

Question # 1

Prove that:

$$(1) \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{7}{25} = \cos^{-1} \frac{253}{325}$$

Solution L.H.S = $\sin^{-1} \frac{5}{13} + \sin^{-1} \frac{7}{25}$

$$\begin{aligned} \because \sin^{-1} A + \sin^{-1} B &= \sin^{-1} \left(A\sqrt{1-B^2} + B\sqrt{1-A^2} \right) \\ &= \sin^{-1} \left(\frac{5}{13} \sqrt{1 - \left(\frac{7}{25}\right)^2} + \frac{7}{25} \sqrt{1 - \left(\frac{5}{13}\right)^2} \right) \\ &= \sin^{-1} \left(\frac{5}{13} \sqrt{1 - \frac{49}{625}} + \frac{7}{25} \sqrt{1 - \frac{25}{169}} \right) \\ &= \sin^{-1} \left(\frac{5}{13} \sqrt{\frac{576}{625}} + \frac{7}{25} \sqrt{\frac{144}{169}} \right) = \sin^{-1} \left(\frac{5}{13} \left(\frac{24}{25}\right) + \frac{7}{25} \left(\frac{12}{13}\right) \right) \\ &= \sin^{-1} \left(\frac{120}{325} + \frac{84}{325} \right) = \sin^{-1} \left(\frac{204}{325} \right) \\ &= \frac{\pi}{2} - \cos^{-1} \left(\frac{204}{325} \right) \qquad \because \sin^{-1} \theta = \frac{\pi}{2} - \cos^{-1} \theta \\ &= \cos^{-1} (0) - \cos^{-1} \left(\frac{204}{325} \right) \qquad \because \frac{\pi}{2} = \cos^{-1}(0) \\ \because \cos^{-1} A - \cos^{-1} B &= \cos^{-1} \left(AB + \sqrt{(1-A^2)(1-B^2)} \right) \\ &= \cos^{-1} \left((0) \left(\frac{204}{325} \right) + \sqrt{(1-(0)^2) \left(1 - \left(\frac{204}{325} \right)^2 \right)} \right) \\ &= \cos^{-1} \left(0 + \sqrt{(1-0) \left(1 - \frac{41616}{105625} \right)} \right) = \cos^{-1} \left(\sqrt{(1) \left(\frac{64009}{105625} \right)} \right) \\ &= \cos^{-1} \left(\sqrt{\frac{64009}{105625}} \right) = \cos^{-1} \frac{253}{325} = \text{R.H.S} \end{aligned}$$

Question # 2

Prove that: $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{5} = \tan^{-1} \left(\frac{9}{19} \right)$

Solution L.H.S = $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{5}$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{\frac{1}{4} + \frac{1}{5}}{1 - \left(\frac{1}{4}\right)\left(\frac{1}{5}\right)} \right) \quad \because \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left(\frac{A + B}{1 - AB} \right) \\
 &= \tan^{-1} \left(\frac{\frac{9}{20}}{1 - \frac{1}{20}} \right) = \tan^{-1} \left(\frac{\frac{9}{20}}{\frac{19}{20}} \right) \\
 &= \tan^{-1} \left(\frac{9}{19} \right) = \text{R.H.S}
 \end{aligned}$$

Question # 3

Prove that:

$$2 \tan^{-1} \frac{2}{3} = \sin^{-1} \frac{12}{13}$$

Solution

Suppose

$$\alpha = \sin^{-1} \frac{12}{13} \dots\dots\dots (i)$$

$$\Rightarrow \sin \alpha = \frac{12}{13}$$

$$\text{Now } \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\text{Now } \tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \sqrt{\frac{1 - \frac{5}{13}}{1 + \frac{5}{13}}} = \sqrt{\frac{\frac{8}{13}}{\frac{18}{13}}} = \sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

$$\Rightarrow \frac{\alpha}{2} = \tan^{-1} \left(\frac{2}{3} \right) \Rightarrow \alpha = 2 \tan^{-1} \frac{2}{3} \dots\dots\dots (ii)$$

from (i) and (ii)

$$2 \tan^{-1} \frac{2}{3} = \sin^{-1} \frac{12}{13} \quad \text{proved.}$$

Question # 4

Prove that:

$$\tan^{-1} \frac{120}{119} = 2 \cos^{-1} \frac{12}{13}$$

Solution

Suppose

$$\alpha = \tan^{-1} \frac{120}{119} \dots\dots\dots (i)$$

$$\Rightarrow \tan \alpha = \frac{120}{119}$$

$$\text{Now } \sec \alpha = \sqrt{1 + \tan^2 \alpha}$$

$$= \sqrt{1 + \left(\frac{120}{119}\right)^2} = \sqrt{1 + \frac{14400}{14161}}$$

$$= \sqrt{\frac{28561}{14161}} = \frac{169}{119}$$

So $\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{169/119} = \frac{119}{169}$

Now $\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}} = \sqrt{\frac{1 + 119/169}{2}} = \sqrt{\frac{288/169}{2}} = \sqrt{\frac{288}{2 \times 169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$$\Rightarrow \frac{\alpha}{2} = \cos^{-1} \frac{12}{13} \Rightarrow \alpha = 2 \cos^{-1} \frac{12}{13} \dots \dots \dots \text{(ii)}$$

From (i) and (ii)

$$\tan^{-1} \frac{120}{119} = 2 \cos^{-1} \frac{12}{13} \quad \text{proved.}$$

Question # 5

Prove that:

$$\sin^{-1} \frac{1}{\sqrt{5}} + \cot^{-1} 3 = \frac{\pi}{4}$$

Solution Suppose

$$\alpha = \sin^{-1} \frac{1}{\sqrt{5}} \dots \dots \dots \text{(i)}$$

$$\Rightarrow \sin \alpha = \frac{1}{\sqrt{5}}$$

Now $\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{1 - \frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

So $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{1/\sqrt{5}}{2/\sqrt{5}} = \frac{1}{2}$

$$\Rightarrow \alpha = \tan^{-1} \frac{1}{2} \dots \dots \dots \text{(ii)}$$

From (i) and (ii)

$$\sin^{-1} \frac{1}{\sqrt{5}} = \tan^{-1} \frac{1}{2}$$

Now $\cot^{-1} 3 = \tan^{-1} \frac{1}{3} \qquad \because \cot^{-1} x = \tan^{-1} \frac{1}{x}$

And L.H.S = $\sin^{-1} \frac{1}{\sqrt{5}} + \cot^{-1} 3$

$$\begin{aligned}
 &= \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \\
 &= \tan^{-1} \left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} \right) = \tan^{-1} \left(\frac{\frac{5}{6}}{1 - \frac{1}{6}} \right) \\
 &= \tan^{-1} \left(\frac{\frac{5}{6}}{\frac{5}{6}} \right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S} \quad \textit{proved.}
 \end{aligned}$$

Question # 6

Prove that:

$$\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} = \sin^{-1} \left(\frac{77}{85} \right)$$

Solution L.H.S = $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17}$

$$\begin{aligned}
 &= \sin^{-1} \left(\frac{3}{5} \sqrt{1 - \left(\frac{8}{17}\right)^2} + \frac{8}{17} \sqrt{1 - \left(\frac{3}{5}\right)^2} \right) \\
 &= \sin^{-1} \left(\frac{3}{5} \sqrt{1 - \frac{64}{289}} + \frac{8}{17} \sqrt{1 - \frac{9}{25}} \right) = \sin^{-1} \left(\frac{3}{5} \sqrt{\frac{225}{289}} + \frac{8}{17} \sqrt{\frac{16}{25}} \right) \\
 &= \sin^{-1} \left(\frac{3}{5} \left(\frac{15}{17} \right) + \frac{8}{17} \left(\frac{4}{5} \right) \right) = \sin^{-1} \left(\frac{45}{85} + \frac{32}{85} \right) \\
 &= \sin^{-1} \left(\frac{77}{85} \right) = \text{R.H.S} \quad \textit{proved.}
 \end{aligned}$$

Question # 7

Prove that:

$$\sin^{-1} \frac{77}{85} - \sin^{-1} \frac{3}{5} = \cos^{-1} \left(\frac{15}{17} \right)$$

Solution L.H.S = $\sin^{-1} \frac{77}{85} - \sin^{-1} \frac{3}{5}$

$$\begin{aligned}
 &= \left(\frac{\pi}{2} - \cos^{-1} \frac{77}{85} \right) - \left(\frac{\pi}{2} - \cos^{-1} \frac{3}{5} \right) \quad \because \sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \\
 &= \frac{\pi}{2} - \cos^{-1} \frac{77}{85} - \frac{\pi}{2} + \cos^{-1} \frac{3}{5} = \cos^{-1} \frac{3}{5} - \cos^{-1} \frac{77}{85} \\
 &= \cos^{-1} \left(\left(\frac{3}{5} \right) \left(\frac{77}{85} \right) + \sqrt{\left(1 - \left(\frac{3}{5} \right)^2 \right) \left(1 - \left(\frac{77}{85} \right)^2 \right)} \right) \\
 &= \cos^{-1} \left(\frac{231}{425} + \sqrt{\left(1 - \frac{9}{25} \right) \left(1 - \frac{5929}{7225} \right)} \right)
 \end{aligned}$$



$$\begin{aligned}
 &= \cos^{-1}\left(\frac{231}{425} + \sqrt{\left(\frac{16}{25}\right)\left(\frac{1296}{7225}\right)}\right) \\
 &= \cos^{-1}\left(\frac{231}{425} + \sqrt{\frac{20736}{180625}}\right) = \cos^{-1}\left(\frac{231}{425} + \frac{144}{425}\right) \\
 &= \cos^{-1}\left(\frac{375}{425}\right) = \cos^{-1}\left(\frac{15}{17}\right) = \text{L.H.S}
 \end{aligned}$$

Question # 8

Prove that:

$$\cos^{-1}\frac{63}{65} + 2 \tan^{-1}\frac{1}{5} = \sin^{-1}\left(\frac{3}{5}\right)$$

Solution Suppose $\alpha = \cos^{-1}\frac{63}{65}$ (i)

$$\Rightarrow \cos \alpha = \frac{63}{65}$$

Now $\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{63}{65}\right)^2} = \sqrt{1 - \frac{3969}{4225}} = \sqrt{\frac{256}{4225}} = \frac{16}{65}$

$$\Rightarrow \alpha = \sin^{-1}\left(\frac{16}{65}\right)$$
 (ii)

So from equation (i) and (ii)

$$\cos^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{16}{65}\right)$$

Now suppose $\beta = \tan^{-1}\frac{1}{5}$ (iii)

$$\Rightarrow \tan \beta = \frac{1}{5}$$

So $\sec \alpha = \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + \left(\frac{1}{5}\right)^2} = \sqrt{1 + \frac{1}{25}} = \sqrt{\frac{26}{25}} = \frac{\sqrt{26}}{5}$

So $\cos \beta = \frac{1}{\sec \beta} = \frac{1}{\frac{\sqrt{26}}{5}} = \frac{5}{\sqrt{26}}$

As $\frac{\sin \beta}{\cos \beta} = \tan \beta \Rightarrow \sin \beta = \tan \beta \cdot \cos \beta$

$$\Rightarrow \sin \beta = \left(\frac{1}{5}\right)\left(\frac{5}{\sqrt{26}}\right) = \frac{1}{\sqrt{26}}$$

$$\Rightarrow \beta = \sin^{-1}\frac{1}{\sqrt{26}}$$
 (iv)

From (iii) and (iv)

$$\begin{aligned} \tan^{-1} \frac{1}{5} &= \sin^{-1} \frac{1}{\sqrt{26}} \\ \text{Now L.H.S} &= \cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} \\ &= \sin^{-1} \frac{16}{65} + 2 \sin^{-1} \frac{1}{\sqrt{26}} = \sin^{-1} \frac{16}{65} + \left(\sin^{-1} \frac{1}{\sqrt{26}} + \sin^{-1} \frac{1}{\sqrt{26}} \right) \\ &= \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{1}{\sqrt{26}} \sqrt{1 - \left(\frac{1}{\sqrt{26}} \right)^2} + \frac{1}{\sqrt{26}} \sqrt{1 - \left(\frac{1}{\sqrt{26}} \right)^2} \right) \\ &= \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{1}{\sqrt{26}} \sqrt{1 - \frac{1}{26}} + \frac{1}{\sqrt{26}} \sqrt{1 - \frac{1}{26}} \right) \\ &= \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{1}{\sqrt{26}} \sqrt{\frac{25}{26}} + \frac{1}{\sqrt{26}} \sqrt{\frac{25}{26}} \right) \\ &= \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{1}{\sqrt{26}} \cdot \frac{5}{\sqrt{26}} + \frac{1}{\sqrt{26}} \cdot \frac{5}{\sqrt{26}} \right) \\ &= \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{5}{26} + \frac{5}{26} \right) = \sin^{-1} \frac{16}{65} + \sin^{-1} \left(\frac{5}{13} \right) \\ &= \sin^{-1} \left(\frac{16}{65} \sqrt{1 - \left(\frac{5}{13} \right)^2} + \frac{5}{13} \sqrt{1 - \left(\frac{16}{65} \right)^2} \right) \\ &= \sin^{-1} \left(\frac{16}{65} \sqrt{1 - \frac{25}{169}} + \frac{5}{13} \sqrt{1 - \frac{256}{4225}} \right) \\ &= \sin^{-1} \left(\frac{16}{65} \sqrt{1 - \frac{25}{169}} + \frac{5}{13} \sqrt{1 - \frac{256}{4225}} \right) = \sin^{-1} \left(\frac{16}{65} \sqrt{\frac{144}{169}} + \frac{5}{13} \sqrt{\frac{3969}{4225}} \right) \\ &= \sin^{-1} \left(\frac{16}{65} \sqrt{\frac{144}{169}} + \frac{5}{13} \sqrt{\frac{3969}{4225}} \right) = \sin^{-1} \left(\frac{16}{65} \left(\frac{12}{13} \right) + \frac{5}{13} \left(\frac{63}{65} \right) \right) \\ &= \sin^{-1} \left(\frac{192}{845} + \frac{315}{845} \right) = \sin^{-1} \left(\frac{3}{5} \right) = \text{R.H.S} \end{aligned}$$

Question # 9

Prove that:

$$\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19} = \frac{\pi}{4}$$

Solution L.H.S = $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19}$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \left(\frac{3}{4}\right)\left(\frac{3}{5}\right)} \right) - \tan^{-1} \frac{8}{19} \\
 &= \tan^{-1} \left(\frac{\frac{27}{20}}{1 - \frac{9}{20}} \right) - \tan^{-1} \frac{8}{19} = \tan^{-1} \left(\frac{\frac{27}{20}}{\frac{11}{20}} \right) - \tan^{-1} \frac{8}{19} \\
 &= \tan^{-1} \left(\frac{27}{11} \right) - \tan^{-1} \left(\frac{8}{19} \right) = \tan^{-1} \left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \left(\frac{27}{11}\right)\left(\frac{8}{19}\right)} \right) \\
 &= \tan^{-1} \left(\frac{\frac{425}{209}}{1 + \frac{216}{209}} \right) = \tan^{-1} \left(\frac{\frac{425}{209}}{1 + \frac{216}{209}} \right) \\
 &= \tan^{-1} \left(\frac{\frac{425}{209}}{\frac{425}{209}} \right) = \tan^{-1} (1) = \frac{\pi}{4} = \text{R.H.S proved.}
 \end{aligned}$$

Question # 10

Do Yourself

Question # 11

Prove that:

$$\tan^{-1} \frac{1}{11} + \tan^{-1} \frac{5}{6} = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2}$$

Solution L.H.S = $\tan^{-1} \frac{1}{11} + \tan^{-1} \frac{5}{6}$
 = *Solve this* (i)

R.H.S = $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2}$
 = *Solve this* (ii)

From (i) and (ii)

L.H.S = R.H.S

Question # 12

Prove that:

$$\tan^{-1} \frac{1}{11} + \tan^{-1} \frac{5}{6} = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2}$$

Solution L.H.S = $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$
 = $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$

Now do yourself as Question # 9.

Question # 13

Show that:

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}$$

Solution Suppose $y = \sin^{-1} x$
 $\Rightarrow \sin y = x$

Since $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$
 $\Rightarrow \cos(\sin^{-1} x) = \sqrt{1-x^2}$ *Proved*

Question # 14

Show that:

$$\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$$

Solution Suppose $y = \cos^{-1} x$

Then $\cos y = x$

Also $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

Now $\sin(2\cos^{-1} x) = \sin(2y)$
 $= 2\sin y \cdot \cos y$
 $= 2\sqrt{1-x^2} \cdot x$
 $= 2x\sqrt{1-x^2}$

Question # 15

Show that:

$$\cos(2\sin^{-1} x) = 1-2x^2$$

Solution Suppose $y = \sin^{-1} x \Rightarrow \sin y = x$

& $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

Now $\cos(2\sin^{-1} x) = \cos 2y$
 $= \cos^2 y - \sin^2 y$
 $= (\sqrt{1-x^2})^2 - x^2 = 1-x^2-x^2$
 $= 1-2x^2$ *Proved*

Question # 16

Show that:

$$\tan^{-1}(-x) = -\tan^{-1} x$$

Solution Suppose $y = \tan^{-1}(-x)$ (i)

$\Rightarrow \tan y = -x \Rightarrow -\tan y = x$

$\Rightarrow \tan(-y) = x \quad \because -\tan \theta = \tan(-\theta)$

$$\begin{aligned} \Rightarrow -y &= \tan^{-1} x \\ \Rightarrow y &= -\tan^{-1} x \dots\dots\dots (ii) \end{aligned}$$

From equation (i) and (ii)

$$\tan^{-1}(-x) = -\tan^{-1} x \qquad \text{Proved}$$

Question # 17

Do yourself as above

Question # 18

Show that:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

Solution Suppose $y = \pi - \cos^{-1} x \dots\dots\dots (i)$

$$\begin{aligned} \Rightarrow \pi - y &= \cos^{-1} x \Rightarrow \cos(\pi - y) = x \\ \Rightarrow \cos \pi \cos y + \sin \pi \sin y &= x \Rightarrow (-1)\cos y + (0)\sin y = x \\ \Rightarrow -\cos y + 0 &= x \Rightarrow -\cos y = x \\ \Rightarrow \cos y &= -x \qquad \Rightarrow y = \cos^{-1}(-x) \dots\dots\dots (ii) \end{aligned}$$

From (i) and (ii)

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Proved}$$

Question # 19

Show that:

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$$

Solution Suppose $y = \sin^{-1} x \Rightarrow \sin y = x$

Now $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

& $\tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1-x^2}}$

Now $\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1-x^2}} \qquad \text{proved}$

Question # 20

Given that $x = \sin^{-1} \frac{1}{2}$, find the values of the following trigonometric functions:

$\sin x, \cos x, \tan x, \cot x, \sec x$ and $\csc x$.

Solution Since $x = \sin^{-1} \frac{1}{2} \Rightarrow \sin x = \frac{1}{2}$

Now $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}},$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}},$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\frac{1}{2}} = 2$$

If you found any error, submit at www.megalecture@gmail.com

