

Question # 1

Find the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ when:

(i) $\sin \alpha = \frac{12}{13}$ (ii) $\cos \alpha = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$

Solution

(i) $\sin \alpha = \frac{12}{13}$; $0 < \alpha < \frac{\pi}{2}$

Since $\cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}$

As α is in the first quadrant so value of \cos is +ive

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha}$$

$$= \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} \Rightarrow \cos \alpha = \frac{5}{13}$$

and $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{12/13}{5/13} \Rightarrow \tan \alpha = \frac{12}{5}$

Now $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
 $= 2 \left(\frac{12}{13}\right) \left(\frac{5}{13}\right) \Rightarrow \sin 2\alpha = \frac{120}{169}$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} \Rightarrow \cos 2\alpha = \frac{119}{169}$$

$$\tan 2\alpha = \frac{2\alpha \tan \alpha}{1 - \tan^2 \alpha}$$

$$= \frac{2\left(\frac{12}{5}\right)}{1 - \left(\frac{12}{5}\right)^2} = \frac{24/5}{1 - 144/25} = \frac{24/5}{-119/25} = -\frac{24}{5} \cdot \frac{25}{119}$$

$$\Rightarrow \tan 2\alpha = \frac{120}{119}$$

(ii) $\cos \alpha = \frac{3}{5}$; $0 < \alpha < \frac{\pi}{2}$

Hint: First find $\sin \alpha$ and $\tan \alpha$ then solve as above

Prove the following identities (**Question 2 – 13**)

Question # 2

$$\cot \alpha - \tan \alpha = 2 \cot 2\alpha$$

Solution L.H.S = $\cot \alpha - \tan \alpha = \frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha}$

$$= \frac{2(\cos^2 \alpha - \sin^2 \alpha)}{2 \sin \alpha \cos \alpha} = \frac{2 \cos 2\alpha}{\sin 2\alpha} = 2 \cot 2\alpha = \text{R.H.S}$$

Question # 3

$$\frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha$$

Solution

$$\text{L.H.S} = \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \frac{2 \sin \alpha \cos \alpha}{2 \cos^2 \alpha}$$

$$= \frac{\sin \alpha}{\cos \alpha} = \tan \alpha = \text{R.H.S}$$

$$\begin{aligned} \therefore \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} \\ \therefore 2 \cos^2 \alpha &= 1 + \cos 2\alpha \end{aligned}$$

Question # 4

$$\frac{1 - \cos \alpha}{\sin \alpha} = \tan \frac{\alpha}{2}$$

Solution

$$\text{L.H.S} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}$$

$$= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \tan \frac{\alpha}{2} = \text{R.H.S}$$

$$\begin{aligned} \therefore \sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\ \therefore 2 \sin^2 \frac{\alpha}{2} &= 1 - \cos \alpha \\ \sin \alpha &= 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \end{aligned}$$

Question # 5

$$\frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} = \sec 2\alpha - \tan 2\alpha$$

Solution

$$\text{L.H.S} = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha}$$

$$= \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \times \frac{\cos \alpha - \sin \alpha}{\cos \alpha - \sin \alpha}$$

$$= \frac{(\cos \alpha - \sin \alpha)^2}{\cos^2 \alpha - \sin^2 \alpha} = \frac{\cos^2 \alpha + \sin^2 \alpha - 2 \sin \alpha \cos \alpha}{\cos 2\alpha}$$

$$= \frac{1 - \sin 2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha} = \sec 2\alpha - \tan 2\alpha = \text{R.H.S}$$

Question 6

$$\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$$

Solution L.H.S = $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$ ∵ $\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$

$$= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$= \sqrt{\frac{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} = \text{R.H.S}$$

$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

Question # 7

$$\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta} = \cot\frac{\theta}{2}$$

Solution L.H.S = $\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta}$

$$= \frac{1}{\sin\theta} + \frac{2}{\sin 2\theta} = \frac{1}{\cos\theta} \left(\frac{1}{\sin\theta} + \frac{2}{2\sin\theta\cos\theta} \right)$$

$$= \frac{1}{\cos\theta} \left(\frac{1}{\sin\theta} + \frac{1}{\sin\theta\cos\theta} \right) = \frac{\cos\theta + 1}{\sin\theta\cos\theta}$$

$$= \frac{\cos\theta + 1}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \cot\frac{\theta}{2} = \text{R.H.S}$$

Question # 8

$$1 + \tan\alpha \tan 2\alpha = \sec 2\alpha$$

Solution L.H.S = $1 + \tan\alpha \tan 2\alpha = 1 + \left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\frac{\sin 2\alpha}{\cos 2\alpha}\right)$

$$= \frac{\cos\alpha\cos 2\alpha + \sin\alpha\sin 2\alpha}{\cos\alpha\cos 2\alpha} = \frac{\cos(2\alpha - \alpha)}{\cos\alpha\cos 2\alpha}$$

$$= \frac{\cos\alpha}{\cos\alpha\cos 2\alpha} = \frac{1}{\cos 2\alpha} = \sec 2\alpha = \text{R.H.S}$$

Question # 9

$$\frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} = \tan 2\theta \tan \theta$$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} \\ &= \frac{2\sin\theta\sin 2\theta}{\cos\theta + 4\cos^3\theta - 3\cos\theta} \\ &= \frac{2\sin\theta\sin 2\theta}{4\cos^3\theta - 2\cos\theta} = \frac{2\sin\theta\sin 2\theta}{2\cos\theta(2\cos^2\theta - 1)} \\ &= \frac{\sin\theta\sin 2\theta}{\cos\theta\cos 2\theta} = \tan\theta \tan 2\theta = \tan 2\theta \tan \theta = \text{R.H.S} \end{aligned}$$

$$\begin{aligned} \therefore \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \therefore \cos^2\theta &= \frac{1 + \cos 2\theta}{2} \\ \therefore 2\cos^2\theta - 1 &= \cos 2\theta \end{aligned}$$

Question # 10

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta \cos\theta - \cos 3\theta \sin\theta}{\sin\theta \cos\theta} \\ &= \frac{\sin(3\theta - \theta)}{\sin\theta \cos\theta} = \frac{\sin 2\theta}{\sin\theta \cos\theta} = \frac{2\sin\theta \cos\theta}{\sin\theta \cos\theta} = 2 = \text{R.H.S} \end{aligned}$$

Question # 11

$$\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = \frac{\cos 3\theta \sin\theta + \sin 3\theta \cos\theta}{\sin\theta \cos\theta} \\ &= \frac{\sin(\theta + 3\theta)}{\sin\theta \cos\theta} = \frac{\sin 4\theta}{\sin\theta \cos\theta} = \frac{2\sin 2\theta \cos 2\theta}{\sin\theta \cos\theta} \\ &= \frac{2(2\sin\theta \cos\theta)\cos 2\theta}{\sin\theta \cos\theta} = 4\cos 2\theta = \text{R.H.S} \end{aligned}$$

Question # 12

$$\frac{\tan \frac{\theta}{2} + \cot \frac{\theta}{2}}{\cot \frac{\theta}{2} - \tan \frac{\theta}{2}} = \sec \theta$$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{\tan \frac{\theta}{2} + \cot \frac{\theta}{2}}{\cot \frac{\theta}{2} - \tan \frac{\theta}{2}} = \frac{\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} + \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}}{\frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}} = \frac{\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}}}{\frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}}} \\ &= \frac{\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}}}{\frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}}} \end{aligned}$$

$$= \frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{1}{\cos \theta} = \sec \theta = \text{R.H.S}$$

Question # 13

$$\frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = 2 \cot 2\theta$$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = \frac{\sin 3\theta \sin \theta + \cos 3\theta \cos \theta}{\sin \theta \cos \theta} \\ &= \frac{\cos(3\theta - \theta)}{\sin \theta \cos \theta} = \frac{\cos 2\theta}{\sin \theta \cos \theta} = \frac{2 \cos 2\theta}{2 \sin \theta \cos \theta} \\ &= \frac{2 \cos 2\theta}{\sin 2\theta} = 2 \cot 2\theta = \text{R.H.S} \end{aligned}$$

Question # 14

Reduce $\sin^4 \theta$ to an expression involving only functions of multiples of θ raised to the first power.

Solution

$$\begin{aligned} \sin^4 \theta &= (\sin^2 \theta)^2 = \left(\frac{1 - \cos 2\theta}{2} \right)^2 \\ &= \frac{1 - 2 \cos 2\theta + \cos^2 2\theta}{4} = \frac{1}{4} (1 - 2 \cos 2\theta + \cos^2 2\theta) \\ &= \frac{1}{4} \left(1 - 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) = \frac{1}{4} \left(\frac{2 - 4 \cos 2\theta + 1 + \cos 4\theta}{2} \right) \\ &= \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta) \end{aligned}$$

Question # 15

Find the values of $\sin \theta$ and $\cos \theta$, without using table or calculator, when θ

- (i) 18° (ii) 3θ (iii) 54° (iv) 72°

Solution

(i) Let $\theta = 18^\circ \Rightarrow 5\theta = 90^\circ \Rightarrow 3\theta + 2\theta = 90^\circ \Rightarrow 2\theta = 90^\circ - 3\theta$

$$\sin 2\theta = \sin(90 - 3\theta)$$

$$\Rightarrow \sin 2\theta = \cos 3\theta$$

$$\Rightarrow 2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\Rightarrow 2 \sin \theta = 4 \cos^2 \theta - 3 \quad \div \text{ing by } \cos \theta$$

$$\Rightarrow 2 \sin \theta = 4(1 - \sin^2 \theta) - 3$$

$$\Rightarrow 2 \sin \theta = 4 - 4 \sin^2 \theta - 3 \Rightarrow 2 \sin \theta = 1 - 4 \sin^2 \theta$$

$$\Rightarrow 4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\because \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\therefore \sin 2\theta = 2 \sin \theta \cos \theta$$

This is quadratic in $\sin \theta$ with $a = 4$, $b = 1$ and $c = -1$

$$\begin{aligned}\sin \theta &= \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)} = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm \sqrt{20}}{8} \\ &= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{8}\end{aligned}$$

Since $\theta = 18^\circ$ lies in the first quadrant so value of sin can not be negative therefore

$$\sin \theta = \frac{-1 + \sqrt{5}}{4} \Rightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5} - 1}{4}} \quad \because \theta = 18^\circ$$

Now

$$\begin{aligned}\cos^2 18^\circ &= 1 - \sin^2 18^\circ \Rightarrow \cos^2 18^\circ = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 \\ \Rightarrow \cos^2 18^\circ &= 1 - \frac{(\sqrt{5})^2 - 2\sqrt{5} + 1}{16} = 1 - \frac{5 - 2\sqrt{5} + 1}{16} \\ &= 1 - \frac{6 - 2\sqrt{5}}{16} = \frac{16 - 6 + \sqrt{5}}{16} = \frac{10 + 2\sqrt{5}}{16} \\ \Rightarrow \cos 18^\circ &= \sqrt{\frac{10 + 2\sqrt{5}}{16}} \Rightarrow \boxed{\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}}\end{aligned}$$

(ii) Since $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

$$\begin{aligned}\Rightarrow \cos 2\theta &= 2\cos^2 \theta - 1 \\ \Rightarrow \cos 2(18) &= 2\cos^2(18) - 1 \\ \Rightarrow \cos 36 &= 2\left(\frac{\sqrt{10 + 2\sqrt{5}}}{4}\right)^2 - 1 \\ &= 2\left(\frac{10 + 2\sqrt{5}}{16}\right) - 1 = \frac{10 + 2\sqrt{5}}{8} - 1 \\ &= \frac{10 + 2\sqrt{5} - 8}{8} = \frac{2 + 2\sqrt{5}}{8} \Rightarrow \boxed{\cos 36^\circ = \frac{1 + \sqrt{5}}{4}}\end{aligned}$$

Now

$$\begin{aligned}\sin^2 36 &= 1 - \cos^2 36 \\ &= 1 - \left(\frac{1 + \sqrt{5}}{4}\right)^2 = 1 - \frac{1 + 2\sqrt{5} + (\sqrt{5})^2}{16} \\ &= 1 - \frac{1 + 2\sqrt{5} + 5}{16} = 1 - \frac{6 + 2\sqrt{5}}{16} \\ &= \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}\end{aligned}$$

$$\Rightarrow \sin 36^\circ = \sqrt{\frac{10 - 2\sqrt{5}}{16}} \Rightarrow \boxed{\sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}}$$

(iii) Now $\sin(90 - 36) = \cos 36^\circ$ $\therefore \sin(90 - \theta) = \cos \theta$

$$\Rightarrow \sin 54^\circ = \cos 36^\circ \Rightarrow \boxed{\sin 54^\circ = \frac{1 + \sqrt{5}}{4}}$$

And $\cos(90 - 36) = \sin 36^\circ$

$$\Rightarrow \cos 54^\circ = \sin 36^\circ \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}}$$

(iv) Now $\sin(90 - 18) = \cos 18^\circ$ $\therefore \sin(90 - \theta) = \cos \theta$

$$\Rightarrow \sin 72^\circ = \cos 18^\circ \Rightarrow \boxed{\sin 72^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}}$$

and $\cos(90 - 18) = \sin 18^\circ$

$$\Rightarrow \cos 72^\circ = \sin 18^\circ \Rightarrow \boxed{\cos 72^\circ = \frac{\sqrt{5} - 1}{4}}$$

Alternative Method for Q # 15 (iii)

Let $\theta = 54^\circ \Rightarrow 5\theta = 270^\circ \Rightarrow 3\theta + 2\theta = 270^\circ \Rightarrow 2\theta = 270^\circ - 3\theta$

$$\sin 2\theta = \sin(270 - 3\theta)$$

$$\Rightarrow \sin 2\theta = \sin(3(90) - 3\theta)$$

$$\Rightarrow \sin 2\theta = -\cos 3\theta$$

$$\Rightarrow 2\sin \theta \cos \theta = -(4\cos^3 \theta - 3\cos \theta)$$

$$\Rightarrow 2\sin \theta \cos \theta = -4\cos^3 \theta + 3\cos \theta$$

$$\Rightarrow 2\sin \theta = -4\cos^2 \theta + 3 \quad \div \text{ing by } \cos \theta$$

$$\Rightarrow 2\sin \theta = -4(1 - \sin^2 \theta) + 3$$

$$\Rightarrow 2\sin \theta = -4 + 4\sin^2 \theta + 3 \Rightarrow 2\sin \theta = 4\sin^2 \theta - 1$$

$$\Rightarrow 4\sin^2 \theta - 2\sin \theta - 1 = 0$$

This is quadratic in $\sin \theta$ with $a = 4$, $b = 1$ and $c = -1$

$$\sin \theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{2 \pm \sqrt{4 + 16}}{8} = \frac{2 \pm \sqrt{20}}{8}$$

$$= \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

Since $\theta = 54^\circ$ lies in the first quadrant so value of sin can not be negative therefore

$$\sin \theta = \frac{1 + \sqrt{5}}{4} \Rightarrow \boxed{\sin 54^\circ = \frac{1 + \sqrt{5}}{4}} \quad \therefore \theta = 54^\circ$$

Now

$$\begin{aligned} \cos^2 54^\circ &= 1 - \sin^2 54^\circ \Rightarrow \cos^2 54^\circ = 1 - \left(\frac{1 + \sqrt{5}}{4}\right)^2 \\ \Rightarrow \cos^2 54^\circ &= 1 - \frac{(\sqrt{5})^2 + 2\sqrt{5} + 1}{16} = 1 - \frac{5 + 2\sqrt{5} + 1}{16} \\ &= 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16} \\ \Rightarrow \cos 54^\circ &= \sqrt{\frac{10 - 2\sqrt{5}}{16}} \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}} \end{aligned}$$

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