ldots$$

$$+\binom{k+1}{k+1-1}a\,x^{k+1-1}+\binom{k+1}{k+1}x^{k+1}$$

$$\Rightarrow\ (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^kx^1+\binom{k+1}{2}a^{k-1}x^2+\ldots$$

$$+\binom{k+1}{k}a\,x^k+\binom{k+1}{k+1}x^{k+1}$$

Multiplying both sides of equation (i) by $(a+x)$

$$(a+x)^k(a+x)=\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(a+x)$$

$$\Rightarrow\ (a+x)^{k+1}=\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(a)$$

$$+\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(x)$$

$$\Rightarrow\ (a+x)^{k+1}=\binom{k}{0}a^{k+1}+\binom{k}{1}a^kx^1+\binom{k}{2}a^{k-1}x^2+\ldots+\binom{k}{k-1}a^2x^{k-1}+\binom{k}{k}a\,x^k$$

$$+\binom{k}{0}a^kx+\binom{k}{1}a^{k-1}x^2+\binom{k}{2}a^{k-2}x^3+\ldots+\binom{k}{k-1}a\,x^k+\binom{k}{k}x^{k+1}$$

$$\Rightarrow\ (a+x)^{k+1}=\binom{k}{0}a^{k+1}+\left(\binom{k}{1}+\binom{k}{0}\right)a^kx^1+\left(\binom{k}{2}+\binom{k}{1}\right)a^{k-1}x^2+\ldots$$

$$+\left(\binom{k}{k}+\binom{k}{k-1}\right)a\,x^{k}+\binom{k}{k}x^{k+1}$$

Since $\binom{n}{0}=\binom{n+1}{0}$, $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$ and $\binom{n}{n}=\binom{n+1}{n+1}$

$$\Rightarrow\ (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^{k}x^{1}+\binom{k+1}{2}a^{k-1}x^{2}+\ldots+\binom{k+1}{k}a\,x^{k}+\binom{k+1}{k+1}x^{k+1}$$

Thus $S(k+1)$ is true when $S(k)$ is true so condition II is satisfied and $S(n)$ is true for all
positive integral value of n.

Question # 1
Using binomial theorem, expand the following:

(i) $(a+2b)^5$ (ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ (iii) $\left(3a-\frac{x}{3a}\right)^4$

(iv) $\left(2a-\frac{x}{a}\right)^7$ (v) $\left(\frac{x}{2y}-\frac{2y}{x}\right)^8$ (vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

Solution

(i)

$$(a+2b)^5=\binom{5}{0}a^5+\binom{5}{1}a^{5-1}(2b)^1+\binom{5}{2}a^{5-2}(2b)^2+\binom{5}{3}a^{5-3}(2b)^3+\binom{5}{4}a^{5-4}(2b)^4+\binom{5}{5}a^{5-5}(2b)^5$$

$$=(1)a^5+(5)a^4(2b)+(10)a^3(4b^2)+(10)a^2(8b^3)+(5)a^1(16b^4)+(1)a^0(32b^5)$$

$$=a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5 \qquad \because\ a^0=1$$

(ii)

$$\left(\frac{x}{2}-\frac{2}{x^2}\right)^6=\binom{6}{0}\left(\frac{x}{2}\right)^6+\binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(-\frac{2}{x^2}\right)^1+\binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(-\frac{2}{x^2}\right)^2+\binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(-\frac{2}{x^2}\right)^3$$

$$+\binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(-\frac{2}{x^2}\right)^4+\binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(-\frac{2}{x^2}\right)^5+\binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(-\frac{2}{x^2}\right)^6$$

$$=(1)\left(\frac{x}{2}\right)^6-(6)\left(\frac{x}{2}\right)^5\left(\frac{2}{x^2}\right)+(15)\left(\frac{x}{2}\right)^4\left(\frac{2}{x^2}\right)^2-(20)\left(\frac{x}{2}\right)^3\left(\frac{2}{x^2}\right)^3$$

$$+(15)\left(\frac{x}{2}\right)^2\left(\frac{2}{x^2}\right)^4-(6)\left(\frac{x}{2}\right)^1\left(\frac{2}{x^2}\right)^5+(1)(1)\left(\frac{2}{x^2}\right)^6$$

$$=\left(\frac{x^6}{64}\right)-6\left(\frac{x^5}{32}\right)\left(\frac{2}{x^2}\right)+15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right)-20\left(\frac{x^3}{8}\right)\left(\frac{8}{x^6}\right)$$

$$+15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)-6\left(\frac{x}{2}\right)\left(\frac{32}{x^{10}}\right)+\left(\frac{64}{x^{12}}\right)$$

$$=\frac{x^6}{64}-\frac{3x^3}{8}+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$$

(iii) *Do yourself*

(iv) *Do yourself*

(v) *Do yourself*

(vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6=\binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6+\binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^{6-1}\left(-\sqrt{\frac{x}{a}}\right)^1+\binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^{6-2}\left(-\sqrt{\frac{x}{a}}\right)^2+\binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^{6-3}$

$\left(-\sqrt{\frac{x}{a}}\right)^3+\binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^{6-4}\left(-\sqrt{\frac{x}{a}}\right)^4+\binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^{6-5}\left(-\sqrt{\frac{x}{a}}\right)^5+\binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^{6-6}\left(-\sqrt{\frac{x}{a}}\right)^6$

$=(1)\left(\sqrt{\frac{a}{x}}\right)^6-(6)\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{x}{a}}\right)^1+(15)\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{x}{a}}\right)^2-(20)\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{x}{a}}\right)^3-$

$\left(\sqrt{\frac{x}{a}}\right)^3+(15)\left(\sqrt{\frac{a}{x}}\right)^2\left(\sqrt{\frac{x}{a}}\right)^4-(6)\left(\sqrt{\frac{a}{x}}\right)^1\left(\sqrt{\frac{x}{a}}\right)^5+(1)\left(\sqrt{\frac{a}{x}}\right)^0\left(\sqrt{\frac{x}{a}}\right)^6 \quad =$

$\left(\sqrt{\frac{a}{x}}\right)^6-6\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^{-1}+15\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^{-2}-20\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^{-3}$

$+15\left(\sqrt{\frac{x}{a}}\right)^{-2}\left(\sqrt{\frac{x}{a}}\right)^4-6\left(\sqrt{\frac{x}{a}}\right)^{-1}\left(\sqrt{\frac{x}{a}}\right)^5+1(1)\left(\sqrt{\frac{x}{a}}\right)^6$

$=\left(\sqrt{\frac{a}{x}}\right)^6-6\left(\sqrt{\frac{a}{x}}\right)^{5-1}+15\left(\sqrt{\frac{a}{x}}\right)^{4-2}-20\left(\sqrt{\frac{a}{x}}\right)^{3-3}+15\left(\sqrt{\frac{x}{a}}\right)^{-2+4}-6\left(\sqrt{\frac{x}{a}}\right)^{-1+5}+1\left(\sqrt{\frac{x}{a}}\right)^6$

$=\left(\sqrt{\frac{a}{x}}\right)^6-6\left(\sqrt{\frac{a}{x}}\right)^4+15\left(\sqrt{\frac{a}{x}}\right)^2-20\left(\sqrt{\frac{a}{x}}\right)^0+15\left(\sqrt{\frac{x}{a}}\right)^2-6\left(\sqrt{\frac{x}{a}}\right)^4+\left(\sqrt{\frac{x}{a}}\right)^6$

$=\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^6-6\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^4+15\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^2-20(1)+15\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^2-6\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^4+\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^6$

$=\left(\frac{a}{x}\right)^3-6\left(\frac{a}{x}\right)^2+15\left(\frac{a}{x}\right)-20+15\left(\frac{x}{a}\right)-6\left(\frac{x}{a}\right)^2+\left(\frac{x}{a}\right)^3$

$=\frac{a^3}{x^3}-6\frac{a^2}{x^2}+15\frac{a}{x}-20+15\frac{x}{a}-6\frac{x^2}{a^2}+\frac{x^3}{a^3}$

Question # 2
Calculate the following by means of binomial theorem:

(i) $(0.97)^3$ (ii) $(2.02)^4$ (iii) $(9.98)^4$ (iv) $(2.1)^5$

Solution (i) $(0.97)^3=(1-0.03)^3$

$=\binom{3}{0}(1)^3+\binom{3}{1}(1)^2(-0.03)+\binom{3}{2}(1)^1(-0.03)^2++\binom{3}{3}(-0.03)^3$

$$= (1)(1) + 3(1)(-0.03) + 3(1)(0.0009) + +(1)(-0.000024)$$
$$= 1 - 0.09 + 0.0027 - 0.000027 = 0.912673$$

(ii) $(2.02)^4 = (2+0.02)^4$ *Now do yourself.*

(iii) $(9.98)^4 = (10-0.02)^4$

$$= \binom{4}{0}(10)^4 + \binom{4}{1}(10)^3(-0.02) + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3$$
$$+\binom{4}{4}(10)^0(-0.02)^4$$
$$= (1)(10000) + 4(1000)(-0.02) + 6(100)(0.0004) + 4(10)(-0.000008)$$
$$+(1)(1)(0.00000016)$$
$$= 10000 - 80 + 0.24 - 0.00032 + 0.00000016 = 9920.23968$$

(iv) $(2.1)^5 = (2+0.1)^5$ *Now do yourself.*

Question # 3

Expand and simplify the following:

(i) $\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4$ (ii) $\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5$

(iii) $(2+i)^5 - (2-i)^5$ (iv) $\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3$

Solution (i) $\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4$

We take

$$\left(a+\sqrt{2}x\right)^4$$
$$= \binom{4}{0}a^4 + \binom{4}{1}a^3\left(\sqrt{2}x\right)^1 + \binom{4}{2}a^2\left(\sqrt{2}x\right)^2 + \binom{4}{3}a^1\left(\sqrt{2}x\right)^3 + \binom{4}{4}a^0\left(\sqrt{2}x\right)^4$$
$$= (1)a^4 + (4)a^3\left(\sqrt{2}x\right) + (6)a^2\left(2x^2\right) + (4)a\left(2\sqrt{2}x^3\right) + (1)(1)\left(4x^4\right)$$
$$\Rightarrow \left(a+\sqrt{2}x\right)^4 = a^4 + 4\sqrt{2}a^3x + 12a^2x^2 + 8\sqrt{2}\,ax^3 + 4x^4 \quad \ldots\ldots\ldots\ldots (i)$$

Replacing $\sqrt{2}$ by $-\sqrt{2}$ in eq. (i)

$$\left(a-\sqrt{2}x\right)^4 = a^4 + 4\left(-\sqrt{2}\right)a^3x + 12a^2x^2 + 8\left(-\sqrt{2}\right)ax^3 + 4x^4$$
$$= a^4 - 4\sqrt{2}a^3x + 12a^2x^2 - 8\sqrt{2}\,ax^3 + 4x^4 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Adding (i) & (ii)

$$\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4 = 2a^4 + 24a^2x^2 + 8x^4$$

(ii) *Do yourself.*

(iii) Since

$$(2+i)^5 = \binom{5}{0}2^5 + \binom{5}{1}2^{5-1}i + \binom{5}{2}2^{5-2}i^2 + \binom{5}{3}2^{5-3}i^3 + \binom{5}{4}2^{5-4}i^4 + \binom{5}{5}2^{5-5}i^5$$
$$= (1)2^5 + (5)2^4 i + (10)2^3 i^2 + (10)2^2 i^3 + (5)2^1 i^4 + (1)2^0 i^5$$
$$= 32 + 80i + 80i^2 + 40i^3 + 10i^4 + i^5 \quad \text{.......... (i)}$$

Replacing i by $-i$ in eq. (i)

$$(2+i)^5 = 32 + 80(-i) + 80(-i)^2 + 40(-i)^3 + 10(-i)^4 + (-i)^5$$
$$= 32 - 80i + 80i^2 - 40i^3 + 10i^4 - i^5 \quad \text{.......(ii)}$$

Subtracting (i) & (ii)

$$(2+i)^5 - (2-i)^5 = 160i + 80i^3 + 2i^5$$
$$= 160i + 80(-1)\cdot i + 2(-1)^2 \cdot i$$
$$= 160i - 80i + 2i = 82i$$

(iv) $\left(x+\sqrt{x^2-1}\right)^3 + \left(x+\sqrt{x^2-1}\right)^3$

Suppose $t = \sqrt{x^2-1}$ then

$$\left(x+\sqrt{x^2-1}\right)^3 + \left(x+\sqrt{x^2-1}\right)^3 = (x+t)^3 + (x+t)^3$$

$$= \left((x)^3 + 3(x)^2(t) + 3(x)(t)^2 + (t)^3\right) + \left((x)^3 + 3(x)^2(-t) + 3(x)(-t)^2 + (-t)^3\right)$$
$$= x^3 + 3x^2t + 3xt^2 + t^3 + x^3 - 3x^2t + 3xt^2 - t^3$$
$$= 2x^3 + 6xt^2$$
$$= 2x^3 + 6x\left(\sqrt{x^2-1}\right)^2 \qquad \because\ t = \sqrt{x^2-1}$$
$$= 2x^3 + 6x\left(x^2-1\right) = 2x^3 + 6x^3 - 6x = 8x^3 - 6x$$

Question # 4

Expand the following in ascending powers of x:

(i) $(2+x-x^2)^4$ (ii) $(1-x+x^2)^4$ (iii) $(1-x-x^2)^4$

Solution (i) $(2+x-x^2)^4$

Put $t = 2+x$ then

$$(2+x-x^2)^4 = (t-x^2)^4$$
$$= \binom{4}{0}(t)^4 + \binom{4}{1}(t)^3(-x^2) + \binom{4}{2}(t)^2(-x^2)^2 + \binom{4}{3}(t)^1(-x^2)^3 + \binom{4}{4}(t)^0(-x^2)^4$$
$$= (1)(t)^4 - (4)(t)^3(x^2) + (6)(t)^2(x^4) - (4)(t)(x^6) + (1)(1)(x^8)$$
$$= t^4 - 4t^3x^2 + 6t^2x^4 - 4tx^6 + x^8 \quad \text{.................... (i)}$$

Now

$$t^4 = (2+x)^4 = \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4$$
$$= (1)(16) + (4)(8)(x) + (6)(4)(x^2) + (4)(2)(x^3) + (1)(1)(x^4)$$
$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

Also

$$t^3 = (2+x)^3 = (2)^3 + (3)(2)^2(x) + (3)(2)^1(x)^2 + (x)^3$$
$$= 8 + 12x + 6x^2 + x^3$$
$$t^2 = (2+x)^2 = 4 + 4x + x^2$$

Putting values of t^4, t^3, t^2 and t in equation (i)

$$(2+x-x^2)^4 = (16+32x+24x^2+8x^3+x^4) - 4(8+12x+6x^2+x^3)x^2$$
$$+6(4+4x+x^2)x^4 - 4(2+x)x^6 + x^8$$
$$= 16+32x+24x^2+8x^3+x^4-32x^2-48x^3-24x^4-4x^5$$
$$+24x^4+24x^5+6x^6-8x^6+4x^7+x^8$$
$$= 16+32-8x^2-40x^3+x^4+20x^5-2x^6-4x^7-x^8$$

(ii) Suppose $t = 1-x$ *Do yourself*

(iii) Suppose $t = 1-x$ *Do yourself*

Question # 5

Expand the following in descending powers of x:

(i) $\left(x^2+x-1\right)^3$ (ii) $\left(x-1-\frac{1}{x}\right)^3$

Solution (i) Suppose $t = x-1$ *Do yourself*

(ii) $\left(x-1-\frac{1}{x}\right)^3$

Suppose $t = x-1$ then

$$\left(t-\frac{1}{x}\right)^3 = (t)^3 + 3(t)^2\left(-\frac{1}{x}\right) + 3(t)\left(-\frac{1}{x}\right)^2 + \left(-\frac{1}{x}\right)^3$$
$$= t^3 - 3t^2\cdot\frac{1}{x} + 3t\cdot\frac{1}{x^2} - \frac{1}{x^3} \quad .... \text{(i)}$$

Now

$$t^3 = \left(x-1\right)^3 = (x)^3 + 3(x)^2(-1) + 3(x)(-1)^2 + (-1)^3$$
$$= x^3 - 3x^2 + 3x - 1$$
$$t^2 = \left(x-1\right)^2 = x^2 - 2x + 1$$

Putting values of t^3, t^2 and t in equation (i)

$$\left(x-1-\frac{1}{x}\right)^3 = (x^3-3x^2+3x-1) - 3(x^2-2x+1)\cdot\frac{1}{x} + 3(x-1)\cdot\frac{1}{x^2} - \frac{1}{x^3}$$
$$= x^3 - 3x^2 + 3x - 1 - 3x + 6 - 3\frac{1}{x} + 3\frac{1}{x} - 3\frac{1}{x^2} - \frac{1}{x^3}$$
$$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

Question # 6

Find the term involving:

(i) x^4 in the expansion of $(3-2x)^7$ (ii) x^{-2} in the expansion of $\left(x-\dfrac{2}{x^2}\right)^{13}$

(iii) a^4 in the expansion of $\left(\dfrac{2}{x}-a\right)^9$ (iv) y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$

Solution (i) Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

Here $a=3$, $x=-2x$, $n=7$ so

$$T_{r+1}=\binom{7}{r}(3)^{7-r}(-2x)^r=\binom{7}{r}(3)^{7-r}(-2)^r(x)^r$$

For term involving x^4 we must have

$$x^r=x^4 \Rightarrow r=4$$

So

$$T_{4+1}=\binom{7}{4}(3)^{7-4}(-2)^4(x)^4$$

$$\Rightarrow T_5=(35)(3)^3(-2)^4(x)^4=(35)(27)(16)(x)^4$$

$$=15120x^4$$

(ii) Since $T_{r+1}=\binom{n}{r}a^{n-r}x^r$

Here $a=x$, $x=-\dfrac{2}{x^2}$, $n=13$ so

$$T_{r+1}=\binom{13}{r}(x)^{13-r}\left(-\frac{2}{x^2}\right)^r=\binom{13}{r}(x)^{13-r}(-2)^r(x)^{-2r}$$

$$=\binom{13}{r}(x)^{13-r-2r}(-2)^r=\binom{13}{r}(x)^{13-3r}(-2)^r$$

For term involving x^{-2} we must have

$$x^{13-3r}=x^{-2} \Rightarrow 13-3r=-2 \Rightarrow -3r=-2-13$$

$$\Rightarrow -3r=-15 \Rightarrow r=5$$

So

$$T_{5+1}=\binom{13}{5}(x)^{13-3(5)}(-2)^5$$

$$\Rightarrow T_6=(1287)(x)^{13-15}(-32)=-41184x^{-2}$$

(iii) Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

Here $a=\frac{2}{x},\ x=-a,\ n=9$ so

$$T_{r+1}=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r(a)^r$$

For term involving a^4 we must have

$$a^r=a^4 \Rightarrow r=4$$

So $$T_{4+1}=\binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-1)^4(a)^4$$

$$\Rightarrow T_5=(126)\left(\frac{2}{x}\right)^5(1)a^4=(126)\left(\frac{32}{x^5}\right)a^4=4032\frac{a^4}{x^5}$$

(iv) Here $a=x,\ x=-\sqrt{y},\ n=11$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{11}{r}(x)^{11-r}\left(-\sqrt{y}\right)^r=\binom{11}{r}(x)^{11-r}\left(-y^{\frac{1}{2}}\right)^r$$

$$=\binom{11}{r}(x)^{11-r}(-1)^r\left(y^{\frac{r}{2}}\right)$$

For term involving y^3 we must have

$$y^{\frac{r}{2}}=y^3 \Rightarrow \frac{r}{2}=3 \Rightarrow r=6$$

So $$T_{6+1}=\binom{11}{6}(x)^{11-6}(-1)^6\left(y^{\frac{6}{2}}\right)$$

$$\Rightarrow T_7=(462)(x)^5(1)\left(y^3\right)=462x^5y^3$$

Question # 7

Find the coefficient of;

(i) x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$ (ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

Solution (i) Here $a=x^2,\ x=-\frac{3}{2x},\ n=10$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{10}{r}\left(x^2\right)^{10-r}\left(-\frac{3}{2x}\right)^r=\binom{10}{r}(x)^{2(10-r)}(-1)^r\frac{(3)^r}{(2)^r(x)^r}$$

$$=\binom{10}{r}(x)^{20-2r}(-1)^r(3)^r(2)^{-r}(x)^{-r}=\binom{10}{r}(x)^{20-2r-r}(-1)^r(3)^r(2)^{-r}$$

$$=\binom{10}{r}(x)^{20-3r}(-1)^r(3)^r(2)^{-r}$$

For term involving x^5 we must have

$$x^{20-3r}=x^5 \;\Rightarrow\; 20-3r=5 \;\Rightarrow\; -3r=5-20$$

$$\Rightarrow\; -3r=-15 \;\Rightarrow\; r=5$$

So $\quad T_{5+1}=\binom{10}{5}(x)^{20-3(5)}(-1)^5(3)^5(2)^{-5}$

$$\Rightarrow T_6=252(x)^{20-15}(-1)^5(3)^5\frac{1}{2^5}=-252(x)^5(243)\frac{1}{32}$$

$$=-\frac{61236}{32}x^5=-\frac{15309}{8}x^5$$

Hence coefficient of $x^5=-\dfrac{15309}{8}$

(ii) Here $a=x^2$, $x=-\dfrac{1}{x}$, $n=2n$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{2n}{r}(x^2)^{2n-r}\left(-\frac{1}{x}\right)^r=\binom{2n}{r}(x)^{2(2n-r)}(-1)^r\frac{1}{x^r}$$

$$=\binom{2n}{r}(x)^{4n-2r}(-1)^r x^{-r}=\binom{2n}{r}(x)^{4n-2r-r}(-1)^r$$

$$=\binom{2n}{r}(x)^{4n-3r}(-1)^r$$

For term involving x^n we must have

$$x^{4n-3r}=x^n \;\Rightarrow\; 4n-3r=n \;\Rightarrow\; -3r=n-4n$$

$$\Rightarrow\; -3r=-3n \;\Rightarrow\; r=n$$

So $\quad T_{n+1}=\binom{2n}{n}(x)^{4n-3n}(-1)^n$

$$=\frac{(2n)!}{(2n-n)!\cdot n!}(x)^n(-1)^n=\frac{(2n)!}{n!\cdot n!}(x)^n(-1)^n$$

$$=(-1)^n\frac{(2n)!}{(n!)^2}x^n$$

Hence coefficient of $x^n=(-1)^n\dfrac{(2n)!}{(n!)^2}$

Question # 8

Find $6th$ term in the expansion of $\left(x^2-\dfrac{3}{2x}\right)^{10}$

Solution Here $a = x^2$, $x = -\dfrac{3}{2x}$, $n = 10$ and $r+1 = 6 \Rightarrow r = 5$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} (x^2)^{10-5} \left(-\frac{3}{2x}\right)^5$$

$$\Rightarrow T_6 = 252 (x^2)^5 \left(-\frac{3^5}{(2x)^5}\right) = 252 x^{10} \left(-\frac{243}{32x^5}\right)$$

$$= -\frac{61236}{32} x^{10-5} = -\frac{15309}{8} x^5$$

Question # 9

Find the term independent of x in the following expansions..

(i) $\left(x - \dfrac{2}{x}\right)^{10}$ (ii) $\left(\sqrt{x} - \dfrac{1}{2x^2}\right)^{10}$ (iii) $(1+x^2)^3 \left(1 + \dfrac{1}{x^2}\right)^4$

Solution (i) *Do yourself as Q # 9* (ii)

(ii) Here $a = \sqrt{x}$, $x = \dfrac{1}{2x^2}$, $n = 10$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{10}{r} (\sqrt{x})^{10-r} \left(\frac{1}{2x^2}\right)^r = \binom{10}{r} \left(x^{\frac{1}{2}}\right)^{10-r} \left(\frac{1}{2^r x^{2r}}\right)$$

$$= \binom{10}{r} (x)^{\frac{1}{2}(10-r)} \frac{1}{2^r} x^{-2r} = \binom{10}{r} (x)^{5-\frac{r}{2}} \frac{1}{2^r} x^{-2r}$$

$$= \binom{10}{r} (x)^{5-\frac{r}{2}-2r} \frac{1}{2^r} = \binom{10}{r} (x)^{5-\frac{5r}{2}} \frac{1}{2^r}$$

For term independent of x we must have

$$x^{5-\frac{5r}{2}} = x^0 \Rightarrow 5 - \frac{5r}{2} = 0 \Rightarrow -\frac{5r}{2} = -5$$

$$\Rightarrow r = (-5)\left(-\frac{2}{5}\right) \Rightarrow r = 2$$

So $$T_{2+1} = \binom{10}{2} (x)^{5-\frac{5(2)}{2}} \frac{1}{2^2}$$

$$\Rightarrow T_3 = 45\,(x)^{5-5} \frac{1}{4} = 45\, x^0 \frac{1}{4}$$

$$= 45\,(1) \frac{1}{4} = \frac{45}{4}$$

(iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4 = (1+x^2)^3\left(\frac{x^2+1}{x^2}\right)^4$

$$= (1+x^2)^3\frac{(x^2+1)^4}{(x^2)^4} = (1+x^2)^3\frac{(1+x^2)^4}{x^8}$$

$$= x^{-8}(1+x^2)^{3+4} = x^{-8}(1+x^2)^7$$

Now $T_{r+1} = x^{-8}\binom{n}{r}a^{n-r}x^r$

Where $n=7,\ a=1,\ x=x^2$

$$T_{r+1} = x^{-8}\binom{7}{r}(1)^{7-r}(x^2)^r = x^{-8}\binom{7}{r}(1)\ x^{2r}$$

$$= \binom{7}{r}x^{2r-8}$$

For term independent of x we must have

$$x^{2r-8} = x^0 \Rightarrow 2r-8=0 \Rightarrow 2r=8 \Rightarrow r=4$$

So

$$T_{4+1} = \binom{7}{4}x^{2(4)-8}$$

$$\Rightarrow T_5 = 35\ x^{8-8} = 35\ x^0 = 35$$

Question # 10

Determine the middle term in the following expensions:

(i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ (ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ (iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

Solution (i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$

Since $n=12$ is an even so middle terms is $\frac{n+2}{2} = \frac{12+2}{2} = 7$

Therefore $r+1=7 \Rightarrow r=7-1=6$

And $a=\frac{1}{x}$, $x=-\frac{x^2}{2}$ and $n=12$

Now

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

$$\Rightarrow T_{6+1} = \binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$\Rightarrow T_7 = 924 \frac{1}{x^6} \frac{x^{12}}{64} = \frac{924}{64} x^{12-6}$$
$$= \frac{231}{16} x^6$$

Thus the middle terms of the given expansion is $\frac{231}{16} x^6$.

(ii) Since $n = 11$ is odd so the middle terms are $\frac{n+1}{2} = \frac{11+1}{2} = 6$ and

$$\frac{n+3}{2} = \frac{11+3}{2} = 7$$

So for first middle term

$$a = \frac{3}{2}x, \quad x = -\frac{1}{3x}, \quad n = 11 \text{ and } r+1 = 6 \quad \Rightarrow \quad r = 5$$

Now

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r \qquad \Rightarrow T_{5+1} = \binom{11}{5}\left(\frac{3}{2}x\right)^{11-5}\left(-\frac{1}{3x}\right)^5$$

Now simplify yourself.

Now for second middle term

$$r+1 = 7 \quad \Rightarrow \quad r = 6$$

so $$T_{6+1} = \binom{11}{6}\left(\frac{3}{2}x\right)^{11-6}\left(-\frac{1}{3x}\right)^6$$ *Now simplify yourself.*

(iii) Since $n = 2m+1$ is odd so there are two middle terms

First middle term = $\frac{n+1}{2} = \frac{2m+1+1}{2} = \frac{2m+2}{2} = m+1$

Second middle terms = $\frac{n+3}{2} = \frac{2m+1+3}{2} = \frac{2m+4}{2} = m+2$

Here $a = 2x$, $\quad x = -\frac{1}{2x}$ and $\quad n = 2m+1$

For first middle term $r+1 = m+1 \Rightarrow \; r = m$.

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\Rightarrow T_{m+1} = \binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^m = \frac{(2m+1)!}{(2m+1-m)!\cdot m!}(2x)^{m+1}\left(-\frac{1}{2x}\right)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m\left(\frac{1}{2}\right)^m\left(\frac{1}{x}\right)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m(2)^{-m}(x)^{-m}$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1-m}(x)^{m+1-m}(-1)^m = \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^1(x)^1(-1)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}2x(-1)^m$$

For second middle term

$$r+1 = m+2 \quad \Rightarrow \quad r = m+2-1 \quad \Rightarrow \quad r = m+1$$

As $\quad T_{r+1} = \binom{n}{r}a^{n-r}x^r$

$$\Rightarrow \quad T_{m+1+1} = \binom{2m+1}{m+1}(2x)^{(2m+1)-(m+1)}\left(-\frac{1}{2x}\right)^{m+1}$$

Now simplify yourself

Question # 11

Find $(2n+1)th$ term of the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

Solution Here $a = x, \quad x = -\frac{1}{2x}$,

Number of term from the end = $2n + 1$

To make it from beginning we take $a = -\frac{1}{2x}$, $x = x$ and $r+1 = 2n+1$

$\Rightarrow \quad r = 2n$

As $\quad T_{r+1} = \binom{n}{r}a^{n-r}x^r$

$$\Rightarrow \quad T_{2n+1} = \binom{3n}{2n}\left(-\frac{1}{2x}\right)^{3n-2n}(x)^{2n} = \frac{(3n)!}{(3n-2n)!\cdot(2n)!}\left(-\frac{1}{2x}\right)^n x^{2n}$$

$$= \frac{(3n)!}{(n)!\cdot(2n)!}(-1)^n\frac{1}{2^n\cdot x^n}x^{2n} = \frac{(3n)!}{n!\cdot(2n)!}(-1)^n\frac{1}{2^n}x^{2n-n}$$

$$= \frac{(-1)^n}{2^n}\frac{(3n)!}{n!\cdot(2n)!}x^n \quad \text{Answer}$$

Note: If there are p term in some expansion and qth term is from the end then the term from the beginning will be $= p-q+1$.

So in above you can use term from the end = $(3n+1)-(2n+1)+1 = n+1$

Question # 12

Show that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5\ldots(2n-1)}{n!}2^n x^n$

Solution Since $2n$ is even so the middle term is $\frac{2n+2}{2} = n+1$ and

$$a = 1, \quad x = x, \quad n = 2n, \quad r+1 = n+1 \Rightarrow \quad r = n$$

Now $\quad T_{r+1} = \binom{n}{r}a^{n-r}x^r$

$$\Rightarrow\ T_{n+1}=\binom{2n}{n}(1)^{2n-n}x^n$$

$$\Rightarrow\ T_{n+1}=\frac{(2n)!}{(2n-n)!\cdot n!}(1)^n x^n=\frac{(2n)!}{n!\cdot n!}x^n$$

$$=\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{n!\cdot n!}x^n$$

$$=\frac{[2n\,(2n-2)(2n-4)\cdot\ldots\cdot 4\cdot 2][(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1]}{n!\cdot n!}x^n$$

$$=\frac{2^n[n\,(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1][(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1]}{n!\cdot n!}x^n$$

$$=\frac{2^n\,n!\,[(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1]}{n!\cdot n!}x^n$$

$$=\frac{2^n\,[1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)]}{n!}x^n$$

$$=\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{n!}2^n x^n$$

Question # 13

Show that:

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}=2^{n-1}$$

Solution Consider

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+\binom{n}{5}x^5+\ldots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n$$

………. (i)

Put $x=1$

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}(1)+\binom{n}{2}(1)^2+\binom{n}{3}(1)^3+\binom{n}{4}(1)^4+\binom{n}{5}(1)^5+\ldots+\binom{n}{n-1}(1)^{n-1}+\binom{n}{n}(1)^n$$

$$\Rightarrow\ 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\binom{n}{5}+\ldots+\binom{n}{n-1}+\binom{n}{n}$$

$$\Rightarrow\ 2^n=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

………. (ii)

Now put $x=-1$ in equation (i)

$$(1-1)^n=\binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+\binom{n}{4}(-1)^4+\binom{n}{5}(-1)^5+\ldots$$

$$\ldots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n$$

If we consider n is even then

$$\Rightarrow\ (0)^n=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\ldots-\binom{n}{n-1}+\binom{n}{n}$$

$$\Rightarrow\quad 0=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]-\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\quad \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]$$

Using it in equation (ii)

$$2^n=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\quad 2^n=2\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\quad \frac{2^n}{2}=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\quad 2^{n-1}=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\quad \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}=2^{n-1}$$

Question # 14

Show that:

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\ldots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$$

Solution

$$\text{L.H.S}=\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\ldots+\frac{1}{n+1}\binom{n}{n}$$

$$=\left[\binom{n}{0}+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{1}{n+1}\binom{n}{n}\right]$$

$$=\frac{n+1}{n+1}\left[1+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{1}{n+1}\cdot 1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\frac{1}{2}\left(\frac{(n+1)\,n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{(n+1)\,n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{(n+1)\,n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{n+1}{n+1}\cdot 1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n-1)!\cdot 2\cdot 1!}\right)+\left(\frac{(n+1)!}{(n-2)!\cdot 3\cdot 2!}\right)+\left(\frac{(n+1)!}{(n-3)!\cdot 4\cdot 3!}\right)+\ldots+1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n+1-2)!\cdot 2!}\right)+\left(\frac{(n+1)!}{(n+1-3)!\cdot 3!}\right)+\left(\frac{(n+1)!}{(n+1-4)!\cdot 4!}\right)+\ldots+1\right]$$

$$=\frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$=\frac{1}{n+1}\left[-1+1+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+2^{n+1}\right]$$

$$= \frac{2^{n+1}-1}{n+1} = \text{R.H.S}$$

Remember

$\binom{n+1}{0}=1$, $\binom{n+1}{1}=n+1$ and $\binom{n+1}{n+1}=1$
