Question # 1

Operation $\oplus$ performed on the two-member set $G=\{0,1\}$ is shown in the adjoining table. Answers the questions:

(i) Name the identity element if it exists?
(ii) What is the inverse of 1?
(iii)Is the set G, under the given operation a group? Abelian and non-abelian?

$\oplus$	0	1
0	1	1
1	1	0

Solutions

i) From the given table we have

$$0+0=0 \quad \text{and} \quad 0+1=1$$

This show that 0 is the identity element.

ii) Since $1+1=0$ (identity element) so the inverse of 1 is 1.

iii) It is clear from table that element of the given set satisfy closure law, associative law, identity law and inverse law thus given set is group under $\oplus$.

Also it satisfies commutative law so it is an abelian group.

Question # 2

The operation $\oplus$ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table, shown that the set is an Abelian group?

Solution

Suppose $G=\{0,1,2,3\}$

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

i) The given table show that each element of the table is a member of G thus closure law holds.

ii) $\oplus$ is associative in G.

iii) Table show that 0 is identity element w.r.t. $\oplus$.

iv) Since $0+0=0$, $1+3=0$, $2+2=0$, $3+1=0$

$\Rightarrow$ $0^{-1}=0$, $1^{-1}=3$, $2^{-1}=2$, $3^{-1}=1$

v) As the table is symmetric w.r.t. to the principal diagonal. Hence commutative law holds.

Question # 3

For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation. From above table solve these (i-v) options.

Solution

(i) As $0\in\mathbb{Q}$, multiplicative inverse of 0 in not in set $\mathbb{Q}$. Therefore the set of rational number is not a group w.r.t to "$\cdot$".

(ii) a- Closure property holds in $\mathbb{Q}$ under $+$ because sum of two rational number is also rational.

b- Associative property holds in $\mathbb{Q}$ under addition.

c- $0\in\mathbb{Q}$ is an identity element.

d- If $a\in\mathbb{Q}$ then additive inverse $-a\in\mathbb{Q}$ such that $a+(-a)=(-a)+a=0$.
Therefore the set of rational number is group under addition.

(iii) *a*- Since for $a,b\in\mathbb{Q}^+$, $ab\in\mathbb{Q}^+$ thus closure law holds.
b- For $a,b,c\in\mathbb{Q}$, $a(bc)=(ab)c$ thus associative law holds.
c- Since $1\in\mathbb{Q}^+$ such that for $a\in\mathbb{Q}^+$, $a\times1=1\times a=a$. Hence 1 is the identity element.
d- For $a\in\mathbb{Q}^+$, $\frac{1}{a}\in\mathbb{Q}^+$ such that $a\times\frac{1}{a}=\frac{1}{a}\times a=1$. Thus inverse of a is $\frac{1}{a}$.
Hence $\mathbb{Q}^+$ is group under addition.

(iv) Since $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,...........\}$
a- Since sum of integers is an integer therefore for $a,b\in\mathbb{Z}$, $a+b\in\mathbb{Z}$.
b- Since $a+(b+c)=(a+b)+c$ thus associative law holds in $\mathbb{Z}$.
c- Since $0\in\mathbb{Z}$ such that for $a\in\mathbb{Z}$, $a+0=0+a=\mathbb{Z}$. Thus 0 an identity element.
d- For $a\in\mathbb{Z}$, $-a\in\mathbb{Z}$ such that $a+(-a)=(-a)+a=0$. Thus inverse of a is $-a$.

(v) Since $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,...........\}$
For any $a\in\mathbb{Z}$ the multiplicative inverse of a is $\frac{1}{a}\notin\mathbb{Z}$. Hence $\mathbb{Z}$ is not a group under multiplication.

Question # 4
Show that the adjoining table represents the sums of the elements of the set $\{E,O\}$.
What is the identity element of this set ? Show that this set is abelian group..
Solution

$\oplus$	E	O
E	E	O
O	O	E

As E + E = E, E + O = O, O + O = E
Thus the table represents the sums of the elements of set $\{E,O\}$.
The identity element of the set is E because
$$E+E=E+E=E \quad \& \quad E+O=O+E=E\ .$$
i) From the table each element belong to the set $\{E,O\}$.
Hence closure law is satisfied.
ii) $\oplus$ is associative in $\{E,O\}$
iii) E is the identity element of w.r.t to $\oplus$
iv) As $O+O=E$ and $E+E=E$, thus inverse of O is O and inverse of E is E.
v) As the table is symmetric about the principle diagonal therefore $\oplus$ is commutative.
Hence $\{E,O\}$ is abelian group under $\oplus$.

Question # 5
Show that the set $\{1,\omega,\omega^2\}$, when $\omega^3=1$ is an abelian group w.r.t. ordinary multiplication.
Solution

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Suppose $G=\{1,\omega,\omega^2\}$

i) A table show that all the entries belong to G.
ii) Associative law holds in G w.r.t. multiplication.
e.g. $1\times(\omega\times\omega^2)=1\times1=1$
$(1\times\omega)\times\omega^2=\omega\times\omega^2=1$
iii) Since $1\times1=1,\ 1\times\omega=\omega\times1=\omega,\ 1\times\omega^2=\omega^2\times1=\omega^2$
Thus 1 is an identity element in G.
iv) Since $1\times1=1\times1=1$, $\omega\times\omega^2=\omega^2\times\omega=1$, $\omega^2\times\omega=\omega\times\omega^2=1$
therefore inverse of 1 is 1, inverse of ω is ω^2, inverse of ω^2 is ω.
v) As table is symmetric about principle diagonal therefore commutative law holds in G.
Hence G is an abelian group under multiplication.

Question # 6
If G is a group under the operation $*$ and $a,b\in G$, find the solutions of the equations: $a*x=b$, $x*a=b$
Solution
Given that G is a group under the operation $*$ and $a,b\in G$ such that
$$a*x=b$$
As $a\in G$ and G is group so $a^{-1}\in G$ such that
$$a^{-1}*(a*x)=a^{-1}*b$$
$\Rightarrow (a^{-1}*a)*x=a^{-1}*b$ as associative law hold in G.
$\Rightarrow e*x=a^{-1}*b$ by inverse law.
$\Rightarrow x=a^{-1}*b$ by identity law.

And for
$$x*a=b$$
$\Rightarrow (x*a)*a^{-1}=b*a^{-1}$ For $a\in G,\ a^{-1}\in G$
$\Rightarrow x*(a*a^{-1})=b*a^{-1}$ as associative law hold in G.
$\Rightarrow x*e=b*a^{-1}$ by inverse law.
$\Rightarrow x=b*a^{-1}$ by identity law.

Question # 7
Show that the set consisting of elements of the form $a+\sqrt{3}b$ (a,b being rational), is an abelian group w.r.t. addition.
Solution
Consider $G=\{a+\sqrt{3}b \mid a,b\in\mathbb{Q}\}$
i) Let $a+\sqrt{3}b,\ c+\sqrt{3}d\in G$, where a, b, c & d are rational.
$$\left(a+\sqrt{3}b\right)+\left(c+\sqrt{3}d\right)=(a+c)+\sqrt{3}(b+d)=a'+\sqrt{3}b'\in G$$
where $a'=a+c$ and $b'=b+d$ are rational as sum of rational is rational.
Thus closure law holds in G under addition.
ii) For $a+\sqrt{3}b,\ c+\sqrt{3}d,\ e+\sqrt{3}f\in G$

$$(a+\sqrt{3}b)+\left((c+\sqrt{3}d)+(e+\sqrt{3}f)\right)=(a+\sqrt{3}b)+\left((c+e)+\sqrt{3}(d+f)\right)$$
$$=\left(a+(c+e)\right)+\sqrt{3}\left(b+(d+f)\right)$$
$$=\left((a+c)+e\right)+\sqrt{3}\left((b+d)+f\right)$$

As associative law hold in $\mathbb{Q}$

$$=\left((a+c)+\sqrt{3}(b+d)\right)+(e+\sqrt{3}f)$$
$$=\left((a+\sqrt{3}b)+(c+\sqrt{3}d)\right)+(e+\sqrt{3}f)$$

Thus associative law hold in G under addition.

iii) $0+\sqrt{3}\cdot 0\in G$ as 0 is a rational such that for any $a+\sqrt{3}b\in G$

$$(a+\sqrt{3}b)+(0+\sqrt{3}\cdot 0)=(a+0)+\sqrt{3}(b+0)=a+\sqrt{3}b$$

And $(0+\sqrt{3}\cdot 0)+(a+\sqrt{3}b)=(0+a)+\sqrt{3}(0+b)=a+\sqrt{3}b$

Thus $0+\sqrt{3}\cdot 0$ is an identity element in G.

iv) For $a+\sqrt{3}b\in G$ where a & b are rational there exit rational $-a$ & $-b$ such that

$$(a+\sqrt{3}b)+\left((-a)+\sqrt{3}(-b)\right)=\left(a+(-a)\right)+\sqrt{3}\left(b+(-b)\right)=0+\sqrt{3}\cdot 0$$

& $\left((-a)+\sqrt{3}(-b)\right)+(a+\sqrt{3}b)=\left((-a)+a\right)+\sqrt{3}\left((-b)+b\right)=0+\sqrt{3}\cdot 0$

Thus inverse of $a+\sqrt{3}b$ is $(-a)+\sqrt{3}(-b)$ exists in G.

v) For $a+\sqrt{3}b,\ c+\sqrt{3}d\in G$

$$\left(a+\sqrt{3}b\right)+\left(c+\sqrt{3}d\right)=(a+c)+\sqrt{3}(b+d)$$
$$=(c+a)+\sqrt{3}(d+b)$$ As commutative law hold in $\mathbb{Q}$.
$$=(c+d\sqrt{3})+(a+\sqrt{3}b)$$

Thus Commutative law holds in G under addition.
And hence G is an abelian group under addition.

Question 8

Determine whether $\left(P(S),*\right)$, where $*$ stands for intersection is a semi group, a monoid or neither. If it is a monoid, specify its identity.

Solution

Let $A,B\in P(S)$ where A & B are subsets of S.
As intersection of two subsets of S is subset of S.
Therefore $A*B=A\cap B\in P(S)$. Thus closure law holds in $P(S)$.

For $A,B,C\in P(S)$

$$A*(B*C)=A\cap(B\cap C)=(A\cap B)\cap C=(A*B)*C$$

Thus associative law holds and $P(S)$.
And hence $\left(P(S),*\right)$ is a semi-group.
For $A\in P(S)$ where A is a subset of S we have $S\in P(S)$ such that

$$A\cap S=S\cap A=A\ .$$

Thus S is an identity element in $P(S)$. And hence $\left(P(S),*\right)$ is a monoid.

Question 9
Complete the following table to obtain a semi-group under $*$

$*$	a	b	c
a	c	a	b
b	a	b	c
c	x_1	x_2	a

Solution
Let x_1 and x_2 be the required elements.
By associative law

$$(a*a)*a = a*(a*a)$$
$$\Rightarrow \quad c*a = a*c$$
$$\Rightarrow \quad x_1 = b$$

Now again by associative law

$$(a*a)*b = a*(a*b)$$
$$\Rightarrow \quad c*b = a*a \quad \Rightarrow \quad x_2 = c$$

Question 10
Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.
Solution Let G be the all non-singular 2×2 matrices over the real field.
i) Let $A, B \in G$ then $A_{2\times2}\times B_{2\times2} = C_{2\times2} \in G$
Thus closure law holds in G under multiplication.
ii) Associative law in matrices of same order under multiplication holds.
therefore for $A, B, C \in G$

$$A\times(B\times C) = (A\times B)\times C$$

iii) $I_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a non-singular matrix such that

$$A_{2\times2}\times I_{2\times2} = I_{2\times2}\times A_{2\times2} = A_{2\times2}$$

Thus $I_{2\times2}$ is an identity element in G
iv) Since inverse of non-singular square matrix exists,
therefore for $A\in G$ there exist $A^{-1}\in G$ such that $AA^{-1} = A^{-1}A = I$.
v) As we know for any two matrices $A, B \in G$, $AB \neq BA$ in general.
Therefore commutative law does not holds in G under multiplication.
Hence the set of all 2×2 non-singular matrices over a real field is a non-abelian group under multiplication.

Book: Exercise 2.8 (Page 78)
Text Book of Algebra and Trigonometry Class XI
Punjab Textbook Board, Lahore.