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TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS III

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Domains and Ranges of Sine and Cosine Functions

Let us consider a unit circle with centre at origin O .

Let $P(x, y)$ be any point on the circle such that $\angle XOP = \theta$ is in standard position. Then

$$\sin \theta = \frac{y}{1} \Rightarrow \sin \theta = y$$

$$\cos \theta = \frac{x}{1} \Rightarrow \cos \theta = x$$

\Rightarrow Corresponding to any real number

θ , there is one and only one

value of x and y i.e., one and only one value for each $\sin \theta$ and $\cos \theta$.

Hence $\sin \theta$ and $\cos \theta$ are functions of θ .

$\therefore \sin \theta$ and $\cos \theta$ are defined for all $\theta \in \mathbb{R}$, the set of real numbers.

\therefore Domain of $\sin \theta = \mathbb{R}$

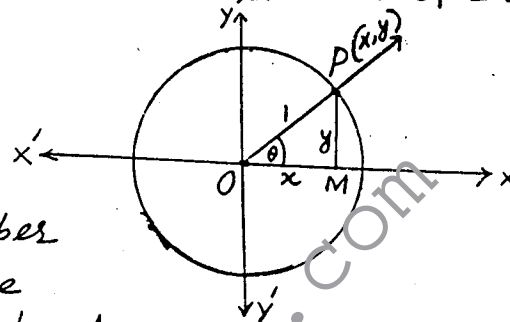
Domain of $\cos \theta = \mathbb{R}$

To find the range, we have

Since $P(x, y)$ is a point on the unit circle with centre at O .

$$\therefore -1 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1$$

$$\Rightarrow -1 \leq \cos \theta \leq 1 \quad \text{and} \quad -1 \leq \sin \theta \leq 1$$



Domains and Ranges of Tangent and Cotangent Functions:

From figure ; $\tan \theta = \frac{y}{x}$, $x \neq 0$

\Rightarrow terminal side \vec{OP} should not coincide with OY or OY' (i.e., Y -axis)

$$\Rightarrow \theta \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

$$\Rightarrow \theta \neq (2n+1) \frac{\pi}{2}, \quad n \in \mathbb{Z}$$

\therefore Domain of $\tan \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq (2n+1) \frac{\pi}{2}$, $n \in \mathbb{Z}$

and range of $\tan \theta = \mathbb{R}$

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$$\text{Now } \cot \theta = \frac{1}{\tan \theta} = \frac{x}{y}, \quad y \neq 0$$

⇒ terminal side \vec{OP} should not coincide with Ox or Ox' (i.e., x -axis).

$$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \dots$$

$$\Rightarrow \theta \neq n\pi, \quad n \in \mathbb{Z}$$

∴ Domain of $\cot \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi, n \in \mathbb{Z}$.

and range of $\cot \theta$ is \mathbb{R}

Domains and Ranges of Secant and Cosecant Functions.

$$\text{From fig. } \sec \theta = \frac{1}{\cos \theta} = \frac{1}{x}, \quad x \neq 0$$

⇒ terminal side \vec{OP} should not coincide with Oy or Oy' (i.e., y -axis)

$$\Rightarrow \theta \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

$$\Rightarrow \theta \neq (2n+1) \frac{\pi}{2}, \quad n \in \mathbb{Z}$$

∴ Domain of $\sec \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq (2n+1) \frac{\pi}{2}, n \in \mathbb{Z}$.

As $\sec \theta$ attains all real values except those between -1 and 1 .

∴ Range of $\sec \theta = \mathbb{R} - \{x \mid -1 < x < 1\}$

$$\text{Now } \operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{1}{y}, \quad y \neq 0$$

⇒ terminal side \vec{OP} should not coincide with Ox or Ox' (i.e., x -axis)

$$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \dots$$

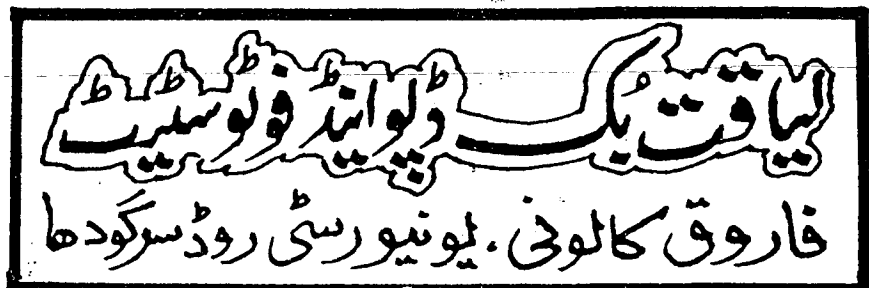
$$\Rightarrow \theta \neq n\pi, \quad n \in \mathbb{Z}$$

∴ Domain of $\operatorname{cosec} \theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi, n \in \mathbb{Z}$

As $\operatorname{cosec} \theta$ attains all real values except those between -1 and 1 .

∴ Range of $\operatorname{cosec} \theta = \mathbb{R} - \{x \mid -1 < x < 1\}$

Now summarizing the above results in the form of a table as:



[3]

Function	Domain	Range
$y = \sin x$	\mathbb{R}	$-1 \leq y \leq 1$
$y = \cos x$	\mathbb{R}	$-1 \leq y \leq 1$
$y = \tan x$	$x \in \mathbb{R}$ but $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	\mathbb{R}
$y = \cot x$	$x \in \mathbb{R}$ but $x \neq n\pi, n \in \mathbb{Z}$	\mathbb{R}
$y = \sec x$	$x \in \mathbb{R}$ but $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	$\mathbb{R} - \{x \mid -1 < x < 1\}$
$y = \csc x$	$x \in \mathbb{R}$ but $x \neq n\pi, n \in \mathbb{Z}$	$\mathbb{R} - \{x \mid -1 < x < 1\}$

Periodic Function

A function f is said to be periodic if for every x belonging to its domain D , there exists a positive number p such that $x+p \in D$ and

$$f(x+p) = f(x).$$

If p is the least positive number satisfying these conditions, then it is called the period of f .

Periodicity: All the

six trigonometric functions repeat their values for each increase or decrease of 2π in x . This behaviour of trigonometric functions is called periodicity.

Theorem

Sine is a periodic function and its period is 2π .

Proof: Let p be the period of sine. Then

$$\sin(\theta+p) = \sin \theta, \forall \theta \in \mathbb{R}$$

putting $\theta=0$ in ①, we get

$$\sin(0+p) = \sin 0 \Rightarrow \sin p = 0$$

$$\Rightarrow p = \sin^{-1}(0)$$

$$\Rightarrow p = 0, \pi, 2\pi, \dots$$

i) If $p = \pi$, then from ①

$$\sin(\theta + \pi) = \sin \theta$$

$$\Rightarrow -\sin \theta = \sin \theta \text{ (not true)}$$

$$\because \sin(\theta + \pi) = -\sin \theta$$

$\therefore \pi$ is not the period of $\sin \theta$

ii) If $p = 2\pi$, then from ①

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\Rightarrow \sin \theta = \sin \theta \text{ (true)}$$

$\therefore 2\pi$ is the period of $\sin \theta$

Theorem: Tangent is a periodic function and its period is π .

Proof: Let p be the period of \tan . Then

$$\tan(\theta+p) = \tan \theta, \forall \theta \in \mathbb{R}$$

Putting $\theta=0$ in ①, we get

$$\tan(0+p) = \tan 0$$

$$\Rightarrow \tan p = 0$$

$$\Rightarrow p = 0, \pi, 2\pi, 3\pi, \dots$$

$p=0$ can't be the period of $\tan \theta$ $\therefore p=0$ is not positive.

If $p=\pi$, then from ①

$$\tan(\theta+\pi) = \tan \theta$$

$$\Rightarrow \tan \theta = \tan \theta \text{ (true)}$$

$\therefore \pi$ is the period of $\tan \theta$

\therefore it is the least +ve number for which $\tan(\theta+\pi) = \tan \theta$.

Similarly we can prove that

i) 2π is the period of $\cos \theta$

ii) 2π is the period of $\operatorname{cosec} \theta$

iii) 2π is the period of $\operatorname{sec} \theta$

iv) π is the period of $\cot \theta$.

* EXERCISE 11.1 *

Find the periods of the following functions.

1) $\sin 3x = \sin(3x+2\pi)$

$$= \sin 3\left(x + \frac{2\pi}{3}\right)$$

$$\therefore \text{period of } \sin 3x = \frac{2\pi}{3} \text{ Ans.}$$

2) $\cos 2x = \cos(2x+2\pi)$

$$= \cos 2\left(x + \pi\right)$$

$$\therefore \text{period of } \cos 2x = \pi \text{ Ans.}$$

3) $\tan 4x = \tan(4x+\pi)$

$$= \tan 4\left(x + \frac{\pi}{4}\right)$$

$$\therefore \text{period of } \tan 4x = \frac{\pi}{4} \text{ Ans.}$$

④ $\cot \frac{x}{2} = \cot\left(\frac{x}{2} + \pi\right)$

$$= \cot \frac{1}{2}(x+2\pi)$$

$$\therefore \text{period of } \cot \frac{x}{2} = 2\pi \text{ Ans.}$$

⑤ $\sin \frac{x}{3} = \sin\left(\frac{x}{3} + 2\pi\right)$

$$= \sin \frac{1}{3}(x+6\pi)$$

$$\therefore \text{period of } \sin \frac{x}{3} = 6\pi \text{ Ans.}$$

⑥ $\operatorname{cosec} \frac{x}{4} = \operatorname{cosec}\left(\frac{x}{4} + 2\pi\right)$

$$= \operatorname{cosec} \frac{1}{4}(x+8\pi)$$

$$\therefore \text{period of } \operatorname{cosec} \frac{x}{4} = 8\pi \text{ Ans.}$$

⑦ $\sin \frac{x}{5} = \sin\left(\frac{x}{5} + 2\pi\right)$

$$= \frac{1}{5} \sin(x+10\pi)$$

$$\therefore \text{period of } \sin \frac{x}{5} = 10\pi \text{ Ans.}$$

⑧ $\cos \frac{x}{6} = \cos\left(\frac{x}{6} + 2\pi\right)$

$$= \cos \frac{1}{6}(x+12\pi)$$

$$\therefore \text{period of } \cos \frac{x}{6} = 12\pi$$

⑨ $\tan \frac{x}{7} = \tan\left(\frac{x}{7} + \pi\right)$

$$= \tan \frac{1}{7}(x+7\pi)$$

$$\therefore \text{period of } \tan \frac{x}{7} = 7\pi \text{ Ans.}$$

⑩ $\cot 8x = \cot(8x+\pi)$

$$= \cot 8\left(x + \frac{\pi}{8}\right)$$

$$\therefore \text{period of } \cot 8x = \frac{\pi}{8} \text{ Ans.}$$

⑪ $\operatorname{sec} 9x = \operatorname{sec}(9x+2\pi)$

$$= \operatorname{sec} 9\left(x + \frac{2\pi}{9}\right)$$

$$\therefore \text{period of } \operatorname{sec} 9x = \frac{2\pi}{9} \text{ Ans.}$$

(12) $\text{Cosec } 10x$

$= \text{Cosec}(10x + 2\pi)$

$= \text{Cosec } 10(x + \frac{2\pi}{10})$

$= \text{Cosec } 10(x + \frac{\pi}{5})$

$\therefore \text{period of } \text{Cosec } 10x = \frac{\pi}{5} \text{ Ans.}$

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(13) $3 \sin x = 3 \sin(x + 2\pi)$

$\therefore \text{period of } 3 \sin x = 2\pi \text{ Ans.}$

(14) $2 \cos x = 2 \cos(x + 2\pi)$

$\therefore \text{period of } 2 \cos x = 2\pi \text{ Ans.}$

(15) $3 \cos \frac{x}{5} = 3 \cos(\frac{x}{5} + 2\pi)$

$= 3 \cos \frac{1}{5}(x + 10\pi)$

$\therefore \text{period of } 3 \cos \frac{x}{5} = 10\pi \text{ Ans.}$

EXERCISE 11.2

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1. i) $y = -\sin x ; x \in [-2\pi, 2\pi]$

x	-360°	-330°	-300°	-270°	-240°	-210°	-180°	-150°	-120°	-90°	-60°	-30°	0°
y	0	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1	0.9	0.5	0
x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°	
y	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1	0.9	0.5	0	

Scale

One big square along x-axis = 100°

One big square along y-axis = 1 unit.

