Conjugate of a Complex Number.

For a complex number $z = x + iy$, the complex number $\bar{z} = \overline{x+iy} = x - iy$ is called conjugate of z.

Here z and $\bar{z}$ have same real part but their imaginary parts differ in sign, so z and $\bar{z}$ are called conjugate of each other.

Thus $5+4i$ and $5-4i$ are conjugate complex numbers.

Note: If $z = \bar{z}$, then z is called self conjugate.

Since for $a \in \mathbb{R}$

$\bar{a} = a$

$\Rightarrow$ Every real number is self conjugate.

Powers of i

$i^2 = -1$ $\quad \because i = \sqrt{-1}$

$i^3 = (i^2).i = (-1).i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^2)^2.i = (-1)^2.i = 1.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = (i^2)^3.i = (-1)^3.i = (-1)i = -i$

$i^8 = (i^2)^4 = (-1)^4 = 1$

and so on.

$\Rightarrow$ Powers of i should be expressed in terms of powers of i^2.

EXERCISE 1.2

① Verify the addition properties of complex numbers.

Solution:

i) Closure Property

For $(a,b), (c,d) \in C$

$(a,b) + (c,d) = (a+c, b+d) \in C$

$\because +$ is closed in $\mathbb{R}$

ii) Associative Property

For $(a,b), (c,d), (e,f) \in C$

$[(a,b)+(c,d)]+(e,f) = (a+c, b+d)+(e,f)$

$= ((a+c)+e, (b+d)+f)$

$= (a+(c+e), b+(d+f))$

$\because$ '+' is associative in $\mathbb{R}$

$= (a,b) + (c+e, d+f)$

$= (a,b) + [(c,d)+(e,f)]$

iii) Additive Identity

$\forall (a,b) \in C$ we have $(0,0) \in C$ such that

$(a,b)+(0,0) = (a+0, b+0) = (a,b)$

$\Rightarrow (0,0)$ is additive Identity in C

iv) Additive Inverse

$\forall (a,b) \in C$ there is $(-a,-b) \in C$ such that

$(a,b)+(-a,-b) = (a-a, b-b)$

$= (0,0)$

$\Rightarrow (a,b)$ and $(-a,-b)$ are additive inverse of each other.

v) Commutative Property

$\forall (a,b), (c,d) \in C$

The set $C = \{x+iy \,/\, x, y \in R,\ i=\sqrt{-1}\}$ is called set of complex numbers.

Operations on Complex Numbers:

Let $z_1 = (x_1, y_1) = x_1 + iy_1$

$z_2 = (x_2, y_2) = x_2 + iy_2$

i) Addition:

$$
\begin{aligned}
z_1 + z_2 &= (x_1, y_1) + (x_2, y_2)\\
&= x_1 + iy_1 + x_2 + iy_2\\
&= x_1 + x_2 + iy_1 + iy_2\\
&= (x_1 + x_2) + i(y_1 + y_2)\\
&= (x_1 + x_2, y_1 + y_2)
\end{aligned}
$$

ii) Subtraction:

$$
\begin{aligned}
z_1 - z_2 &= (x_1, y_1) - (x_2, y_2)\\
&= (x_1 + iy_1) - (x_2 + iy_2)\\
&= x_1 + iy_1 - x_2 - iy_2\\
&= x_1 - x_2 + iy_1 - iy_2\\
&= (x_1 - x_2) + i(y_1 - y_2)\\
&= (x_1 - x_2, y_1 - y_2)
\end{aligned}
$$

iii) Multiplication

$$
\begin{aligned}
z_1 z_2 &= (x_1, y_1) \cdot (x_2, y_2)\\
&= (x_1 + iy_1) \cdot (x_2 + iy_2)\\
&= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2\\
&= x_1x_2 + ix_1y_2 + ix_2y_1 + (-1)y_1y_2\\
&= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2\\
&= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\\
&= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)
\end{aligned}
$$

iv) Division:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{x_1 + iy_1}{x_2 + iy_2}\\
&= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}\\
&= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{x_2^2 - i^2y_2^2}\\
&= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - (-1)y_1y_2}{x_2^2 - (-1)y_2^2}\\
&= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2 + y_2^2}\\
&= \frac{x_1x_2 + y_1y_2 + ix_2y_1 - ix_1y_2}{x_2^2 + y_2^2}\\
&= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}\\
&= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)\\
&= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2},\ \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)
\end{aligned}
$$

v) Equality:

$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2,\ y_1 = y_2$

vi) Scalar Multiplication

If $z = (x, y) = x + iy$ be a complex number and k is any real constant then

$$
\begin{aligned}
kz = k(x, y) &= k(x + iy)\\
&= kx + iky\\
&= (kx, ky)
\end{aligned}
$$

Note that: Every real number a can be written as $a = a + i\cdot 0$

$\therefore$ Set of real numbers is subset of set of complex numbers.

$(a,b)+(c,d)=(a+c\,,b+d)$
$=(c+a\,,d+b)$ $\because$ '+' is commutative in $\mathbb{R}$
$=(c,d)+(a,b)$

② Verify the multiplication properties of the complex numbers.

Solution:

i) Closure Property

$\forall\ (a,b),(c,d)\in C$

$(a,b)\cdot(c,d)=(ac-bd\,,ad+bc)\in C$
$\because$ '$\times$' is closed in $\mathbb{R}$

ii) Associative Property

For $(a,b),(c,d),(e,f)\in C$
Consider
$[(a,b)\cdot(c,d)]\cdot(e,f)=(ac-bd,ad+bc)\cdot(e,f)$
$=\big((ac-bd)e-(ad+bc)f\,,(ac-bd)f+(ad+bc)e\big)$
$=(ace-bde-adf-bcf\,,acf-bdf+ade+bce)$ → ①
Now Consider
$(a,b)\cdot[(c,d)\cdot(e,f)]=(a,b)\cdot[(ce-df,cf+de)]$
$=\big(a(ce-df)-b(cf+de),a(cf+de)+b(ce-df)\big)$
$=(ace-adf-bcf-bde\,,acf+ade+bce-bdf)$ → ②
$\therefore$ From ① & ②, we get
$[(a,b)\cdot(c,d)]\cdot(e,f)=(a,b)\cdot[(c,d)\cdot(e,f)]$

iii) Multiplicative Identity:

$\forall\ (a,b)\in C$ we have $(1,0)\in C$ such that
$(a,b)\cdot(1,0)=(a-0\,,0+b)$
$=(a,b)$
$\Rightarrow (1,0)$ is multiplicative identity in C.

iv) Multiplicative Inverse

$\forall (a,b)\in C$ there is
$\dfrac{1}{(a,b)}=\dfrac{1}{a+ib}=\dfrac{1}{a+ib}\times\dfrac{a-ib}{a-ib}$
$=\dfrac{a-ib}{a^2-i^2b^2}=\dfrac{a-ib}{a^2-(-1)b^2}=\dfrac{a-ib}{a^2+b^2}$
$=\dfrac{a}{a^2+b^2}-i\dfrac{b}{a^2+b^2}\in C$ such that
$(a,b)\cdot\dfrac{1}{(a,b)}=1=1+i.0=(1,0)$
$\Rightarrow (a,b)$ and $\dfrac{1}{(a,b)}$ are multiplicative inverse of each other.

v) Commutative Property

$\forall (a,b),(c,d)\in C$
$(a,b)\cdot(c,d)=(ac-bd\,,ad+bc)$
$=(ca-db\,,da+cb)$ $\because$ '$\times$' is commutative in $\mathbb{R}$
$=(c,d)\cdot(a,b)$

③ Verify the distributive law of complex numbers.

$(a,b)[(c,d)+(e,f)]=(a,b)(c,d)+(a,b)(e,f)$
L.H.S. $=(a,b)[(c,d)+(e,f)]$
$=(a,b)(c+e\,,d+f)$
$=\big(a(c+e)-b(d+f),\ a(d+f)+b(c+e)\big)$
$=(ac+ae-bd-bf\,,ad+af+bc+be)$ → ①
R.H.S. $=(a,b)\cdot(c,d)+(a,b)\cdot(e,f)$
$=(ac-bd,ad+bc)+(ae-bf,af+be)$
$=(ac-bd+ae-bf\,,ad+bc+af+be)$ → ②
$\therefore$ From ① & ②
L.H.S. $=$ R.H.S.

④ Simplify the following

i) $i^9 = (i^2)^4 \cdot i = (-1)^4 \cdot i = 1 \cdot i = i$ Ans

ii) $i^{14} = (i^2)^7 = (-1)^7 = -1$ Ans.

iii) $(-i)^{19} = [(-1)\cdot(i)]^{19} = (-1)^{19} \cdot i^{19} = -i^{19}$
$= -(i^2)^9 \cdot i = -(-1)^9 \cdot i$
$= -(-1)\ i = i$ Ans.

iv) $(-1)^{-\frac{21}{2}} = \dfrac{1}{(-1)^{+\frac{21}{2}}} = \dfrac{1}{[(-1)^{\frac{1}{2}}]^{21}} = \dfrac{1}{i^{21}}$
$= \dfrac{1}{(i^2)^{10} \cdot i} = \dfrac{1}{(-1)^{10} \cdot i} = \dfrac{1}{1 \cdot i}$
$= \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1}$
$= -i$ Ans.

⑤ Write in terms of i

i) $\sqrt{-1}\, b = i b$ Ans.

ii) $\sqrt{-5} = \sqrt{(-1)(5)} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$ Ans.

iii) $\sqrt{-\dfrac{16}{25}} = \sqrt{(-1)\cdot\dfrac{16}{25}} = \sqrt{-1}\sqrt{\dfrac{16}{25}} = i\dfrac{4}{5}$
$= \dfrac{4}{5} i$ Ans.

iv) $\sqrt{\dfrac{1}{-4}}$ Method I
$= \dfrac{\sqrt{1}}{\sqrt{-4}} = \dfrac{1}{\sqrt{(-1)(4)}} = \dfrac{1}{\sqrt{-1}\sqrt{4}} = \dfrac{1}{i \cdot 2}$
$= \dfrac{1}{2i} = \dfrac{1}{2i} \times \dfrac{i}{i} = \dfrac{i}{2i^2} = \dfrac{i}{2(-1)} = \dfrac{i}{-2}$
$= -\dfrac{i}{2}$ Ans

⑥ $(7,9)+(3,-5) = (7+3, 9-5) = (10,4)$ Ans.

⑦ $(8,-5)-(-7,4) = (8+7, -5-4)$
$= (15,-9)$ Ans.

⑧ $(2,6)\cdot(3,7)$
$= (2\cdot3 - 6\cdot7,\ 2\cdot7 + 6\cdot3)$
$= (6-42,\ 14+18)$
$= (-36,\ 32)$ Ans.

⑨ $(5,-4)\cdot(-3,-2)$
$= ((5)(-3)-(-4)(-2),\ (5)(-2)+(-4)(-3))$
$= (-15-8,\ -10+12) = (-23, 2)$ Ans.

⑩ $(0,3)\cdot(0,5)$
$= ((0)(0)-(3)(5),\ (0)(5)+(3)(0))$
$= (0-15,\ 0+0) = (-15, 0)$ Ans.

⑪ $(2,6) \div (3,7) = \dfrac{(2,6)}{(3,7)}$
$= \dfrac{2+6i}{3+7i} = \dfrac{2+6i}{3+7i} \times \dfrac{3-7i}{3-7i}$
$= \dfrac{6-14i+18i-42i^2}{(3)^2-(7i)^2} = \dfrac{6+4i-42(-1)}{9-49i^2}$
$= \dfrac{6+4i+42}{9-49(-1)} = \dfrac{48+4i}{9+49} = \dfrac{48+4i}{58}$
$= \dfrac{48}{58} + \dfrac{4}{58}i = \dfrac{24}{29} + \dfrac{2}{29}i$
$= \left(\dfrac{24}{29}, \dfrac{2}{29}\right)$ Ans.

⑫ $(5,-4) \div (-3,-8)$
$= \dfrac{(5,-4)}{(-3,-8)} = \dfrac{5-4i}{-3-8i} = \dfrac{5-4i}{-3-8i} \times \dfrac{-3+8i}{-3+8i}$
$= \dfrac{-15+40i+12i-32i^2}{(-3)^2-(8i)^2}$
$= \dfrac{-15+52i-32(-1)}{9-64i^2} = \dfrac{-15+52i+32}{9+64}$
$= \dfrac{17+52i}{73} = \dfrac{17}{73} + \dfrac{52}{73}i$
$= \left(\dfrac{17}{73}, \dfrac{52}{73}\right)$ Ans.

⑬ Let the two conjugate complex numbers be
$z = x+iy$ and $\bar{z} = x-iy$ where $x, y \in \mathbb{R}$

Sum $= z + \bar{z}$
$= x+iy+x-iy$
$= 2x \in \mathbb{R} \quad \because x \in \mathbb{R}$

Amir Mahmood

Product $= z\cdot\bar{z}$

$= (x+iy)(x-iy)$

$= x^2 - i^2y^2$

$= x^2 - (-1)y^2 = x^2+y^2 \in \mathbb{R}$

$\because x, y \in \mathbb{R}$

(4) i) $(-4,7)$

Let $z = (-4,7)$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(-4,7)} = \frac{1}{-4+7i} = \frac{1}{-4+7i} \times \frac{-4-7i}{-4-7i}$

$= \frac{-4-7i}{(-4)^2-(7i)^2} = \frac{-4-7i}{16-49i^2}$

$= \frac{-4-7i}{16-49(-1)} = \frac{-4-7i}{16+49} = \frac{-4-7i}{65}$

$= -\frac{4}{65} - \frac{7}{65}i = \left(-\frac{4}{65}, -\frac{7}{65}\right)$ Ans.

ii) Let $z = (\sqrt{2}, -\sqrt{5})$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(\sqrt{2},-\sqrt{5})} = \frac{1}{\sqrt{2}-\sqrt{5}i}$

$= \frac{1}{\sqrt{2}-\sqrt{5}i} \times \frac{\sqrt{2}+\sqrt{5}i}{\sqrt{2}+\sqrt{5}i} = \frac{\sqrt{2}+\sqrt{5}i}{(\sqrt{2})^2-(\sqrt{5}i)^2}$

$= \frac{\sqrt{2}+\sqrt{5}i}{2-5i^2} = \frac{\sqrt{2}+\sqrt{5}i}{2-5(-1)} = \frac{\sqrt{2}+\sqrt{5}i}{2+5}$

$= \frac{\sqrt{2}+\sqrt{5}i}{7} = \frac{\sqrt{2}}{7} + \frac{\sqrt{5}}{7}i$

$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$ Ans.

iii) Let $z = (1,0)$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(1,0)} = \frac{1}{1+i\cdot 0} = \frac{1}{1+0} = \frac{1}{1} = 1$

$= 1 + i\cdot 0 = (1,0)$ Ans.

(15) i) a^2+4b^2

$= a^2 - (-1)4b^2 = a^2 - i^2 2^2 b^2$

$= (a)^2 - (i2b)^2$

$= (a+i2b)(a-i2b)$

$= (a+2bi)(a-2bi)$ Ans.

ii) $9a^2+16b^2$

$= 9a^2 - (-1)16b^2$

$= (3a)^2 - i^2 4^2 b^2 = (3a)^2 - (i4b)^2$

$= (3a+i4b)(3a-i4b)$

$= (3a+4bi)(3a-4bi)$ Ans.

iii) $3x^2+3y^2$

$= 3(x^2+y^2)$

$= 3[x^2-(-1)y^2]$

$= 3[(x)^2 - i^2y^2] = 3[(x)^2-(iy)^2]$

$= 3(x+iy)(x-iy)$ Ans.

(16) i) $\frac{2-7i}{4+5i} = \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$

$= \frac{8-10i-28i+35i^2}{(4)^2-(5i)^2}$

$= \frac{8-38i+35(-1)}{16-25i^2} = \frac{8-38i-35}{16+25}$

$= \frac{-27-38i}{41} = -\frac{27}{41} - \frac{38}{41}i$ Ans.

ii) $\frac{(-2+3i)^2}{1+i} = \frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$

$= \frac{4+9i^2-12i}{1+i} = \frac{4-9-12i}{1+i} = \frac{-5-12i}{1+i}$

$= \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-5+5i-12i+12i^2}{1^2-i^2} = \frac{-5-7i-12}{1-(-1)}$

$= \frac{-17-7i}{1+1} = \frac{-17-7i}{2} = -\frac{17}{2} - \frac{7}{2}i$ Ans

iii) $\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1^2-i^2}$

$= \frac{i-(-1)}{1-(-1)} = \frac{i+1}{2} = \frac{1+i}{2}$

$= \frac{1}{2} + \frac{i}{2}$ Ans.