Question # 1
Express the following products as sums or differences:

(i) $2\sin 3\theta\cos\theta$ (ii) $2\cos 5\theta\cos 3\theta$ (iii) $\sin 5\theta\cos 2\theta$

(iv) $2\sin 7\theta\sin 2\theta$ (v) $\cos(x+y)\sin(x-y)$ (vi) $\cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)$

(vii) $\sin 12^\circ\sin 46^\circ$ (viii) $\sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)$

Solution

(i) Since $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

Put $\alpha=3\theta$ and $\beta=\theta$

$$2\sin 3\theta\cos\theta=\sin(3\theta+\theta)+\sin(3\theta-\theta)$$
$$=\sin 4\theta+\sin 2\theta$$

(ii) Since $2\cos\alpha\cos\beta=\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)$

Put $\alpha=5\theta$ and $\beta=3\theta$

$$2\cos 5\theta\cos 3\theta=\cos(5\theta+3\theta)-\cos(5\theta-3\theta)$$
$$=\cos 8\theta-\cos 2\theta$$

(iii) Since $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

Put $\alpha=5\theta$ and $\beta=2\theta$

$$2\sin 5\theta\cos 2\theta=\sin(5\theta+2\theta)+\sin(5\theta-2\theta)$$
$$=\sin 7\theta+\sin 3\theta$$
$$\Rightarrow\ \sin 5\theta\cos 2\theta=\frac{1}{2}\left(\sin 7\theta+\sin 3\theta\right)$$

(iv) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha=7\theta$ and $\beta=2\theta$

$$-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$$
$$-2\sin 7\theta\sin 2\theta=\cos(7\theta+2\theta)-\cos(7\theta-2\theta)$$
$$=\cos 9\theta-\cos 5\theta$$
$$2\sin 7\theta\sin 2\theta=\cos 5\theta-\cos 9\theta$$

(v) Since $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

Put $\alpha=x+y$, $\beta=x-y$

$$2\cos(x+y)\sin(x-y)=\sin(x+y+x-y)-\sin(x+y-x+y)$$
$$=\sin 2x-\sin 2y$$
$$\Rightarrow\ \cos(x+y)\sin(x-y)=\frac{1}{2}(\sin 2x-\sin 2y)$$

(vi) Since $2\cos\alpha\cos\beta=\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)$

Put $\alpha=2x+30^\circ$ and $\beta=2x-30^\circ$

$$2\cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)=\cos\left(2x+30^\circ+2x-30^\circ\right)+\cos\left(2x+30^\circ-2x+30^\circ\right)$$
$$=\cos\left(4x\right)+\cos\left(60^\circ\right)$$
$$\Rightarrow \cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)=\frac{1}{2}\left(\cos 4x+\cos 60^\circ\right)$$

(vii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha=12^\circ$ and $\beta=46^\circ$

$$-2\sin 12^\circ\sin 46^\circ=\cos(12+46)-\cos(12-46)$$
$$=\cos 58-\cos(-34)$$
$$=\cos 58-\cos 34 \quad \because \cos(-\theta)=\cos\theta$$
$$\Rightarrow \sin 12^\circ\sin 46^\circ=-\frac{1}{2}\left(\cos 58-\cos 34\right)$$
$$=\frac{1}{2}\left(\cos 34-\cos 58\right)$$

(viii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha=x+45^\circ$ and $\beta=x-45^\circ$

$$-2\sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)=\cos\left\{(x+45^\circ)+(x-45^\circ)\right\}-\cos\left\{(x+45^\circ)-(x-45^\circ)\right\}$$
$$=\cos 2x-\cos 90^\circ$$
$$\Rightarrow \sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)=\cos 90^\circ-\frac{1}{2}\cos 2x$$

Question # 2

Express the following sum or difference as product:

(i) $\sin 5\theta+\sin 3\theta$ (ii) $\sin 8\theta-\sin 4\theta$ (iii) $\cos 6\theta+\cos 3\theta$

(iv) $\cos 7\theta-\cos\theta$ (v) $\cos 12^\circ+\cos 48^\circ$ (vi) $\sin(x+30^\circ)+\sin(x-30^\circ)$

Solution

(i) Since $\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha=5\theta$, $\beta=3\theta$

$$\sin 5\theta+\sin 3\theta=2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$
$$=2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)=2\sin 4\theta\ \cos\theta$$

(ii) Since $\sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha=8\theta$, $\beta=4\theta$

$$\sin 8\theta-\sin 4\theta=2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$$
$$=2\cos 6\theta\ \sin 2\theta$$

(iii) *Do yourself*

(iv) Since $\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha=7\theta$, $\beta=\theta$

$$\cos7\theta-\cos\theta=-2\sin\left(\frac{7\theta+\theta}{2}\right)\sin\left(\frac{7\theta-\theta}{2}\right)=-2\sin\left(\frac{8\theta}{2}\right)\sin\left(\frac{6\theta}{2}\right)$$
$$=-2\sin4\theta\sin3\theta$$

(v) Since $\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha=12$, $\beta=48$

$$\cos12+\cos48=2\cos\left(\frac{12+48}{2}\right)\cos\left(\frac{12-48}{2}\right)$$
$$=2\cos\left(\frac{60}{2}\right)\cos\left(\frac{-36}{2}\right)=2\cos30\cos(-18)$$
$$=2\cos30^\circ\cos18^\circ \qquad \because \cos(-\theta)=\cos\theta$$

(vi) Since $\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha=x+30$, $\beta=x-30$

$$\sin(x+30)+\sin(x-30)=2\sin\left(\frac{x+30+x-30}{2}\right)\cos\left(\frac{x+30-x+30}{2}\right)$$
$$=2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{60}{2}\right)=2\sin x\cos30$$

Question # 3

Prove the following identities:

(i) $\frac{\sin3x-\sin x}{\cos x-\cos3x}=\cot2x$ (ii) $\frac{\sin8x+\sin2x}{\cos8x+\cos2x}=\tan5x$

(iii) $\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\cot\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right)$

Solution

(i) L.H.S $=\frac{\sin3x-\sin x}{\cos x-\cos3x}$

$$=\frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}=\frac{\cos\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)}{-\sin\left(\frac{4x}{2}\right)\sin\left(\frac{-2x}{2}\right)}$$

$$= \frac{\cos(2x)\sin(x)}{+\sin(2x)\sin(x)} = \cot 2x \quad = \text{R.H.S}$$

(ii) *Do yourself*

(iii) L.H.S $= \dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$

$$= \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} = \cot\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right) = \text{R.H.S}$$

Question # 4

Prove that:

(i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

(ii) $\sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}\cos 2\theta$

(iii) $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

Solution

(i) L.H.S $= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$$= \left(\cos 100^\circ + \cos 20^\circ\right) + \cos 140^\circ$$

$$= 2\cos\left(\frac{100+20}{2}\right)\cos\left(\frac{100-20}{2}\right) + \cos 140^\circ$$

$$= 2\cos 60^\circ \cos 40^\circ + \cos 140^\circ \quad = 2\left(\frac{1}{2}\right)\cos 40^\circ + \cos 140^\circ$$

$$= \cos 140^\circ + \cos 40^\circ \quad = 2\cos\left(\frac{140+40}{2}\right)\cos\left(\frac{140-40}{2}\right)$$

$$= 2\cos 90^\circ \cos 50^\circ \quad = 2(0)\cos 50^\circ \quad = 0 = \text{R.H.S}$$

(ii) L.H.S $= \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right)$

$$= \left(\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right)\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cos\theta\right)^2 - \left(\frac{1}{\sqrt{2}}\sin\theta\right)^2 \quad = \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta$$

$$= \frac{1}{2}\left(\cos^2\theta - \sin^2\theta\right) \quad = \frac{1}{2}\cos 2\theta \quad = \text{R.H.S}$$

(iii) L.H.S $= \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta}$

$$= \frac{(\sin 7\theta + \sin\theta) + (\sin 5\theta + \sin 3\theta)}{(\cos 7\theta + \cos\theta) + (\cos 5\theta + \cos 3\theta)}$$

$$= \frac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right) + 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right) + 2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$$

$$= \frac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta}$$

$$= \frac{2\sin 4\theta(\cos 3\theta + \cos\theta)}{2\cos 4\theta(\cos 3\theta + \cos\theta)} \quad = \frac{\sin 4\theta}{\cos 4\theta} \quad = \tan 4\theta \quad = \text{R.H.S}$$

Question # 5

Prove that:

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$ (ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$

Solution

(i) L.H.S $= \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$$= \cos 20^\circ \cos 40^\circ \left(\frac{1}{2}\right)\cos 80^\circ \quad = \frac{1}{2}\cos 80^\circ \cos 40^\circ \cos 20^\circ$$

$$= \frac{1}{4}(2\cos 80^\circ \cos 40^\circ)\cos 20^\circ \quad = \frac{1}{4}\big(\cos(80+40) + \cos(80-40)\big)\cos 20^\circ$$

$$= \frac{1}{4}(\cos 120^\circ + \cos 40^\circ)\cos 20^\circ \quad = \frac{1}{4}\left(-\frac{1}{2} + \cos 40^\circ\right)\cos 20^\circ$$

$$= -\frac{1}{8}\cos 20^\circ + \frac{1}{4}\cos 40^\circ \cos 20^\circ \quad = -\frac{1}{8} + \frac{1}{8}(2\cos 40^\circ \cos 20^\circ)$$

$$= -\frac{1}{8}\cos 20^\circ + \frac{1}{8}(\cos(40+20) + \cos(40-20))$$

$$= -\frac{1}{8}\cos 20^\circ + \frac{1}{8}(\cos 60 + \cos 20) \quad = -\frac{1}{8}\cos 20^\circ + \frac{1}{8}\left(\frac{1}{2} + \cos 20\right)$$

$$= -\frac{1}{8}\cos 20^\circ + \frac{1}{16} + \frac{1}{8}\cos 20^\circ = \frac{1}{16} \quad = \text{R.H.S}$$

(ii) L.H.S $= \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$

$$= \sin\frac{180^\circ}{9}\sin\frac{2(180^\circ)}{9}\sin\frac{(180^\circ)}{3}\sin\frac{4(180^\circ)}{9} \qquad \because \pi = 180^\circ$$

$$= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \quad = \sin 20^\circ \sin 40^\circ \frac{\sqrt{3}}{2}\sin 80^\circ$$

$$= \frac{\sqrt{3}}{2}\sin 80^\circ \sin 40^\circ \sin 20^\circ \quad = -\frac{\sqrt{3}}{4}(-2\sin 80^\circ \sin 40^\circ)\sin 20^\circ$$

$$= -\frac{\sqrt{3}}{4}\left(\cos(80+40)-\cos(80-40)\right)\sin 20^\circ$$

$$= -\frac{\sqrt{3}}{4}\left(\cos 120^\circ - \cos 40^\circ\right)\sin 20^\circ \quad = -\frac{\sqrt{3}}{4}\left(-\frac{1}{2}-\cos 40^\circ\right)\sin 20^\circ$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{4}\cos 40^\circ \sin 20^\circ \ = \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(2\cos 40^\circ \sin 20^\circ\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\sin(40+20)-\sin(40-20)\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\sin 60^\circ - \sin 20^\circ\right) \ = \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2}-\sin 20^\circ\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{3}{16} - \frac{\sqrt{3}}{8}\sin 20^\circ \ = \frac{3}{16} \quad = \text{R.H.S}$$

(iii) *Do yourself as above*
