

Principle of Mathematical Induction

A given statement $S(n)$ is true for each positive integer n if two below conditions hold

Condition I: $S(1)$ is true i.e. $S(n)$ is true for $n = 1$ and

Condition II: $S(k + 1)$ is true whenever $S(k)$ is true for any positive integer k ,

Then $S(n)$ is true for all positive integers.

Use mathematical induction to prove the following formulae for every positive integer n

Question # 1

$$1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$$

Solution Suppose $S(n): 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$

Put $n = 1$

$$S(1): 1 = 1(2(1) - 1) \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 + 5 + 9 + \dots + (4k - 3) = k(2k - 1) \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$\begin{aligned} S(k + 1): 1 + 5 + 9 + \dots + (4(k + 1) - 3) &= (k + 1)(2(k + 1) - 1) \\ \Rightarrow 1 + 5 + 9 + \dots + (4k + 1) &= (k + 1)(2k + 2 - 1) \\ &= (k + 1)(2k + 1) \\ &= 2k^2 + 2k + k + 1 \\ &= 2k^2 + 3k + 1 \end{aligned}$$

Adding $4k + 1$ on both sides of equation (i)

$$\begin{aligned} 1 + 5 + 9 + \dots + (4k - 3) + (4k + 1) &= k(2k - 1) + 4k + 1 \\ \Rightarrow 1 + 5 + 9 + \dots + (4k + 1) &= 2k^2 - k + 4k + 1 \\ &= 2k^2 + 3k + 1 \end{aligned}$$

Thus $S(k + 1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 2

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Solution Suppose

$$S(n): 1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Put $n = 1$

$$S(1): 1 = (1)^2 \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 + 3 + 5 + \dots + (2k - 1) = k^2 \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

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$$S(k+1): 1+3+5+\dots+(2(k+1)-1) = (k+1)^2$$

$$\Rightarrow 1+3+5+\dots+(2k+1) = (k+1)^2$$

Adding $2k+1$ on both sides of equation (i)

$$1+3+5+\dots+(2k-1)+(2k+1) = k^2 + 2k + 1$$

$$\Rightarrow 1+3+5+\dots+(2k+1) = (k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 3

$$1+4+7+\dots+(3n-2) = \frac{n(3n-1)}{2}$$

Solution Suppose

$$S(n): 1+4+7+\dots+(3n-2) = \frac{n(3n-1)}{2}$$

Put $n = 1$

$$S(1): 1 = \frac{1(3(1)-1)}{2} \Rightarrow 1 = \frac{2}{2} \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1+4+7+\dots+(3k-2) = \frac{k(3k-1)}{2} \dots\dots\dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1+4+7+\dots+(3(k+1)-2) = \frac{(k+1)(3(k+1)-1)}{2}$$

$$\begin{aligned} \Rightarrow 1+4+7+\dots+(3k+1) &= \frac{(k+1)(3k+3-1)}{2} \\ &= \frac{(k+1)(3k+2)}{2} \end{aligned}$$

Adding $3k+1$ on both sides of equation (i)

$$1+4+7+\dots+(3k-2)+(3k+1) = \frac{k(3k-1)}{2} + 3k+1$$

$$\begin{aligned} \Rightarrow 1+4+7+\dots+(3k+1) &= \frac{k(3k-1)+2(3k+1)}{2} \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{3k^2 + 3k + 2k + 2}{2} \\ &= \frac{3k(k+1) + 2(k+1)}{2} \end{aligned}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 4

$$1 + 2 + 4 + \dots + 2^n = 2^n - 1$$

Solution Suppose

$$S(n): 1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$$

Put $n = 1$

$$S(1): 1 = 2^1 - 1 \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 + 2 + 4 + \dots + 2^{k-1} = 2^k - 1 \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1 + 2 + 4 + \dots + 2^{k+1-1} = 2^{k+1} - 1$$

$$\Rightarrow 1 + 2 + 4 + \dots + 2^k = 2^{k+1} - 1$$

adding 2^k on both sides of equation (i)

$$1 + 2 + 4 + \dots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$$

$$\Rightarrow 1 + 2 + 4 + \dots + 2^k = 2(2^k) - 1 \qquad \because 2^k + 2^k = 2(2^k)$$

$$= 2^{k+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 5

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = 2 \left[1 - \frac{1}{2^n} \right]$$

Solution Suppose

$$S(n): 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = 2 \left(1 - \frac{1}{2^n} \right)$$

Put $n = 1$

$$S(1): 1 = 2 \left(1 - \frac{1}{2^1} \right) \Rightarrow 1 = 2 \left(\frac{1}{2} \right) \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k-1}} = 2 \left(1 - \frac{1}{2^k} \right) \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k+1-1}} = 2 \left(1 - \frac{1}{2^{k+1}} \right)$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k} = 2 - \frac{2}{2^{k+1}}$$

$$= 2 - \frac{2}{2^k \cdot 2}$$

$$= 2 - \frac{1}{2^k}$$

Adding $\frac{1}{2^k}$ on both sides of equation (i)

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1-1}} = 2 \left(1 - \frac{1}{2^k} \right) + \frac{1}{2^k}$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k+1-1}} = 2 - \frac{2}{2^k} + \frac{1}{2^k}$$

$$= 2 - \frac{1}{2^k} (2 - 1)$$

$$= 2 - \frac{1}{2^k} (1) = 2 - \frac{1}{2^k}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 6

Do yourself as Question # 1

Question # 7

$$2 + 6 + 18 + \dots + 2 \times 3^{n-1} = 3^n - 1$$

Solution Suppose

$$S(n): 2 + 6 + 18 + \dots + 2 \times 3^{n-1} = 3^n - 1$$

Put $n = 1$

$$S(1): 2 = 3^1 - 1 \Rightarrow 2 = 2$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 2 + 6 + 18 + \dots + 2 \times 3^{k-1} = 3^k - 1 \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 2 + 6 + 18 + \dots + 2 \times 3^{k+1-1} = 3^{k+1} - 1$$

Adding 2×3^k on both sides of equation (i)

$$2 + 6 + 18 + \dots + 2 \times 3^{k-1} + 2 \times 3^k = 3^k - 1 + 2 \times 3^k$$

$$\Rightarrow 2 + 6 + 18 + \dots + 2 \times 3^{k+1-1} = 3(3^k) - 1 \qquad \because 3^k + 2 \times 3^k = 3(3^k)$$

$$= 3^{k+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 8

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}$$

Solution Suppose

$$S(n): 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}$$

Put $n = 1$

$$S(1): 1 \times 3 = \frac{1(1+1)(4(1)+5)}{6} \Rightarrow 3 = \frac{(2)(9)}{6} \Rightarrow 3 = 3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + k \times (2k + 1) = \frac{k(k+1)(4k+5)}{6} \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + (k+1) \times (2(k+1)+1) = \frac{(k+1)(k+1+1)(4(k+1)+5)}{6}$$

$$\Rightarrow 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + (k+1) \times (2k+3) = \frac{(k+1)(k+2)(4k+9)}{6}$$

Adding $(k+1) \times (2k+3)$ on both sides of equation (i)

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + k \times (2k+1) + (k+1) \times (2k+3) = \frac{k(k+1)(4k+5)}{6} + (k+1) \times (2k+3)$$

$$\Rightarrow 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + (k+1) \times (2k+3) = (k+1) \left(\frac{k(4k+5)}{6} + (2k+3) \right)$$

$$= (k+1) \left(\frac{k(4k+5) + 6(2k+3)}{6} \right)$$

$$= (k+1) \left(\frac{4k^2 + 5k + 12k + 18}{6} \right)$$

$$= (k+1) \left(\frac{4k^2 + 17k + 18}{6} \right)$$

$$= (k+1) \left(\frac{4k^2 + 17k + 18}{6} \right)$$

$$= (k+1) \left(\frac{4k^2 + 8k + 9k + 18}{6} \right)$$

$$= (k+1) \left(\frac{4k(k+2) + 9(k+2)}{6} \right)$$

$$= (k+1) \left(\frac{(k+2)(4k+9)}{6} \right)$$

$$= \frac{(k+1)(k+2)(4k+9)}{6}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 9

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$$

Solution

Do yourself as Question # 8

Question # 10

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + (2n-1) \times 2n = \frac{n(n+1)(4n-1)}{3}$$

Solution Do yourself as Question # 8

Question # 11

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

Solution Suppose

$$S(n): \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

Put $n = 1$

$$S(1): \frac{1}{1 \times 2} = 1 - \frac{1}{1+1} \Rightarrow \frac{1}{2} = 1 - \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1} \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{(k+1)(k+1+1)} = 1 - \frac{1}{k+1+1}$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+2}$$

Adding $\frac{1}{(k+1)(k+2)}$ on both sides of equation (i)

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\Rightarrow \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+1} \left(1 - \frac{1}{k+2} \right)$$

$$= 1 - \frac{1}{k+1} \left(\frac{k+2-1}{k+2} \right)$$

$$= 1 - \frac{1}{k+1} \left(\frac{k+1}{k+2} \right)$$

$$= 1 - \frac{1}{k+2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 12

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Solution Suppose

$$S(n): \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Put $n = 1$

$$S(1): \frac{1}{1 \times 3} = \frac{1}{2(1)+1} \Rightarrow \frac{1}{3} = \frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1}$$

$$\Rightarrow \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

Adding $\frac{1}{(2k+1)(2k+3)}$ on both sides of equation (i)

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$\Rightarrow \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{1}{2k+1} \left(k + \frac{1}{(2k+3)} \right)$$

$$= \frac{1}{2k+1} \left(\frac{k(2k+3)+1}{2k+3} \right)$$

$$= \frac{1}{2k+1} \left(\frac{2k^2+3k+1}{2k+3} \right)$$

$$= \frac{1}{2k+1} \left(\frac{2k^2+2k+k+1}{2k+3} \right)$$

$$= \frac{1}{2k+1} \left(\frac{2k(k+1)+1(k+1)}{2k+3} \right)$$

$$= \frac{1}{2k+1} \left(\frac{(2k+1)(k+1)}{2k+3} \right)$$

$$= \left(\frac{k+1}{2k+3} \right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 13

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

Solution

Do yourself as Question # 12

Question # 14

$$r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r} \quad (r \neq 1)$$

Solution Suppose $S(n): r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r}$

Put $n = 1$

$$S(1): r = \frac{r(1-r^1)}{1-r} \Rightarrow r = r$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): r + r^2 + r^3 + \dots + r^k = \frac{r(1-r^k)}{1-r} \dots \dots \dots \text{(i)}$$

The statement for $n = k + 1$ becomes

$$S(k+1): r + r^2 + r^3 + \dots + r^{k+1} = \frac{r(1-r^{k+1})}{1-r}$$

Adding r^{k+1} on both sides of equation (i)

$$\begin{aligned} r + r^2 + r^3 + \dots + r^k + r^{k+1} &= \frac{r(1-r^k)}{1-r} + r^{k+1} \\ \Rightarrow r + r^2 + r^3 + \dots + r^{k+1} &= \frac{r(1-r^k) + r^{k+1}(1-r)}{1-r} \\ &= \frac{r - r^{k+1} + r^{k+1} - r^{k+2}}{1-r} \\ &= \frac{r - r^{k+2}}{1-r} \\ &= \frac{r(1-r^{k+1})}{1-r} \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 15

$$a + (a + d) + (a + 2d) + \dots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

Solution Suppose

$$S(n): a + (a + d) + (a + 2d) + \dots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

Put $n = 1$

$$S(1): a = \frac{1}{2}[2a + (1-1)d] \Rightarrow a = \frac{1}{2}[2a + (0)d] \Rightarrow a = \frac{1}{2}[2a] = a$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): a + (a + d) + (a + 2d) + \dots + [a + (k - 1)d] = \frac{k}{2} [2a + (k - 1)d] \dots\dots\dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k + 1): a + (a + d) + (a + 2d) + \dots + [a + (k + 1 - 1)d] = \frac{k + 1}{2} [2a + (k + 1 - 1)d]$$

$$\Rightarrow a + (a + d) + (a + 2d) + \dots + [a + kd] = \frac{k + 1}{2} [2a + kd]$$

Adding $a + kd$ on both sides of equation (i)

$$a + (a + d) + (a + 2d) + \dots + [a + (k - 1)d] + [a + kd] = \frac{k}{2} [2a + (k - 1)d] + [a + kd]$$

$$\begin{aligned} \Rightarrow a + (a + d) + (a + 2d) + \dots + [a + kd] &= \frac{k}{2} [2a + kd - d] + [a + kd] \\ &= \frac{k[2a + kd - d] + 2[a + kd]}{2} \\ &= \frac{2ak + k^2d - kd + 2a + 2kd}{2} \\ &= \frac{2ak + k^2d + kd + 2a}{2} \\ &= \frac{2ak + 2a + k^2d + kd}{2} \\ &= \frac{2a(k + 1) + kd(k + 1)}{2} \\ &= \frac{(k + 1)(2a + kd)}{2} \\ &= \frac{k + 1}{2} [2a + kd] \end{aligned}$$

Thus $S(k + 1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 16

$$1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + n \cdot \underline{n} = \underline{n + 1} - 1$$

Solution

Suppose $S(n): 1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + n \cdot \underline{n} = \underline{n + 1} - 1$

Put $n = 1$

$$S(1): 1 \cdot \underline{1} = \underline{1 + 1} - 1 \Rightarrow 1 = \underline{2} - 1 \Rightarrow 1 = 2 - 1 \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + k \cdot \underline{k} = \underline{k + 1} - 1 \dots\dots\dots (i)$$

The statement for $n = k + 1$ becomes

$$\begin{aligned} S(k + 1): 1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + (k + 1) \cdot \underline{k + 1} &= \underline{k + 1 + 1} - 1 \\ \Rightarrow 1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + (k + 1) \cdot \underline{k + 1} &= \underline{k + 2} - 1 \end{aligned}$$

Adding $(k + 1) \cdot \underline{k + 1}$ on both sides of equation (i)

$$1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + k \cdot \underline{k} + (k + 1) \cdot \underline{k + 1} = \underline{k + 1} - 1 + (k + 1) \cdot \underline{k + 1}$$

$$\begin{aligned} \Rightarrow 1 \cdot \underline{1} + 2 \cdot \underline{2} + 3 \cdot \underline{3} + \dots + (k+1) \cdot \underline{k+1} &= \underline{k+1} + \underline{k+1(k+1)} - 1 \\ &= \underline{k+1} (1+k+1) - 1 \\ &= \underline{k+1} (k+2) - 1 \\ &= (k+2) \underline{k+1} - 1 \\ &= \underline{k+2} - 1 \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 17

$a_n = a_1 + (n-1)d$ When, $a_1, a_1 + d, a_1 + 2d, \dots$ from an A.P.

Solution

Suppose $S(n): a_n = a_1 + (n-1)d$

Put $n = 1$

$S(1): a_1 = a_1 + (1-1)d \Rightarrow a_1 = a_1 + 0d = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$S(k): a_k = a_1 + (k-1)d \dots \dots \dots$ (i)

The statement for $n = k + 1$ becomes

$S(k+1): a_{k+1} = a_1 + (k+1-1)d$
 $= a_1 + (k)d$

Adding d on both sides of equation (i)

$$\begin{aligned} a_k + d &= a_1 + (k-1)d + d && \because a_2 = a_1 + d \\ \Rightarrow a_{k+1} &= a_1 + (k-1+1)d && a_3 = a_2 + d \\ \Rightarrow a_{k+1} &= a_1 + (k)d && \therefore a_{k+1} = a_k + d \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 18

$a_n = a_1 r^{n-1}$ When, $a_1, a_1 r, a_1 r^2, \dots$ from an G.P.

Solution Suppose $S(n): a_n = a_1 r^{n-1}$

Put $n = 1$

$S(1): a_1 = a_1 r^{1-1} \Rightarrow a_1 = a_1 r^0 = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$S(k): a_k = a_1 r^{k-1} \dots \dots \dots$ (i)

The statement for $n = k + 1$ becomes

$S(k+1): a_{k+1} = a_1 r^{k+1-1}$
 $= a_1 r^k$

Multiplying r on both sides of equation (i)

$$\begin{aligned} a_k \cdot r &= a_1 r^{k-1} \cdot r^1 && \because a_2 = a_1 r \\ \Rightarrow a_{k+1} &= a_1 r^{k-1+1} && a_3 = a_2 r \\ &&& \therefore a_{k+1} = a_k r \end{aligned}$$

$$\Rightarrow a_{k+1} = a_1 r^k$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 19

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2 - 1)}{3}$$

Solution

Suppose $S(n): 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2 - 1)}{3}$

Put $n = 1$

$$S(1): 1^2 = \frac{1(4(1)^2 - 1)}{3} \Rightarrow 1 = \frac{1(4-1)}{3} \Rightarrow 1 = \frac{3}{3} = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(4k^2 - 1)}{3} \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$\begin{aligned} S(k+1): 1^2 + 3^2 + 5^2 + \dots + (2(k+1)-1)^2 &= \frac{(k+1)(4(k+1)^2 - 1)}{3} \\ \Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 &= \frac{(k+1)(4(k^2 + 2k + 1) - 1)}{3} \\ &= \frac{(k+1)(4k^2 + 8k + 4 - 1)}{3} \\ &= \frac{(k+1)(4k^2 + 8k + 3)}{3} \\ &= \frac{4k^3 + 8k^2 + 3k + 4k^2 + 8k + 3}{3} \\ &= \frac{4k^3 + 12k^2 + 11k + 3}{3} \end{aligned}$$

Adding $(2k+1)^2$ on both sides of equation (i)

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 &= \frac{k(4k^2 - 1)}{3} + (2k+1)^2 \\ \Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 &= \frac{k(4k^2 - 1) + 3(2k+1)^2}{3} \\ &= \frac{k(4k^2 - 1) + 3(4k^2 + 4k + 1)}{3} \\ &= \frac{4k^3 - k + 12k^2 + 12k + 3}{3} \end{aligned}$$

$$= \frac{4k^3 + 12k^2 + 11k + 3}{3}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 20

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$$

Solution

Suppose $S(n): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$

Put $n = 1$

L.H.S = $\binom{3}{3} = 1$

R.H.S = $\binom{1+3}{4} = \binom{4}{4} = 1$

L.H.S = R.H.S

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+2}{3} = \binom{k+3}{4} \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+1+2}{3} = \binom{k+1+3}{4}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+3}{3} = \binom{k+4}{4}$$

Adding $\binom{k+3}{3}$ on both sides of equation (i)

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+2}{3} + \binom{k+3}{3} = \binom{k+3}{4} + \binom{k+3}{3}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+3}{3} = \binom{k+3+1}{4}$$

$$\because \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+3}{3} = \binom{k+4}{4}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 21

Prove by mathematical induction that for all positive integral values of n .

- (i) $n^2 + n$ is divisible by 2 (ii) $5^n - 2^n$ is divisible by 3
 (iii) $5^n - 1$ is divisible by 4 (iv) $8 \times 10^n - 2$ is divisible by 6
 (v) $n^3 - n$ is divisible by 6

Solution

(i) Suppose $S(n): n^2 + n$

Put $n = 1$

$$S(1): 1^2 + 1 = 2$$

$S(1)$ is clearly divisible by 2, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): k^2 + k$$

Then there exists quotient Q such that

$$k^2 + k = 2Q$$

The statement for $n = k + 1$

$$S(k+1): (k+1)^2 + k + 1$$

$$= k^2 + 2k + 1 + k + 1$$

$$= k^2 + k + 2k + 2$$

$$= 2Q + 2k + 2$$

$$= 2(Q + k + 1)$$

$$\therefore k^2 + k = 2Q$$

$$\begin{array}{r} Q \\ 2 \overline{)k^2 + k} \\ \underline{k^2 + k} \\ 0 \end{array}$$

Clearly $S(k+1)$ is divisible by 2.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

(ii) Suppose $S(n): 5^n - 2^n$

Put $n = 1$

$$S(1): 5^1 - 2^1 = 3$$

$S(1)$ is clearly divisible by 3, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): 5^k - 2^k$$

Then there exists quotient Q such that

$$5^k - 2^k = 3Q$$

The statement for $n = k + 1$

$$S(k+1): 5^{k+1} - 2^{k+1}$$

$$= 5 \cdot 5^k - 2 \cdot 2^k$$

$$= 5 \cdot 5^k - 5 \cdot 2^k + 5 \cdot 2^k - 2 \cdot 2^k$$

$$= 5(5^k - 2^k) + 2^k(5 - 2)$$

$$= 5(3Q) + 2^k \cdot 3$$

$$\therefore 5^k - 2^k = 3Q$$

$$= 3(5Q + 2^k)$$

Clearly $S(k+1)$ is divisible by 3.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement



is true for $\forall n \in \mathbb{Z}^+$.

(iii) Same as Question # 21 (ii)

$$\begin{aligned} \text{Hint: } S(k+1) &: 5^{k+1} - 1 \\ &= 5 \cdot 5^k - 1 = 5 \cdot 5^k - 5 + 5 - 1 \\ &= 5(5^k - 1) + 4 = 5(4Q) - 4 \qquad \because 5^k - 1 = 4Q \end{aligned}$$

(iv) Suppose $S(n): 8 \times 10^n - 2$

Put $n = 1$

$$S(1): 8 \times 10^1 - 2 = 80 - 2 = 78 = 6 \times 13$$

$S(1)$ is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): 8 \times 10^k - 2$$

Then there exists quotient Q such that

$$8 \times 10^k - 2 = 6Q$$

The statement for $n = k + 1$

$$S(k+1): 8 \times 10^{k+1} - 2$$

$$= 8 \times 10 \cdot 10^k - 2$$

$$= 8 \times 10 \cdot 10^k - 2 \cdot 10 + 2 \cdot 10 - 2 \qquad \text{-ing \& +ing } 2 \cdot 10$$

$$= 10(8 \times 10^k - 2) + 20 - 2$$

$$= 10(6Q) + 18$$

$$\because 8 \times 10^k - 2 = 6Q$$

$$= 6(10Q + 3)$$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

(v) Suppose $S(n): n^3 - n$

Put $n = 1$

$$S(1): 1^3 - 1 = 0$$

$S(1)$ i.e. 0 is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): k^3 - k$$

Then there exists quotient Q such that

$$k^3 - k = 6Q$$

The statement for $n = k + 1$

$$S(k+1): (k+1)^3 - (k+1)$$

$$= k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= k^3 + 3k^2 + 3k - k$$

$$= (k^3 - k) + 3(k^2 + k)$$

$$= 6Q + 3(2Q')$$

$$= 6Q + 6Q'$$

Since $n^2 + n$ is divisible by 2

Therefore $n^2 + n = 2Q'$

Or $k^2 + k = 2Q'$



Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

Question # 22

$$\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2} \left(1 - \frac{1}{3^n} \right)$$

Solution Suppose $S(n): \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2} \left(1 - \frac{1}{3^n} \right)$

Put $n = 1$

$$S(1): \frac{1}{3} = \frac{1}{2} \left(1 - \frac{1}{3^1} \right) \Rightarrow \frac{1}{3} = \frac{1}{2} \left(\frac{2}{3} \right) \Rightarrow \frac{1}{3} = \frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k} = \frac{1}{2} \left(1 - \frac{1}{3^k} \right) \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{k+1}} = \frac{1}{2} \left(1 - \frac{1}{3^{k+1}} \right)$$

Adding $\frac{1}{3^k}$ on both sides of equation (i)

$$\begin{aligned} \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k} + \frac{1}{3^{k+1}} &= \frac{1}{2} \left(1 - \frac{1}{3^k} \right) + \frac{1}{3^{k+1}} \\ \Rightarrow \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{k+1}} &= \frac{1}{2} - \frac{1}{2 \cdot 3^k} + \frac{1}{3 \cdot 3^k} \\ &= \frac{1}{2} - \frac{1}{3^k} \left(\frac{1}{2} - \frac{1}{3} \right) \\ &= \frac{1}{2} - \frac{1}{3^k} \left(\frac{3-2}{6} \right) = \frac{1}{2} - \frac{1}{3^k} \left(\frac{1}{6} \right) \\ &= \frac{1}{2} \left(1 - \frac{1}{3^k} \left(\frac{1}{3} \right) \right) \\ &= \frac{1}{2} \left(1 - \frac{1}{3^{k+1}} \right) \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 23

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

Solution Suppose

$$S(n): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

Put $n = 1$

$$S(1): 1^2 = \frac{(-1)^{1-1} \cdot 1(1+1)}{2} \Rightarrow 1 = \frac{(-1)^0 \cdot 2}{2} \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} \cdot k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k+1-1} \cdot (k+1)^2 = \frac{(-1)^{k+1-1} \cdot (k+1)(k+1+1)}{2}$$

$$\Rightarrow 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^k \cdot (k+1)^2 = \frac{(-1)^k \cdot (k+1)(k+2)}{2}$$

Adding $(-1)^k \cdot (k+1)^2$ on both sides of equation (i)

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} \cdot k^2 + (-1)^k \cdot (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} + (-1)^k \cdot (k+1)^2$$

$$\Rightarrow 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^k \cdot (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1) + 2(-1)^k \cdot (k+1)^2}{2}$$

$$= \frac{(-1)^k (k+1) [(-1)^{-1} k + 2(k+1)]}{2}$$

$$= \frac{(-1)^k (k+1) [-k + 2k + 2]}{2}$$

$$= \frac{(-1)^k (k+1)(k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 24

$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2 (2n^2 - 1)$$

Solution Suppose $S(n): 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2 (2n^2 - 1)$

Put $n = 1$

$$S(1): 1^3 = 1^2 (2(1)^2 - 1) \Rightarrow 1 = 1(2 - 1) \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1^3 + 3^3 + 5^3 + \dots + (2k-1)^3 = k^2 (2k^2 - 1) \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$\begin{aligned}
 S(k+1): 1^3 + 3^3 + 5^3 + \dots + (2(k+1)-1)^3 &= (k+1)^2 (2(k+1)^2 - 1) \\
 \Rightarrow 1^3 + 3^3 + 5^3 + \dots + (2k+1)^3 &= (k^2 + 2k + 1)(2(k^2 + 2k + 1) - 1) \\
 &= (k^2 + 2k + 1)(2k^2 + 4k + 2 - 1) \\
 &= (k^2 + 2k + 1)(2k^2 + 4k + 1) \\
 &= 2k^4 + 4k^3 + 2k^2 + 4k^3 + 8k^2 + 4k + k^2 + 2k + 1 \\
 &= 2k^4 + 8k^3 + 11k^2 + 6k + 1
 \end{aligned}$$

Adding $(2k+1)^3$ on both sides of equation (i)

$$\begin{aligned}
 S(k): 1^3 + 3^3 + 5^3 + \dots + (2k-1)^3 + (2k+1)^3 &= k^2 (2k^2 - 1) + (2k+1)^3 \\
 \Rightarrow 1^3 + 3^3 + 5^3 + \dots + (2k+1)^3 &= k^2 (2k^2 - 1) + (2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + (1)^3 \\
 \Rightarrow 1^3 + 3^3 + 5^3 + \dots + (2k+1)^3 &= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\
 &= 2k^4 + 8k^3 + 11k^2 + 6k + 1
 \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n .

Question # 25

$x+1$ is a factor of $x^{2n} - 1; (x \neq -1)$

Solution Suppose $S(n): x^{2n} - 1$

Put $n=1$

$$S(1): x^{2(1)} - 1 = x^2 - 1 = (x-1)(x+1)$$

$x+1$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): x^{2k} - 1$$

Then there exists quotient Q such that

$$x^{2k} - 1 = (x+1)Q$$

The statement for $n = k+1$

$$\begin{aligned}
 S(k+1): x^{2(k+1)} - 1 &= x^{2k+2} - 1 \\
 &= x^{2k+2} - x^{2k} + x^{2k} - 1 \quad \text{+ing and -ing } x^{2k} \\
 &= x^{2k}(x^2 - 1) + (x^{2k} - 1) \\
 &= x^{2k}(x-1)(x+1) + (x+1)Q \quad \because x^{2k} - 1 = (x+1)Q \\
 &= (x+1)(x^{2k}(x-1) + Q)
 \end{aligned}$$

Clearly $x+1$ is a factor of $S(k+1)$.

Since the truth for $n = k$ implies the truth for $n = k+1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

Question # 26

$x - y$ is a factor of $x^n - y^n; (x \neq y)$

Solution Suppose $S(n): x^n - y^n$

Put $n=1$

$$S(1): x^1 - y^1 = x - y$$

$x - y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): x^k - y^k$$

Then there exists quotient Q such that

$$x^k - y^k = (x - y)Q$$

The statement for $n = k + 1$

$$\begin{aligned} S(k+1): x^{k+1} - y^{k+1} \\ &= x \cdot x^k - y \cdot y^k \\ &= x \cdot x^k - x \cdot y^k + x \cdot y^k - y \cdot y^k && \text{-ing \& +ing } xy^k \\ &= x(x^k - y^k) + y^k(x - y) \\ &= x(x - y)Q + y^k(x - y) && \because x^k - y^k = (x - y)Q \end{aligned}$$

Clearly $x - y$ is a factor of $S(k + 1)$.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

Question # 27

$x + y$ is a factor of $x^{2n-1} + y^{2n-1}; (x \neq y)$

Solution Suppose $S(n): x^{2n-1} + y^{2n-1}$

Put $n = 1$

$$S(1): x^{2(1)-1} + y^{2(1)-1} = x^1 + y^1 = x + y$$

$x + y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): x^{2k-1} + y^{2k-1}$$

Then there exists quotient Q such that

$$x^{2k-1} + y^{2k-1} = (x + y)Q$$

The statement for $n = k + 1$

$$\begin{aligned} S(k+1): x^{2(k+1)-1} + y^{2(k+1)-1} \\ &= x^{2k+2-1} + y^{2k+2-1} \\ &= x^{2k+2-1} - x^{2k-1}y^2 + x^{2k-1}y^2 + y^{2k+2-1} && \text{+ing and -ing } x^{2k-1}y^2 \\ &= x^{2k-1}(x^2 - y^2) + y^2(x^{2k-1} + y^{2k-1}) \\ &= x^{2k-1}(x - y)(x + y) + y^2(x + y)Q && \because x^{2k-1} + y^{2k-1} = (x + y)Q \\ &= (x + y)(x^{2k-1}(x - y) + y^2 Q) \end{aligned}$$

Clearly $x + y$ is a factor of $S(k + 1)$.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall n \in \mathbb{Z}^+$.

Principle of Extended Mathematical Induction

A given statement $S(n)$ is true for $n \geq i$ if the following two conditions hold

Condition I: $S(i)$ is true i.e. $S(n)$ is true for $n = i$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k ,

Then $S(n)$ is true for all positive integers

Question # 28

Use mathematical induction to show that

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1 \text{ for all non-negative integers } n.$$

Solution Suppose $S(n): 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

Put $n = 0$

$$S(1): 1 = 2^{0+1} - 1 = \Rightarrow 1 = 2 - 1 \Rightarrow 1 = 1$$

Note: Non- negative number are 0,1,2,3,.....

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1 \dots \dots \dots (i)$$

The statement for $n = k + 1$ becomes

$$\begin{aligned} S(k+1): 1 + 2 + 2^2 + \dots + 2^{k+1} &= 2^{k+1+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

Adding 2^{k+1} on both sides of equation (i)

$$\begin{aligned} 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ \Rightarrow 1 + 2 + 4 + \dots + 2^{k+1} &= 2(2^{k+1}) - 1 && \because 2^{k+1} + 2^{k+1} = 2(2^{k+1}) \\ &= 2^{k+1+1} - 1 \\ &= 2^{k+1+1} - 1 \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all non-negative integers n .

Question # 29

If A and B are square matrices and $AB = BA$, then show by mathematical induction that $AB^n = B^nA$ for any positive integer n .

Solution Suppose $S(n): AB^n = B^nA$

Put $n = 1$

$$S(1): AB^1 = B^1A \Rightarrow AB = BA$$

$S(1)$ is true as we have given $AB = BA$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): AB^k = B^kA \dots \dots \dots (i)$$

The statement for $n = k + 1$

$$S(k+1): AB^{k+1} = B^{k+1}A$$

Post-multiplying equation (i) by B .

$$\begin{aligned} (AB^k)B &= (B^kA)B \\ \Rightarrow A(B^k B) &= B^k(AB) && \text{by associative law} \\ \Rightarrow AB^{k+1} &= B^k(BA) && \because AB = BA \text{ (given)} \\ &= (B^k B)A = B^{k+1}A \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integers n .

Question # 30

Prove by Principle of mathematical induction that $n^2 - 1$ is divisible by 8 when n is an odd positive integer.

Solution Suppose $S(n): n^2 - 1$

Put $n = 1$

$$S(1): (1)^2 - 1 = 0$$

$S(1)$ is clearly divisible by 8, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$ where k is odd.

$$S(k): k^2 - 1$$

Then there exists quotient Q such that

$$k^2 - 1 = 8Q$$

As $k + 2$ is the next odd integer after k The statement for $n = k + 2$

$$S(k + 2): (k + 2)^2 - 1$$

$$= k^2 + 4k + 4 - 1$$

$$= k^2 - 1 + 4k + 4$$

$$= 8Q + 4(k + 1) \quad \because k^2 - 1 = 8Q$$

Since k is odd therefore $k + 1$ is even so there exists integer t such that $k + 1 = 2t$

$$\Rightarrow S(k + 2) = 8Q + 4(2t)$$

$$= 8Q + 8t$$

Clearly $S(k + 2)$ is divisible by 8 so condition II is satisfied.

Therefore the given statement is true for odd positive integers.

Question # 31

Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integral $n \geq 0$ if x is a positive number.

Solution Suppose $S(n): \ln x^n = n \ln x$

Put $n = 1$

$$S(1): \ln x^1 = (1) \ln x \Rightarrow \ln x = \ln x$$

$S(1)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): \ln x^k = k \ln x \dots \dots \dots (i)$$

The statement for $n = k + 1$

$$S(k + 1): \ln x^{k+1} = (k + 1) \ln x$$

Now adding $\ln x$ on both sides of equation (i)

$$\ln x^k + \ln x = k \ln x + \ln x$$

$$\Rightarrow \ln x^k \cdot x = (k + 1) \ln x \quad \because \ln x + \ln y = \ln xy$$

$$\Rightarrow \ln x^{k+1} = (k + 1) \ln x$$

Thus $S(k + 1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all $n \in \mathbb{Z}^+$.

Question # 32

$n! > 2^n - 1$ for integral values of $n \geq 4$.

Solution Suppose $S(n): n! > 2^n - 1$; $n \geq 4$

Put $n = 4$

$$S(4): 4! > 2^4 - 1 \Rightarrow 24 > 16 - 1 \Rightarrow 24 > 15$$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): k! > 2^k - 1 \dots \dots \dots (i)$$

The statement for $n = k + 1$

$$S(k+1): (k+1)! > 2^{k+1} - 1$$

Multiplying both sides of equation (i) by $k + 1$

$$\begin{aligned} (k+1)k! &> (k+1)(2^k - 1) \\ \Rightarrow (k+1)! &> (k+1+2-2)(2^k - 1) && \because (k+1)k! = (k+1)! \\ \Rightarrow (k+1)! &> (k-1+2)(2^k - 1) \\ \Rightarrow (k+1)! &> k \cdot 2^k - k - 2^k + 1 + 2 \cdot 2^k - 2 \\ \Rightarrow (k+1)! &> (k \cdot 2^k - 2^k - k) + 2^{k+1} - 1 \\ \Rightarrow (k+1)! &> 2^{k+1} - 1 && \because k \cdot 2^k - 2^k - k \geq 0 \quad \forall k \geq 4 \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 33

$n^2 > n + 3$ for integral values of $n \geq 3$.

Solution Suppose $S(n): n^2 > n + 3$; $n \geq 3$

Put $n = 3$

$$S(3): 3^2 > 3 + 3 \Rightarrow 9 > 6$$

$S(3)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): k^2 > k + 3 \dots \dots \dots (i)$$

The statement for $n = k + 1$

$$S(k+1): (k+1)^2 > k+1+3 \Rightarrow (k+1)^2 > k+4$$

Adding $2k + 1$ on both sides of equation (i)

$$\begin{aligned} k^2 + 2k + 1 &> k + 3 + 2k + 1 \\ \Rightarrow (k+1)^2 &> k + 4 + 2k \\ \Rightarrow (k+1)^2 &> k + 4 && \text{ignoring } 2k \text{ as } 2k > 0 \end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 34

$4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

Solution Suppose $S(n): 4^n > 3^n + 2^{n-1}$; $n \geq 2$

Put $n = 2$

$$S(2): 4^2 > 3^2 + 2^{2-1} \Rightarrow 16 > 9 + 2 \Rightarrow 16 > 11$$

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$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): 4^k > 3^k + 2^{k-1} \dots\dots\dots (i)$$

The statement for $n = k + 1$

$$S(k+1): 4^{k+1} > 3^{k+1} + 2^{k+1-1}$$

$$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k$$

Multiplying both sides of equation (i) by 4.

$$4(4^k) > 4(3^k + 2^{k-1})$$

$$\Rightarrow 4^{k+1} > 4 \cdot 3^k + 4 \cdot 2^{k-1}$$

$$\Rightarrow 4^{k+1} > (3+1) \cdot 3^k + (2+2) \cdot 2^{k-1}$$

$$\Rightarrow 4^{k+1} > 3 \cdot 3^k + 3^k + 2 \cdot 2^{k-1} + 2 \cdot 2^{k-1}$$

$$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k + (3^k + 2^k)$$

$$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k \qquad \text{ignoring } 3^k + 2^k \text{ as } 3^k + 2^k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 35

$3^n < n!$ for integral values of $n \geq 6$.

Solution Suppose $S(n): 3^n < n!$; $n > 6$

Put $n = 7$

$$S(7): 3^7 < 7! \Rightarrow 2187 < 5040$$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): 3^k < k! \dots\dots\dots (i)$$

The statement for $n = k + 1$

$$S(k+1): 3^{k+1} < (k+1)!$$

Multiplying both sides of equation (i) by $k + 1$.

$$(k+1)3^k < (k+1)k!$$

$$\Rightarrow ((k-2)+3)3^k < (k+1)k!$$

$$\Rightarrow (k-2)3^k + 3^{k+1} < (k+1)k!$$

$$\Rightarrow 3^{k+1} < (k+1)k! \qquad \because (k-2)3^k > 0 \quad \forall k > 6$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n > 6$.

Question # 36

$n! > n^2$ for integral values of $n \geq 4$.

Solution Suppose $S(n): n! > n^2$; $n \geq 4$

Put $n = 4$

$$S(4): 4! > 4^2 \Rightarrow 24 > 16$$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): k! > k^2 \dots\dots\dots (i)$$

The statement for $n = k + 1$

$$S(k+1): (k+1)! > (k+1)^2$$

Multiplying both sides of equation (i) by $k + 1$.

$$(k+1)k! > (k+1)k^2$$

$$\Rightarrow (k+1)! > (k+1)(k+1) \quad \because k+1 < k^2 \quad \forall k \geq 4$$

$$\Rightarrow (k+1)! > (k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 37

$3 + 5 + 7 + \dots\dots\dots + (2n + 5) = (n + 2)(n + 4)$ for integral values of $n \geq -1$.

Solution Suppose $S(n): 3 + 5 + 7 + \dots\dots\dots + (2n + 5) = (n + 2)(n + 4) \quad ; n \geq -1$

Put $n = -1$

$$S(-1): 3 = (-1 + 2)(-1 + 4) \Rightarrow 3 = (1)(3) \Rightarrow 3 = 3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 3 + 5 + 7 + \dots\dots\dots + (2k + 5) = (k + 2)(k + 4) \dots\dots\dots (i)$$

The statement for $n = k + 1$ becomes

$$S(k+1): 3 + 5 + 7 + \dots\dots\dots + (2(k+1) + 5) = ((k+1) + 2)((k+1) + 4)$$

$$\Rightarrow 3 + 5 + 7 + \dots\dots\dots + (2k + 7) = (k + 3)(k + 5)$$

Adding $(2k + 7)$ on both sides of equation (i)

$$S(k): 3 + 5 + 7 + \dots\dots\dots + (2k + 5) + (2k + 7) = (k + 2)(k + 4) + (2k + 7)$$

$$\Rightarrow 3 + 5 + 7 + \dots\dots\dots + (2k + 7) = k^2 + 2k + 4k + 8 + 2k + 7$$

$$= k^2 + 8k + 15$$

$$= k^2 + 5k + 3k + 15$$

$$= k(k + 5) + 3(k + 5)$$

$$= (k + 5)(k + 3)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq -1$.

Question # 38

$1 + nx \leq (1 + x)^n$ for integral values of $n \geq -1$.

Solution Suppose $S(n): 1 + nx \leq (1 + x)^n \quad ; n \geq 2$

Put $n = 2$

$$S(2): 1 + 2x \leq (1 + x)^2 \Rightarrow 1 + 2x \leq 1 + 2x + x^2$$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): 1 + kx \leq (1 + x)^k \dots\dots\dots (i)$$

The statement for $n = k + 1$

$$S(k+1): 1 + (k+1)x \leq (1 + x)^{k+1}$$

Multiplying both sides of equation (i) by $1 + x$.

$$(1+kx)(1+x) \leq (1+x)^k(1+x)$$

$$\Rightarrow 1+kx+x+kx^2 \leq (1+x)^{k+1}$$

$$\Rightarrow 1+kx+x \leq (1+x)^{k+1} \qquad \because kx^2 > 0$$

$$\Rightarrow 1+(k+1)x \leq (1+x)^{k+1}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 2$

If you found any error, please report us at www.megalecture@gmail.com

