Question # 1

(i) Let x be a cube root of 8 then

Review:
$\omega = \dfrac{-1+\sqrt{-3}}{2}$
$\omega^2 = \dfrac{-1-\sqrt{-3}}{2}$

$$x = (8)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = 8$$
$$\Rightarrow \quad x^3 - 8 = 0 \qquad \Rightarrow \quad (x)^3 - (2)^3 = 0$$
$$\Rightarrow \quad (x-2)(x^2+2x+4) = 0$$
$$\Rightarrow \quad x - 2 = 0 \quad \text{or} \quad x^2 + 2x + 4 = 0$$
$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$
$$= \frac{-2 \pm 2\sqrt{-3}}{2} = 2\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$
$$\Rightarrow \quad x = 2\left(\frac{-1+\sqrt{-3}}{2}\right) \quad \text{or} \quad x = 2\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$\Rightarrow \quad x = 2\omega \quad \text{or} \quad x = 2\omega^2$$

Hence cube root of 8 are $2, 2\omega$ and $2\omega^2$.

(ii) Hint

Considering x as a cube root of -8 and Solving as above you will get the following values of x

$$x = -2\ , \quad x = \frac{2+2\sqrt{-3}}{2}\ , \quad x = \frac{2-2\sqrt{-3}}{2}$$
$$\Rightarrow \quad x = -2\left(\frac{-1-\sqrt{-3}}{2}\right) \ , \quad x = -2\left(\frac{-1+\sqrt{-3}}{2}\right)$$
$$\Rightarrow \quad x = -2\omega^2 \ , \quad x = -2\omega$$

Hence cube root of -8 are $-2, -2\omega$ and $-2\omega^2$.

(iii) *Do yourself as (iv) below.*

(iv) Let x be a cube root of -27 then

$$x = (-27)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = -27$$
$$\Rightarrow \quad x^3 + 27 = 0 \qquad \Rightarrow \quad (x)^3 + (3)^3 = 0$$
$$\Rightarrow \quad (x+3)(x^2 - 3x + 9) = 0$$
$$\Rightarrow \quad x + 3 = 0 \quad \text{or} \quad x^2 - 3x + 9 = 0$$
$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{3 \pm \sqrt{9-36}}{2} = \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm 3\sqrt{-3}}{2}$$
$$\Rightarrow \quad x = \frac{3+3\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{3-3\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = -3\left(\frac{-1-\sqrt{-3}}{2}\right) \quad \text{or} \quad x = -3\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad x = -3\omega^2 \qquad \text{or} \qquad x = -3\omega$$

Hence cube root of -27 are $-3, -3\omega$ and $-3\omega^2$.

(v) Let x be a cube root of 64 then

$$x = (64)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = 64$$

$$\Rightarrow \quad x^3 - 64 = 0 \qquad \Rightarrow \quad (x)^3 - (4)^3 = 0$$

$$\Rightarrow \quad (x-4)(x^2+4x+16) = 0$$

$$\Rightarrow \quad x - 4 = 0 \quad \text{or} \quad x^2 + 4x + 16 = 0$$

$$\Rightarrow \quad x = 4 \quad \text{or} \quad x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-64}}{2} = \frac{-4 \pm \sqrt{-48}}{2}$$

$$= \frac{-4 \pm 4\sqrt{-3}}{2} \qquad \because \ 48 = 16 \times 3$$

$$\Rightarrow \quad x = \frac{-4+4\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{-4-4\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = 4\left(\frac{-1+\sqrt{-3}}{2}\right) \quad \text{or} \quad x = 4\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad x = 4\omega \qquad \text{or} \qquad x = 4\omega^2$$

Hence cube root of 64 are $4, 4\omega$ and $4\omega^2$.

Question # 2

(i)
$$\left(1+\omega-\omega^2\right)^8 = \left(1+\omega+w^2-2w^2\right)^8$$

$$= \left(0-2w^2\right)^8 \qquad \because \ 1+\omega+\omega^2 = 0$$

$$= (-2)^8(w^2)^8 = 256\,\omega^{16}$$

$$= 256\,\omega^{15}\cdot\omega = 256\left(\omega^3\right)^5\cdot\omega$$

$$= 256\,(1)^5\cdot\omega = 256\,\omega \quad \text{Answer} \qquad \because \ \omega^3 = 1$$

(ii)
$$\omega^{28} + w^{29} + 1 = \omega^{27}\cdot\omega + \omega^{27}\cdot\omega^2 + 1$$

$$= \left(\omega^3\right)^9\cdot\omega + \left(\omega^3\right)^9\cdot\omega^2 + 1$$

$$= (1)^9\cdot\omega + (1)^9\cdot\omega^2 + 1 \qquad \because \ \omega^3 = 1$$

$$= \omega + \omega^2 + 1$$

$$= 0 \quad \text{Answer} \qquad \because \ 1+\omega+\omega^2 = 0$$

(iii)
$$\left(1+\omega-\omega^2\right)\left(1-\omega+\omega^2\right)$$

$$= \left(1+\omega+\omega^2-2\omega^2\right)\left(1+\omega+\omega^2-2\omega\right) \qquad \because \ 1+\omega+\omega^2 = 0$$

$$= \left(0-2\omega^2\right)(0-2\omega) = \left(-2\omega^2\right)(-2\omega)$$

$$= 4\,\omega^3 = 4(1) = 4 \quad \text{Answer} \qquad \because \ \omega^3 = 1$$

(iv) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^7+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$

$=\omega^7+(\omega^2)^7$

$=\omega^7+\omega^{14}$

$=\omega^6\cdot\omega+\omega^{12}\cdot\omega^2=\left(\omega^3\right)^3\cdot\omega+\left(\omega^3\right)^4\cdot\omega^2$

$=(1)^3\cdot\omega+(1)^4\cdot\omega^2$

$=\omega+\omega^2=-1$ Answer

$\because\ \omega=\dfrac{-1+\sqrt{-3}}{2}$, $\omega^2=\dfrac{-1-\sqrt{-3}}{2}$

$\because\ 1+\omega+\omega^2=0$

(v) $\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5$

$=\left(2\cdot\dfrac{-1+\sqrt{-3}}{2}\right)^5+\left(2\cdot\dfrac{-1-\sqrt{-3}}{2}\right)^5$

$=(2\cdot\omega)^5+\left(2\cdot\omega^2\right)^5$

$=32\omega^5+32\omega^{10}=32\omega^3\cdot\omega^2+32\omega^9\cdot\omega^1$

$=32(1)\cdot\omega^2+32(1)\cdot\omega$

$=32(\omega+\omega^2)$

$=32(-1)=-32$

$\because\ \omega=\dfrac{-1+\sqrt{-3}}{2}$, $\omega^2=\dfrac{-1-\sqrt{-3}}{2}$

$\because\ \omega^9=\left(\omega^3\right)^3=(1)^3=1$

$\because\ 1+\omega+\omega^2=0$

{ Question # 3

(i) R.H.S$=(x-y)(x-\omega y)\left(x-\omega^2y\right)$

$=(x-y)\left[x\left(x-\omega^2y\right)-\omega y\left(x-\omega^2y\right)\right]$

$=(x-y)\left[x^2-\omega^2xy-\omega xy+\omega^3y^2\right]$

$=(x-y)\left[x^2-\left(\omega^2+\omega\right)xy+(1)y^2\right]$ $\qquad\because\ \omega^3=1$

$=(x-y)\left[x^2-(-1)xy+y^2\right]$

$=(x-y)\left[x^2+xy+y^2\right]$

$=x^3-y^3$ = L.H.S

$\because\ 1+\omega+\omega^2=0$

$\therefore\ \omega+\omega^2=-1$

(ii) R.H.S$=(x+y+z)\left(x+\omega y+\omega^2z\right)\left(x+\omega^2y+\omega z\right)$

$=(x+y+z)[x^2+\omega^2xy+\omega xz+\omega xy+\omega^3y^2+\omega^2yz$
$+\omega^2xz+\omega^4yz+\omega^3z^2]$

$=(x+y+z)[x^2+(\omega^2+\omega)xy+(\omega+\omega^2)xz+(\omega^2+\omega^4)yz$
$+\omega^3y^2+\omega^3z^2]$

$=(x+y+z)[x^2+(-1)xy+(-1)xz+(\omega^2+\omega)yz+(1)y^2+(1)z^2]$

$\because\ \omega^4=\omega\ \ \&\ \ \omega+\omega^2=-1$

$=(x+y+z)[x^2+y^2+z^2-xy+(-1)yz-xz]$

$=(x+y+z)[x^2+y^2+z^2-xy-yz-xz]$

$=x^3+y^3+z^3-3xyz$ = L.H.S

(iii) L.H.S$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$............$2n$ factors

$=[(1+\omega)(1+\omega^2)][(1+\omega^4)(1+\omega^8)]$............ n factors

$=[(1+\omega)(1+\omega^2)][(1+\omega^3\cdot\omega)(1+\omega^6\cdot\omega^2)]$............ n factors

$=[(1+\omega)(1+\omega^2)][(1+\omega^3\cdot\omega)(1+(\omega^3)^2\cdot\omega^2)]$............ n factors

$= [(1+\omega)(1+\omega^2)][(1+1\cdot\omega)(1+(1)^2\cdot\omega^2)]$............ n factors

$= [(1+\omega)(1+\omega^2)][(1+\omega)(1+\omega^2)]$............ n factors

$= \left[(1+\omega)(1+\omega^2)\right]^n = \left[1+\omega+\omega^2+\omega^3\right]^n$

$= [0+1]^n \qquad \because\ 1+\omega+\omega^2 = 0\ ,\ \omega^3 = 1$

$= [1]^n \quad = 1 \quad = \text{R.H.S}$

Question # 4 (i)

Let $x^2 + x + 1 = 0$ ....(i)

Since ω is root of (i) therefore

$\omega^2 + \omega + 1 = 0$...........$(ii)$

To prove ω^2 is root of (i)

Consider $\left(\omega^2\right)^2 + \omega^2 + 1 = \omega^4 + 2\omega^2 + 1 - \omega^2$

$= \left(\omega^2+1\right)^2 - \omega^2 = \left(\omega^2+1+\omega\right)\left(\omega^2+1-\omega\right)$

$= (0)\left(\omega^2+1-\omega\right) \qquad$ from (i)

$\Rightarrow \left(\omega^2\right)^2 + \omega^2 + 1 = 0$ ....(iii)

$\Rightarrow \omega^2$ is the root of the equation (i).

Now subtracting (ii) from (iii)

$$\begin{array}{l} \left(\omega^2\right)^2 + \omega^2 \qquad + 1 = 0 \\ \qquad\quad \underset{-}{\omega^2} \underset{-}{+\omega} \underset{-}{+1} = 0 \\ \hline \omega^4 \qquad\quad -\omega \qquad = 0 \end{array}$$

$\Rightarrow \omega\left(\omega^3 - 1\right) = 0$

$\Rightarrow \omega^3 - 1 = 0 \qquad$ as $\quad \omega \neq 0$

$\Rightarrow \boxed{\omega^3 = 1}$

Question # 5

Let x be a cube root of -1 then

$x = (-1)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = -1$

$\Rightarrow x^3 + 1 = 0 \qquad \Rightarrow (x)^3 + (1)^3 = 0$

$\Rightarrow (x+1)(x^2 - x + 1) = 0$

$\Rightarrow x + 1 = 0$ or $x^2 - x + 1 = 0$

$\Rightarrow x = -1$ or $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = \dfrac{1+\sqrt{-3}}{2}$ or $x = \dfrac{1-\sqrt{-3}}{2}$

$\Rightarrow x = \dfrac{1+\sqrt{3}i}{2}$ or $x = \dfrac{1-\sqrt{3}i}{2}$

Hence complex cube root of -1 are $\dfrac{1+\sqrt{3}i}{2}$ and $\dfrac{1-\sqrt{3}i}{2}$.

Question # 6

Since 2ω and $2\omega^2$ are roots of required equation, therefore

$$(x-2\omega)\left(x-2\omega^2\right)=0$$

$$\Rightarrow\ x^2-2\omega x-2\omega^2x+4\omega^3=0$$

$$\Rightarrow\ x^2-2x(\omega+\omega^2)+4(1)=0 \qquad \because\ \omega^3=1$$

$$\Rightarrow\ x^2-2x(-1)+4=0 \qquad \because\ 1+\omega+\omega^2=0$$

$$\Rightarrow\ x^2+2x+4=0$$

is the required equation.

Question # 7

(i) Let x be a fourth root of 16 then

$$x=(16)^{\frac{1}{4}} \quad \Rightarrow\ x^4=16$$

$$\Rightarrow\ x^4-16=0 \quad \Rightarrow\ \left(x^2\right)^2-(4)^2=0$$

$$\Rightarrow\ \left(x^2+4\right)\left(x^2-4\right)=0$$

$$\Rightarrow\ x^2+4=0 \quad \text{or} \quad x^2-4=0$$

$$\Rightarrow\ x^2=-4 \quad \text{or} \quad x^2=4$$

$$\Rightarrow\ x=\pm\sqrt{-4} \quad \text{or} \quad x=\pm\sqrt{4}$$

$$\Rightarrow\ x=\pm 2i \quad \text{or} \quad x=\pm 2$$

Hence the four fourth root of 16 are $2,-2,2i,-2i$.

(ii) *Do yourself as above.* ***Hint:*** $81=(9)^2$

(iii) Let x be a fourth root of 625 then

$$x=(625)^{\frac{1}{4}} \quad \Rightarrow\ x^4=625$$

$$\Rightarrow\ x^4-625=0 \quad \Rightarrow\ \left(x^2\right)^2-(25)^2=0$$

$$\Rightarrow\ \left(x^2+25\right)\left(x^2-25\right)=0$$

$$\Rightarrow\ x^2+25=0 \quad \text{or} \quad x^2-25=0$$

$$\Rightarrow\ x^2=-25 \quad \text{or} \quad x^2=25$$

$$\Rightarrow\ x=\pm\sqrt{-25} \quad \text{or} \quad x=\pm\sqrt{25}$$

$$\Rightarrow\ x=\pm 5i \quad \text{or} \quad x=\pm 5$$

Hence the four fourth root of 625 are $5,-5,5i,-5i$.

Question # 8

(i) $2x^4-32=0$

$$\Rightarrow\ 2(x^4-16)=0 \quad \Rightarrow\ x^4-16=0$$

Now do you as in Question # 7 (i)

(ii) $3y^5-243y=0$

$$\Rightarrow\ 3y\left(y^4-81\right)=0$$

$$\Rightarrow\ 3y=0 \quad \text{or} \quad y^4-81=0$$

$$\Rightarrow \quad y = 0 \quad \text{or} \quad \left(y^2\right)^2 - (9)^2 = 0$$
$$\Rightarrow \quad \left(y^2 + 9\right)\left(y^2 - 9\right) = 0$$
$$\Rightarrow \quad y^2 + 9 = 0 \quad \text{or} \quad y^2 - 9 = 0$$
$$\Rightarrow \quad y^2 = -9 \quad \text{or} \quad y^2 = 9$$
$$\Rightarrow \quad y = \pm\sqrt{-9} \quad \text{or} \quad y = \pm\sqrt{9}$$
$$\Rightarrow \quad y = \pm 3i \quad \text{or} \quad y = \pm 3$$

Hence S.Set $= \left\{0, \pm 3, \pm 3i\right\}$

(iii) $x^3 + x^2 + x + 1 = 0$

$$\Rightarrow \quad x^2(x+1) + 1(x+1) = 0$$
$$\Rightarrow \quad (x+1)(x^2+1) = 0$$
$$\Rightarrow \quad x + 1 = 0 \quad \text{or} \quad x^2 + 1 = 0$$
$$\Rightarrow \quad x = -1 \quad \text{or} \quad x^2 = -1 \quad \Rightarrow \quad x = \pm i$$

Hence S.Set $= \{-1, \pm i\}$

(iv) $5x^5 - 5x = 0$

$$\Rightarrow \quad 5x(x^4 - 1) = 0$$
$$\Rightarrow \quad 5x = 0 \quad \text{or} \quad x^4 - 1 = 0$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad \left(x^2\right)^2 - (1)^2 = 0$$
$$\Rightarrow \quad \left(x^2 + 1\right)\left(x^2 - 1\right) = 0$$
$$\Rightarrow \quad x^2 + 1 = 0 \quad \text{or} \quad x^2 - 1 = 0$$
$$\Rightarrow \quad x^2 = -1 \quad \text{or} \quad x^2 = 1$$
$$\Rightarrow \quad x = \pm i \quad \text{or} \quad x = \pm 1$$

Hence S.Set $= \left\{0, \pm 1, \pm i\right\}$
