Exercise 3.4 (Solutions)

TEXTBOOK OF ALGEBRA AND TRIGONOMETRY FOR CLASS XI

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EXERCISE 3.4

1. $A=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}$ $B=\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$

Show That $(A+B)$ is Symmetric

Sol. $A+B=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}+\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$

$=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}$

$(A+B)^t=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}=A+B$

Hence $(A+B)$ is symmetric Matrix

2. If $A=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}$

Then show That

i) $A+A^t$ is symmetric

Sol. $A+A^t=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}+\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$

$=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}$

$(A+A^t)^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}=A+A^t$

$\Rightarrow (A+A^t)$ is symmetric

ii) $A-A^t=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}-\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$

$=\begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}$

$(A-A^t)^t=\begin{bmatrix}0&1&-1\\-1&0&4\\1&-4&0\end{bmatrix}=-\begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}$

$=-(A-A^t)$

$\Rightarrow$ $A-A^t$ is skew Symmetric

3 i) If A is square matrix of order 3 Show That $A+A^t$ is symmetric

Sol. $(A+A^t)^t=A^t+(A^t)^t$

$=A^t+A=A+A^t$

$\Rightarrow$ $A+A^t$ is symmetric

ii) $A-A^t$ is skew Symmetric

$(A-A^t)^t=A^t-(A^t)^t=A^t-A$

$=-(A-A^t)$

$\Rightarrow$ $A-A^t$ is skew Symmetric

4. If the matrices A and B are symmetric matrix $AB=BA$ show that AB is symmetric

Sol $A^t=A$ $B^t=B$

To prove $(AB)^t=AB$

L.H.S $(AB)^t=B^tA^t=BA=AB=$ R.H.S

Hence AB is symmetric

5. Show that AA^t and A^tA are Symmetric for any matrix of order 2×3

Sol. let $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$

$AA^t=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}\begin{bmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{bmatrix}$

$=\begin{bmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}\\a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2\end{bmatrix}$

$(AA^t)^t=\begin{bmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13}\\a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2\end{bmatrix}$

$=AA^t$

Hence AA^t is symmetrix

$A^tA=\begin{bmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$

$=\begin{bmatrix}a_{11}^2+a_{21}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23}\\a_{21}a_{11}+a_{22}a_{21} & a_{12}^2+a_{22}^2 & a_{12}a_{13}+a_{22}a_{23}\\a_{11}a_{13}+a_{21}a_{23} & a_{12}a_{13}+a_{22}a_{23} & a_{13}^2+a_{23}^2\end{bmatrix}$

$(A^tA)^t=\begin{bmatrix}a_{11}^2+a_{21}^2 & a_{21}a_{11}+a_{22}a_{21} & a_{13}a_{11}+a_{23}a_{21}\\a_{11}a_{12}+a_{21}a_{22} & a_{12}^2+a_{22}^2 & a_{13}a_{12}+a_{23}a_{22}\\a_{11}a_{13}+a_{21}a_{23} & a_{12}a_{13}+a_{22}a_{23} & a_{13}^2+a_{23}^2\end{bmatrix}$

$=A^tA$

Thus A^tA is symmetric

6. i) If $A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$

Show that $A+(\bar{A})^t$ is Hermitian

To prove $\left(\overline{A+(\bar{A})^t}\right)^t = A+(\bar{A})^t$

Sol. $\bar{A}=\begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$ $\quad (\bar{A})^t=\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$$A+(\bar{A})^t=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}=\begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\overline{A+(\bar{A})^t}=\begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t=\begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$= A+(\bar{A})^t$$

Thus $A+(\bar{A})^t$ is Hermitian

ii) $A-(\bar{A})^t$ is skew Hermitian

To prove $\left(\overline{A-(\bar{A})^t}\right)^t = -\left(A-(\bar{A})^t\right)$

$$A-(\bar{A})^t=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}-\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}=\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\overline{A-(\bar{A})^t}=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$$

$$\left(\overline{A-(\bar{A})^t}\right)^t=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$$

$$=-\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}=-\left(A-(\bar{A})^t\right)$$

Thus $A-(\bar{A})^t$ is skew Hermitian.

7. If A is symmetric or skew symmetric show that A^2 symmetric

Sol. If A is symmetric $A^t=A$

To prove $(A^2)^t = A^2$

$(A^2)^t=(A\cdot A)^t=A^tA^t=AA=A^2$

Thus A^2 is symmetric

If A is skew symmetric $A^t=-A$

$(A^2)^t=(A\cdot A)^t=A^tA^t=(-A)(-A)=A^2$

Thus A^2 is symmetric

8. If $A=\begin{pmatrix} 1 \\ 1-i \\ -i \end{pmatrix}$ $\quad A(\bar{A})^t=?$

Sol $\bar{A}=\begin{pmatrix} 1 \\ 1+i \\ i \end{pmatrix}$ $\quad (\bar{A})^t=\begin{pmatrix} 1 & 1+i & i \end{pmatrix}$

$$A(\bar{A})^t=\begin{pmatrix} 1 \\ 1-i \\ -i \end{pmatrix}\begin{pmatrix} 1 & 1+i & i \end{pmatrix}$$

$$A(\bar{A})^t=\begin{pmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -1-i^2 \\ i & i-i^2 & -i^2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 1-i & -i \\ 1+i & 2 & -i+1 \\ i & 1+i & 1 \end{pmatrix}$$

9. Find the inverse by matrices, Row, Column

i) $A=\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$$\text{Cofactors of } A=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$A_{11}=(-1)^{1+1}\begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix}=(-1)^2(-4-0)=-4$

$A_{12}=(-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix}=(-1)^3(0+0)=0$

$A_{13}=(-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}=(-1)^4(0-4)=-4$

$A_{21}=(-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix}=(-1)^3(4-6)=2$

$A_{22}=(-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix}=(-1)^4(2-6)=-4$

$A_{23}=(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix}=(-1)^5(-2+4)=-2$

$A_{31}=(-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix}=(-1)^4(0-6)=-6$

$A_{32}=(-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}=(-1)^5(0+0)=0$

$A_{33}=(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}=(-1)^6(-2-0)=-2$

$$\text{Cofactors of } A=\begin{pmatrix} -4 & 0 & -4 \\ 2 & -4 & -2 \\ -6 & 0 & -2 \end{pmatrix}$$

$$\text{Adj}A=(\text{Cofactors of }A)^t=\begin{pmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{pmatrix}$$

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

$=1(-4)+2(0)+(-3)(-4)$

$=-4+0+12=8$

$$A^{-1}=\frac{1}{|A|}\text{Adj}A=\frac{1}{8}\begin{pmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{pmatrix}=\begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

Find Inverse of A by Column operation

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ \hline 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By } C_2 - 2C_1 \to C_2', \; C_3 + 3C_1 \to C_3'$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -4 \\ \hline 1 & 1 & 3 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{By } -\tfrac{1}{2}C_2 \to C_2' \quad \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \\ \hline 1 & 1 & -3/4 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/4 \end{bmatrix} \text{By } -\tfrac{1}{4}C_1 \to C_1'$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix} \quad \text{By } C_1 + 2C_3 \to C_1', \; C_2 + C_3 \to C_2'$$

Hence the inverse of matrix A is $\begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$

10. Find the rank of the matrices

(i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$

Sol. $\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix}$ By $R_2 - 2R_1 \to R_2'$, $R_3 - 3R_1 \to R_3'$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 4 & -1 & -3 \end{bmatrix}$ By $-\frac{1}{4}R_2 \to R_2'$, $\frac{1}{2}R_3 \to R_3'$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ By $R_1 + R_2 \to R_1'$, $R_3 - 4R_2 \to R_3'$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ By $-\frac{1}{4}R_3 \to R_3'$

Hence the rank of given matrix is 3

(ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \begin{array}{l} R_2-2R_1 \to R_2' \\ R_3-R_1 \to R_3' \\ R_4-3R_1 \to R_4' \end{array} \quad \underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} R_2-R_3 \to R_2' \\ \frac{1}{5}R_3 \to R_3' \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{By } \begin{array}{l} R_1+4R_2 \to R_1', \quad R_4-R_2 \to R_4' \\ R_3-2R_2 \to R_3 \end{array}$$

Hence The rank of given matrix is 2

(iii) $$\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$$

Sol. $$\begin{bmatrix} 1 & -1 & 3 & 0 & -1 \\ 3 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \text{By } R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix} \text{By } \begin{array}{l} R_2-3R_1 \to R_2' \\ R_3-2R_1 \to R_3' \\ R_4-2R_1 \to R_4' \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix} \text{By } R_2 \leftrightarrow R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 6 & 30 & 54 \end{bmatrix} \text{By } \begin{array}{l} R_1-2R_2 \to R_1' \\ R_3+R_2 \to R_3' \\ R_4+7R_2 \to R_4' \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 1 & 5 & 9 \end{bmatrix} \quad \frac{1}{6}R_4 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6 & 11 & 16 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & -19 & -38 \end{bmatrix} \begin{array}{l} R_1+R_3 \to R_1' \\ R_4-6R_3 \to R_4' \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad -\frac{1}{19}R_4 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1+4R_4 \to R_1' \\ R_2-3R_4 \to R_2' \\ R_3-5R_4 \to R_3' \end{array}$$

Hence the rank of given matrix is 4.

9 (ii) Let $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

Sol. Cofactors of $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$

$B_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = (-1)^2(-2-0) = 1(-2) = -2$

$B_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^3(0-3) = -1(-3) = 3$

$B_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (-1)^4(0+1) = 1(1) = 1$

$B_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = (-1)^3(4+0) = -1(4) = -4$

$B_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = (-1)^4(2+1) = 1(3) = 3$

$B_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^5(0-2) = -1(-2) = 2$

$B_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = (-1)^4(6-1) = 1(5) = 5$

$B_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = (-1)^5(3+0) = -1(3) = -3$

$B_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = (-1)^6(-1-0) = 1(-1) = -1$

Cofactors of $B = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 3 & 2 \\ 5 & -3 & -1 \end{bmatrix}$

$Adj\,B = (\text{Cofactors of } B)^t = \begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}$

$|B| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 1(-2-0) - 2(0-3) - 1(0+1)$

$= 1(-2) - 2(-3) - 1(1) = -2 + 6 - 1 = 3$

$B^{-1} = \frac{1}{|B|} Adj\,B = \frac{1}{3}\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -2/3 & -4/3 & 5/3 \\ 1 & 1 & -1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$

Now B^{-1} by Row operations

$B = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right] \quad R_3 - R_1 \to R_3'$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right] \quad -R_2 \to R_2'$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right] \quad R_1 - 2R_2 \to R_1',\ R_3 + 2R_2 \to R_3'$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right] \quad -\frac{1}{3}R_3 \to R_3'$