

Formula for the sum

- i) $\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- ii) $\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- iii) $\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$
- iv) $\sum_{k=1}^n (1) = 1 + 1 + 1 + \dots + 1 \text{ (} n \text{ times)} = n$

Sum the following series up to n terms.

Question # 1

$$1 \times 1 + 2 \times 4 + 3 \times 7 + \dots$$

Solution

$$1 \times 1 + 2 \times 4 + 3 \times 7 + \dots$$

If T_k denotes the k th term of the series then

$$T_k = k(3k - 2) = 3k^2 - 2k$$

$$\begin{aligned} &1 + (k - 1)3 \\ &= 1 + 3k - 3 \\ &= 3k - 2 \end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned} S_n &= \sum_{k=1}^n (3k^2 - 2k) \\ &= 3 \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k = 3 \left(\frac{n(n+1)(2n+1)}{6} \right) - 2 \left(\frac{n(n+1)}{2} \right) \\ &= \frac{n(n+1)(2n+1)}{2} - n(n+1) \\ &= \frac{n(n+1)}{2} (2n+1 - 2) = \frac{n(n+1)(2n-1)}{2} \quad \text{Answer} \end{aligned}$$

Question # 2

$$1 \times 3 + 3 \times 6 + 5 \times 9 + \dots$$

Solution

$$1 \times 3 + 3 \times 6 + 5 \times 9 + \dots$$

If T_k denotes the k th term of the series then

$$T_k = (2k - 1)(3k) = 6k^2 - 3k$$

$$\begin{aligned} &1 + (k - 1)2 \\ &= 1 + 2k - 2 \\ &= 2k - 1 \\ &3 + (k - 1)3 \\ &= 3 + 3k - 3 \\ &= 3k \end{aligned}$$

Now do yourself as above

Question # 3

$$1 \times 4 + 2 \times 7 + 3 \times 10 + \dots$$

Solution

Do yourself as Question # 1

Question # 4

$$3 \times 5 + 5 \times 9 + 7 \times 13 + \dots$$

Solution

Do yourself as Question # 1

Question # 5

$$1^2 + 3^2 + 5^2 + \dots$$

Solution

$$1^2 + 3^2 + 5^2 + \dots$$

$$\begin{aligned} & 1 + (k-1)2 \\ & = 1 + 2k - 2 \\ & = 2k - 1 \end{aligned}$$

If T_k denotes the k th term of the series then

$$T_k = (2k - 1)^2 = 4k^2 - 4k + 1$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned} S_n &= \sum_{k=1}^n (4k^2 - 4k + 1) \\ &= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + \sum_{k=1}^n (1) = 4 \left(\frac{n(n+1)(2n+1)}{6} \right) - 4 \left(\frac{n(n+1)}{2} \right) + n \\ &= \frac{2n(n+1)(2n+1)}{3} - 2(n(n+1)) + n \\ &= n \left(\frac{2(n+1)(2n+1)}{3} - 2(n+1) + 1 \right) = n \left(\frac{2(2n^2 + 2n + n + 1)}{3} - 2n - 2 + 1 \right) \\ &= n \left(\frac{2(2n^2 + 3n + 1)}{3} - 2n - 1 \right) = n \left(\frac{4n^2 + 6n + 2 - 6n - 3}{3} \right) \\ &= n \left(\frac{4n^2 - 1}{3} \right) = \frac{n}{3} (4n^2 - 1) \quad \text{Answer} \end{aligned}$$

Question # 6

$$2^2 + 5^2 + 8^2 + \dots$$

Solution

$$2^2 + 5^2 + 8^2 + \dots$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned} & 2 + (k-1)3 \\ & = 2 + 3k - 3 \\ & = 3k - 1 \end{aligned}$$

$$T_k = (3k - 1)^2 = 9k^2 - 6k + 1$$

Let S_n be the sum of first n term of the series then

Now do yourself as above

Question # 7

$$2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + \dots$$

Solution

$$2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + \dots$$

If T_k denotes the k th term of the series then

$$T_k = (2k)(k)^2 = 2k^3 \qquad 2 + (k-1)2$$

Let S_n denotes the sum of first n terms of the series then

$$= 2 + 2k - 2$$

$$= 2k$$

$$S_n = \sum_{k=1}^n (2k^3) = 2 \sum_{k=1}^n k^3$$

$$= 2 \left(\frac{n(n+1)}{2} \right)^2 = 2 \frac{n^2(n+1)^2}{4} = \frac{n^2(n+1)^2}{2} \quad \text{Answer}$$

$$1 + (k-1)1$$

$$= 1 + k - 1$$

$$= k$$

Question # 8

$$3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + \dots$$

Solution

$$3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + \dots$$

If T_k denotes the k th term of the series then

$$3 + (k-1)2$$

$$= 3 + 2k - 2$$

$$= 2k + 1$$

$$T_k = (2k+1)(k+1)^2 = (2k+1)(k^2 + 2k + 1)$$

$$= 2k^3 + 4k^2 + 2k + k^2 + 2k + 1$$

$$= 2k^3 + 5k^2 + 4k + 1$$

$$2 + (k-1)1$$

$$= 2 + k - 1$$

$$= k + 1$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^n (2k^3 + 5k^2 + 4k + 1)$$

$$= 2 \sum_{k=1}^n k^3 + 5 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + \sum_{k=1}^n (1)$$

$$= 2 \left(\frac{n(n+1)}{2} \right)^2 + 5 \left(\frac{n(n+1)(2n+1)}{6} \right) + 4 \frac{n(n+1)}{2} + n$$

$$= 2 \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1) + n$$

$$= n \left(\frac{n(n^2 + 2n + 1)}{2} + \frac{5(2n^2 + 2n + n + 1)}{6} + 2(n+1) + 1 \right)$$

$$= n \left(\frac{n^3 + 2n^2 + n}{2} + \frac{5(2n^2 + 3n + 1)}{6} + 2n + 2 + 1 \right)$$

$$= n \left(\frac{n^3 + 2n^2 + n}{2} + \frac{10n^2 + 15n + 5}{6} + 2n + 3 \right)$$

$$= n \left(\frac{3n^3 + 6n^2 + 3n + 10n^2 + 15n + 5 + 12n + 18}{6} \right)$$

$$= \frac{n}{6} (3n^3 + 16n^2 + 30n + 23) \quad \text{Answer}$$

Question # 9

$$2 \times 4 \times 7 + 3 \times 6 \times 10 + 4 \times 8 \times 13 + \dots$$

Solution

$$2 \times 4 \times 7 + 3 \times 6 \times 10 + 4 \times 8 \times 13 + \dots$$

$$2 + (k-1)1$$

$$= 2 + k - 1$$

$$= k + 1$$

If T_k denotes the k th term of the series then

$$\begin{aligned}
 T_k &= 2(k+1)(k+1)(3k+4) = 2(k+1)^2(3k+4) && 7 + (k-1)3 \\
 &= 2(k^2 + 2k + 1)(3k+4) && = 7 + 3k - 3 \\
 &= 3k^3 + 6k^2 + 3k + 4k^2 + 8k + 4 && = 3k + 4 \\
 &= 3k^3 + 10k^2 + 11k + 4 && 4 + (k-1)2 \\
 & && = 4 + 2k - 2 \\
 & && = 2k + 2 = 2(k+1)
 \end{aligned}$$

Now do yourself.

Question # 10

$$1 \times 4 \times 6 + 4 \times 7 \times 10 + 7 \times 10 \times 14 + \dots$$

Solution

$$1 \times 4 \times 6 + 4 \times 7 \times 10 + 7 \times 10 \times 14 + \dots \qquad 1 + (k-1)3$$

If T_k denotes the k th term of the series then

$$\begin{aligned}
 T_k &= (3k-2)(3k+1)2(2k+1) = 2(3k-2)(3k+1)(2k+1) && = 1 + 3k - 3 \\
 &= 2(3k-2)(6k^2 + 2k + 3k + 1) = 2(3k-2)(6k^2 + 5k + 1) && = 3k - 2 \\
 &= 2(18k^3 + 15k^2 + 3k - 12k^2 - 10k - 2) && 4 + (k-1)3 \\
 &= 2(18k^3 + 3k^2 - 7k - 2) && = 4 + 3k - 3 \\
 & && = 3k + 1
 \end{aligned}$$

Now do yourself.

$$\begin{aligned}
 & 6 + (k-1)4 \\
 & = 6 + 4k - 4 \\
 & = 4k + 2 = 2(2k+1)
 \end{aligned}$$

Question # 11

$$1 + (1+2) + (1+2+3) + \dots$$

Solution

$$1 + (1+2) + (1+2+3) + \dots$$

If T_k denotes the k th term of the series then

$$T_k = 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} = \frac{1}{2}(k^2 + k)$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned}
 S_n &= \sum_{k=1}^n \left(\frac{1}{2}(k^2 + k) \right) = \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{2} \sum_{k=1}^n k \\
 &= \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{1}{2} \left(\frac{n(n+1)}{2} \right) \\
 &= \frac{n(n+1)}{4} \left(\frac{(2n+1)}{3} + 1 \right) = \frac{n(n+1)}{4} \left(\frac{2n+1+3}{3} \right) \\
 &= \frac{n(n+1)}{4} \left(\frac{2n+4}{3} \right) = \frac{n(n+1)(2n+4)}{12} \quad \text{Answer}
 \end{aligned}$$

Question # 12

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$$

Solution

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$$

If T_k denotes the k th term of the series then

$$T_k = 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{1}{6}k(2k^2 + 2k + k + 1) = \frac{1}{6}k(2k^2 + 3k + 1)$$

$$= \frac{1}{6}(2k^3 + 3k^2 + k) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^n \left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k \right) = \frac{1}{3} \sum_{k=1}^n k^3 + \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{6} \sum_{k=1}^n k$$

$$= \frac{1}{3} \left(\frac{n^2(n+1)^2}{4} \right) + \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{1}{6} \left(\frac{n(n+1)}{2} \right)$$

$$= \frac{n(n+1)}{12} (n(n+1) + (2n+1) + 1) = \frac{n(n+1)}{12} (n^2 + n + 2n + 1 + 2)$$

$$= \frac{n(n+1)(n^2 + 3n + 3)}{12} \quad \text{Answer}$$

Question # 13

$$2 + (2 + 5) + (2 + 5 + 8) + \dots$$

Solution

$$2 + (2 + 5) + (2 + 5 + 8) + \dots$$

If T_k denotes the k th term of the series then

$$\therefore S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

$$T_k = 2 + 5 + 8 + \dots + \text{up to } k \text{ terms}$$

$$= \frac{k}{2} [2(2) + (k-1)(3)] = \frac{k}{2} [4 + 3k - 3] = \frac{k}{2} [3k + 1] = \frac{3}{2}k^2 + \frac{1}{2}k$$

Now do yourself

Question # 14

Sum the series

(i) $1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2$

(ii) $1^2 - 3^2 + 5^2 - 7^2 + \dots + (4n-3)^2 - (4n-1)^2$

(iii) $\frac{1^2}{1} + \frac{1^2 + 2^2}{2} + \frac{1^2 + 2^2 + 3^2}{3} + \dots$

Solution

(i)

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2$$

If T_k denotes the k th term of the series then

$$T_k = (2k-1)^2 - (2k)^2 = 4k^2 - 4k + 1 - 4k^2 = -4k + 1$$

Now do yourself

(ii) $1^2 - 3^2 + 5^2 - 7^2 + \dots + (4n-3)^2 - (4n-1)^2$

If T_k denotes the k th term of the series then

$$T_k = (4k-3)^2 - (4k-1)^2 = (16k^2 - 24k + 9) - (16k^2 - 8k + 1)$$

$$= 16k^2 - 24k + 9 - 16k^2 + 8k - 1 = -16k + 8$$

Now do yourself

(iii) $\frac{1^2}{1} + \frac{1^2 + 2^2}{2} + \frac{1^2 + 2^2 + 3^2}{3} + \dots$ to n terms

If T_k denotes the k th term of the series then

$$\begin{aligned} T_k &= \frac{1^2 + 2^2 + 3^2 + \dots + k^2}{k} \\ &= \frac{k(k+1)(2k+1)/6}{k} = \frac{k(k+1)(2k+1)}{6k} = \frac{(k+1)(2k+1)}{6} \\ &= \frac{2k^2 + 2k + k + 1}{6} = \frac{2k^2 + 3k + 1}{6} = \frac{2}{6}k^2 + \frac{3}{6}k + \frac{1}{6} \\ &= \frac{1}{3}k^2 + \frac{1}{2}k + \frac{1}{6} \end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\frac{1}{3}k^2 + \frac{1}{2}k + \frac{1}{6} \right) = \frac{1}{3} \sum_{k=1}^n k^2 + \frac{1}{2} \sum_{k=1}^n k + \frac{1}{6} \sum_{k=1}^n (1) \\ &= \frac{1}{3} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{1}{2} \left(\frac{n(n+1)}{2} \right) + \frac{n}{6} \\ &= \frac{n(n+1)(2n+1)}{18} + \frac{n(n+1)}{4} + \frac{n}{3} = \frac{n}{2} \left(\frac{(n+1)(2n+1)}{9} + \frac{n+1}{2} + \frac{1}{3} \right) \\ &= \frac{n}{2} \left(\frac{(2n^2 + 2n + n + 1)}{9} + \frac{n+1}{2} + \frac{1}{3} \right) = \frac{n}{2} \left(\frac{2n^2 + 3n + 1}{9} + \frac{n+1}{2} + \frac{1}{3} \right) \\ &= \frac{n}{2} \left(\frac{4n^2 + 6n + 2 + 9n + 9 + 6}{18} \right) = \frac{n}{2} \left(\frac{4n^2 + 15n + 17}{18} \right) \\ &= \frac{n}{36} (4n^2 + 15n + 17) \quad \text{Answer} \end{aligned}$$

Question # 15

Find the sum of n term of the series whose n th term are given.

- (i) $3n^2 + n + 1$ (ii) $n^2 + 4n + 1$

Solution

(i)

Since $T_n = 3n^2 + n + 1$

Therefore $T_k = 3k^2 + k + 1$

Now do yourself

(ii)

Do yourself

Question # 16

Given n th term of the series, find the sum to $2n$ term.

- (i) $3n^2 + 2n + 1$ (ii) $n^3 + 2n + 3$

Solution

(i)

Do yourself as below (Q # 16 (ii))

(ii) Since $T_n = n^3 + 2n + 3$

Therefore $T_k = k^3 + 2k + 3$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned} S_n &= \sum_{k=1}^n (k^3 + 2k + 3) = \sum_{k=1}^n k^3 + 2 \sum_{k=1}^n k + 3 \sum_{k=1}^n (1) \\ &= \frac{n^2(n+1)^2}{4} + 2 \frac{n(n+1)}{2} + 3n = n \left(\frac{n(n+1)^2}{4} + n + 1 + 3 \right) \\ &= n \left(\frac{n(n^2 + 2n + 1)}{4} + n + 4 \right) = n \left(\frac{n^3 + 2n^2 + n}{4} + n + 4 \right) \\ &= n \left(\frac{n^3 + 2n^2 + n + 4n + 16}{4} \right) \\ &= \frac{n}{4} (n^3 + 2n^2 + 5n + 16) \end{aligned}$$

Now for sum of first $2n$ terms put $n = 2n$

$$\begin{aligned} S_{2n} &= \frac{2n}{4} ((2n)^3 + 2(2n)^2 + 5(2n) + 16) \\ &= \frac{n}{2} (8n^3 + 8n^2 + 10n + 16) = \frac{2n}{2} (4n^3 + 4n^2 + 5n + 8) \\ &= n(4n^3 + 4n^2 + 5n + 8) \text{ Answer} \end{aligned}$$

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