13 SCHEDULING

Objectives

After studying this chapter you should

- be able to apply a scheduling algorithm to Critical Path Analysis problems;
- appreciate that this does not always produce the optimum solution;
- be able to design methods for solving packing problems;
- be able to use the branch-and-bound method for solving the knapsack problem.

13.0 Introduction

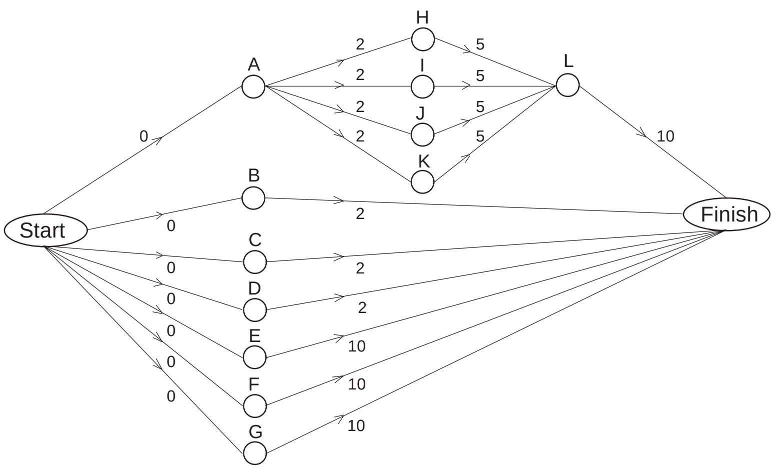
In the previous chapters it was possible to find the critical path for complex planning problems, but no consideration was given to how many workers would be available to undertake the activities, or indeed to how many workers would be needed for each activity.

You will see how scheduling methods can be applied to Critical Path Analysis problems but, importantly, it will be shown that such methods do not necessarily give the **optimal** solution every time.

You will also see how this scheduling problem is related to bin-filling problems, and to a similar problem called the knapsack problem in which the items carried not only have particular **weight**, but also have an appropriate **value**.

Activity 1

Find the critical path for the activity network shown below.



Suppose that each activity can be undertaken by a single worker, and that there are just 2 workers available. Also, that once an activity has been started by a worker, it must be completed by that same worker with no stoppages.

Design a schedule for these two workers so that the complete project is completed in the **minimum** time possible.

Does the minimum completion time depend on the number of workers available?

13.1 Scheduling

As in Activity 1, the following **operating rules** will be assumed:

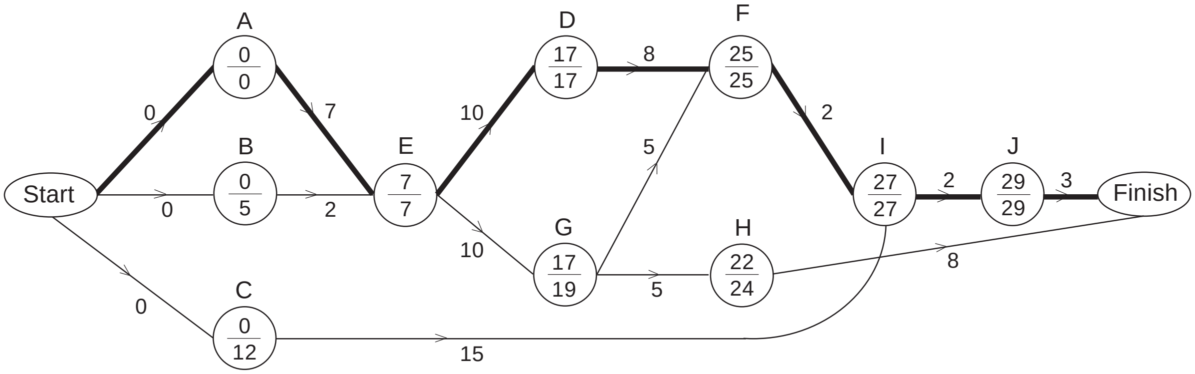
1. Each activity requires only **one** worker.
2. No worker may be idle if there is an activity that can be started.
3. Once a worker starts an activity, it must be continued by that worker until it is completed.

The **objective** will be to:

'Complete the project as soon as possible with the available number of workers.'

The main example from Chapter 12, which related to the construction of a garage, will be used to illustrate the problem.

The activity network, earliest and latest starting times, and the critical path (bold line), are shown below.

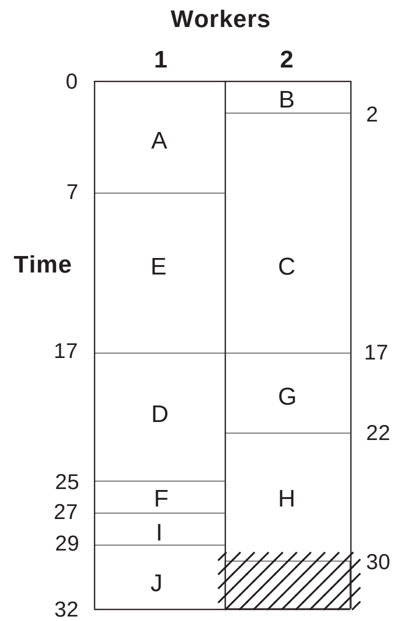


Suppose there are two workers available for the complete project. What is needed is a set procedure in order to decide who does what.

Can you think what would be a suitable procedure for allocating workers to activities?

The method that will be adopted can be summarised as follows.

> At any stage, when a worker becomes free, consider all the activities which have not yet been started but which can now be started. Assign to the worker the most 'critical' one of these (i.e. the one whose latest starting time is the smallest). If there are no activities which can be started at this stage you may have to wait until the worker can be assigned a job.



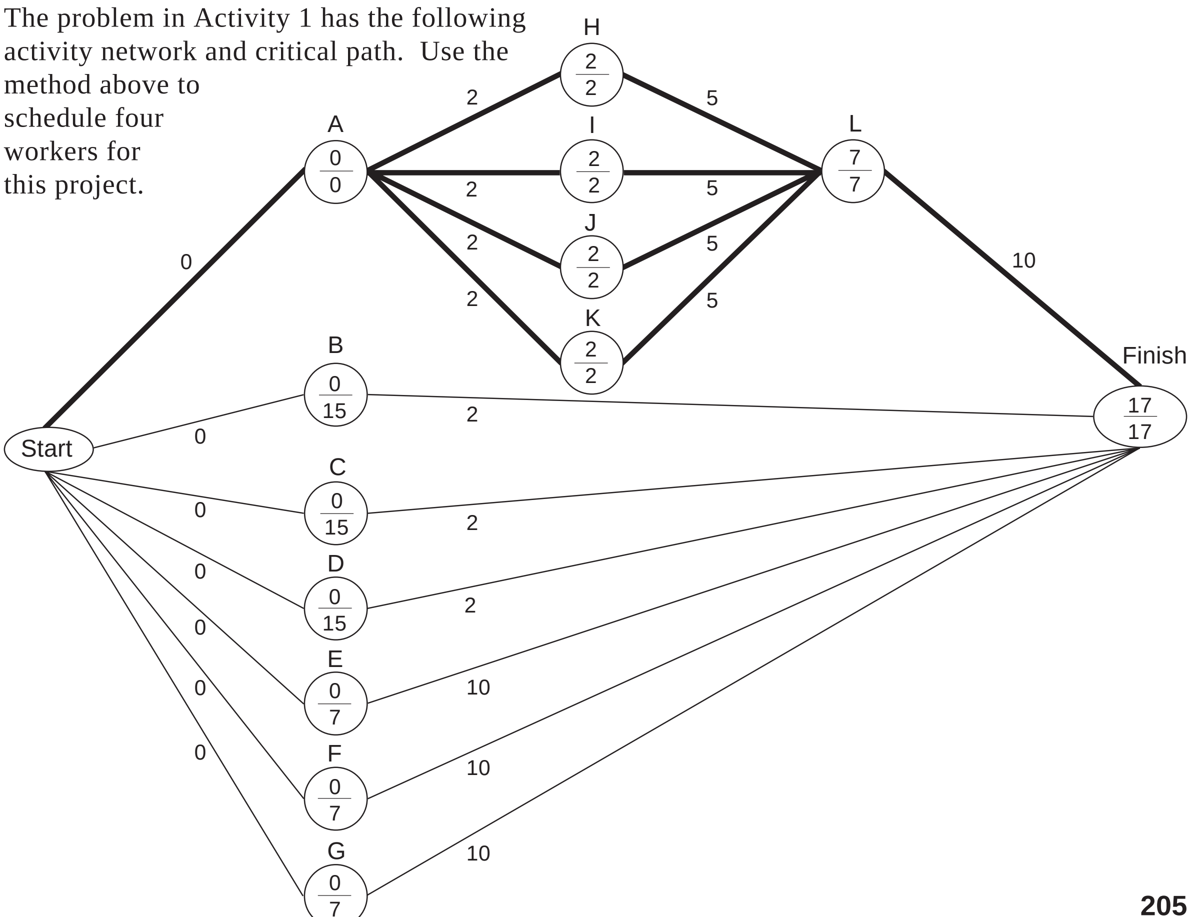
Using this as a basis for decisions, the solution shown opposite is obtained.

Note that worker 1 completes all the activities on the critical path, though, at time $t = 17$, workers 1 and 2 could have swopped over.

Since the whole project is completed in time 32 days, which you already know to be the minimum completion time, you can be assured that this method has produced an optimum solution. However, this is not always the case, as you will see in the next example.

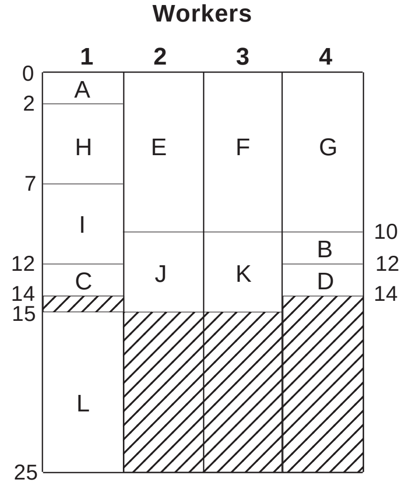
Example

The problem in Activity 1 has the following activity network and critical path. Use the method above to schedule four workers for this project.



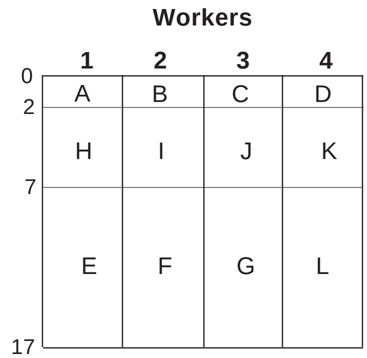
Solution

Applying the method as before results in the schedule shown in the first diagram opposite. This gives a completion time of $t = 25$.



Is this solution optimal?

It should not take too long to find a schedule which completes the project in time 17. A possible solution is shown in the second diagram .



As no worker is ever idle, and they all finish at time 17, this must be an optimum solution - you cannot do better! So the algorithm does not always produce the optimum solution. Currently no procedure exists which always guarantees to give the optimum scheduling solution, except the method of exhaustion where every possible schedule is tested.

Activity 2

A possible revision of the method is to :

> **Evaluate** for each activity, the sum of the earliest and latest starting times, and **rank** the activities in ascending order according to this sum.

Then activities are assigned according to this ranking, taking the precedence relations into account. Use this method to schedule the project above, again using four workers. Does it produce the optimal solution?

Activity 3

Design your own method of scheduling. Try it out on the two examples above.

Exercise 13A

1. Use the first scheduling method to find a solution to Question 3 in Section 12.4, using two workers. Does this produce an optimum schedule?

2. Find a schedule for the problem given in Question 3, Exercise 12A, using 2 workers and the two scheduling algorithms given in this section.

   Does either of these methods provide an optimal solution?

3. A possible modification to the method in this section is as follows :

   'Evaluate for each activity the **product** of the earliest and latest starting times, and rank the activities in ascending order according to these numbers. Assign activities using this ranking, taking the precedence relations into account.'

   Use this method to find possible schedules for the garage construction problem in this section. Does this method always give an optimum solution?

13.2 Bin packing

In the previous section, you saw how to schedule activities for a given number of workers in order to complete the project in minimum time. In this section, the problem is turned round and essentially asks for the minimum number of workers required to complete the project within a given time. It will be assumed that there are no precedence relations. The difficulties will be illustrated in the following problem.

A project consists of the following activities (with no precedence relations):

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (in days)	8	7	4	9	6	9	5	5	6	7	8

What is the minimum number of workers required to complete the project in 15 days?

Find a lower bound to the minimum number of workers needed.

Activity 4

Show that there is a solution to this problem which uses only five workers.

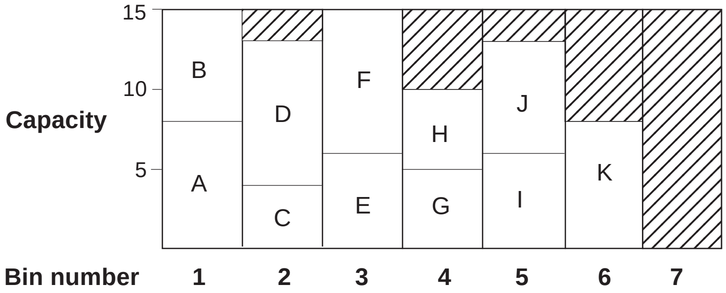
You should obtain a solution without too much difficulty. It is more difficult, however, to find an **algorithm** to solve such problems. The problem considered here is one involving **bin packing**. If you replace workers by bins, each having a maximum capacity of 15 units, the problem is to use the minimum number of bins.

Think about how a precise method could be designed to solve problems of this type.

One possible method is known as **first-fit packing** :

> Number the bins, then always place the next item in the lowest numbered bin which can take that item.

Applying this method to the problem above gives the following solution.



Using this method, six bins are needed, but you should have found a solution in Activity 4 which needs just five bins. So this method does not necessarily produce the optimum solution.

How can the first-fit method be improved?

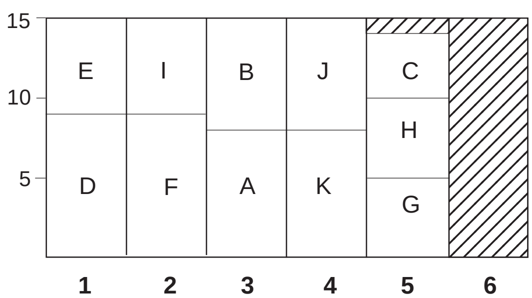
Looking at the way the method works, it seems likely that it might be improved by just reordering the items into decreasing order of size, so that the items of largest size are packed first. Then you have the **first-fit decreasing method** :

1. Reorder the items in decreasing order of size.
2. Apply the first fit procedure to the reordered list.

You will see how this works using the same problem as above. First reorder the activities in decreasing size.

Activity	**D**	**F**	**A**	**K**	**B**	**J**	**E**	**I**	**G**	**H**	**C**
Duration (in days)	9	9	8	8	7	7	6	6	5	5	4

and then apply the method to give the solution below.



This clearly gives an optimal solution.

Will this method always give an optimal solution?

Bin-packing problems occur in a variety of contexts. As you have already seen, one context is that of determining the minimum number of workers to complete a project in a specified time period. Other examples occur in:

Plumbing in which it is required to minimise the number of pipes of standard length required to cut a specified number of different lengths of pipe.

Advertising on television, in which case the bins are the standard length breaks between programmes, with the problem of trying to pack a specified list of adverts into the smallest number of breaks.

Example

A builder has piping of standard length 12 metres.

The following sections of various lengths are required

Section	A	B	C	D	E	F	G	H	I	J	K	L
Length (in metres)	2	2	3	3	3	3	4	4	4	6	7	7

Find a way of cutting these sections from the standard 12 m lengths so that a minimum number of lengths is used. Use

(a) first-fit method,

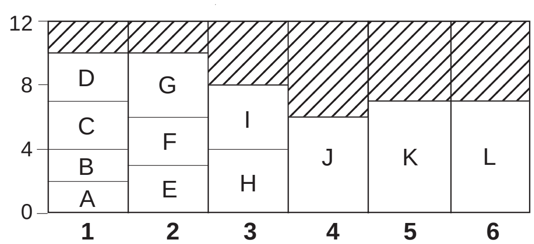
(b) first-fit decreasing method,

(c) trial and error,

to find a solution.

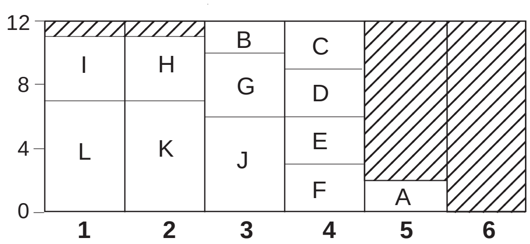
Solution

(a) First-fit method

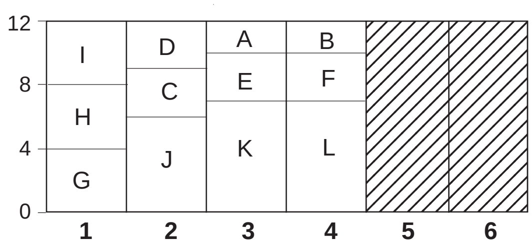


(b) First-fit decreasing method

Section	L	K	J	I	H	G	F	E	D	C	B	A
Length (in metres)	7	7	6	4	4	4	3	3	3	3	2	2



(c) Trial and Error



Note that even the first-fit decreasing method does not necessarily give the optimum solution, as shown above. An indication of the number of bins required can be obtained by evaluating

$$\frac{\text{sum of all sizes}}{\text{bin size}}$$

and noting the smallest integer that is greater than (or equal to) this number. This integer is a **lower bound** to the number of bins required. However you cannot always obtain a solution with this number.

Example

Find an optimum solution for fitting items of size

7, 6, 6, 6, 4, 3

into bins of size 11.

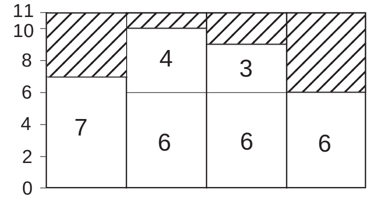
Solution

Noting that

$$\frac{\text{sum of all sizes}}{\text{bin size}} = \frac{7+6+6+6+4+3}{11} = \frac{32}{11} = 2\tfrac{10}{11},$$

it can be see that three is a lower bound to the number of bins required. But it is clear that four bins will in fact be needed and that no solution exists using just three bins.

A possible solution is given opposite.



Exercise 13B

1. A project consists of eight activities whose durations are as follows. There are no precedence relations.

Activity	A	B	C	D	E	F	G	H
Duration (hours)	1	2	3	4	4	3	2	1

It is required to find the minimum number of workers needed to finish the project in 5 hours.

Find the answers to this problem given by

(a) the first-fit packing method;

(b) the first-fit decreasing method.

2. A plumber uses pipes of standard length 10 m and wishes to cut out the following lengths

Length (m)	10	9	8	7	6	5	4	3	2	1
Number	0	0	2	3	1	1	0	2	3	0

Use the first-fit decreasing method to find how many standard lengths are needed to meet this order. Does this method give an optimum solution? If not, find an optimum solution.

3. Determine the minimum number of sheets of metal required, 10 m by 10 m, to meet the following order, and how they should be cut. (Assume no wastage in cutting.)

Size	Number
$3\times1\text{m}^2$	60
$4\times2\text{ m}^2$	49
$7\times5\text{ m}^2$	12

Develop a **general** method of solving 2-dimensional packing problems of this type.

*13.3 Knapsack problem

For the bin-filling problem, the aim was to pack items of different sizes into a minimum number of bins. Now suppose that there is just **one** bin, but that each item has a value associated with it. Thus the question is what items should be packed in order to maximise the total value of the items packed. The problem is known as the **knapsack** problem as it can be interpreted in terms of a hiker who can only carry a certain total weight in his/her knapsack (rucksack). The hiker has a number of items that he/she would like to take, each of which has a particular value. The problem is to decide which items should be packed so that the total value is a maximum, subject to the weight restriction.

This type of problem will be solved using a technique called the **branch and bound method**. How it works will be shown using the following particular problem.

Suppose a traveller wishes to buy some books for his journey. He estimates the time it will take to read each of five books and notes the cost of each one :

Book	A	B	C	D	E
Cost (£)	4	6	3	2	5
Reading time (hours)	5	9	4	4	4

Which of these books should he buy to maximise his total reading time without spending more than £8?

Activity 5

By trial and error, find the solution to the traveller's problem.

As you have probably seen, with just a few items it is easy enough to find the optimum solution. However, in the example above, if there was a choice of, say, 10 or 15 books, the problem of finding the optimum solution would now be far more complex.

The first step in the branch and bound method is to list the items in decreasing order of reading time per unit cost.

Item	**A**	**B**	**C**	**D**	**E**	
Cost	4	6	3	2	5	('weight')
Reading time	5	9	4	4	4	('value')
Reading time per unit cost	1.25	1.5	1.33	2	0.8	('value per unit weight')

Reordering,

Number	**1**	**2**	**3**	**4**	**5**
Item	D	B	C	A	E
Cost (w)	2	6	3	4	5
Value (v)	4	9	4	5	4
Value per unit cost	2	1.5	1.33	1.25	0.8

A solution **vector** of the form $(x_1, x_2, x_3, x_4, x_5)$ will be used to denote a possible solution where

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is bought} \\ 0 & \text{if item } i \text{ is not bought} \end{cases}$$

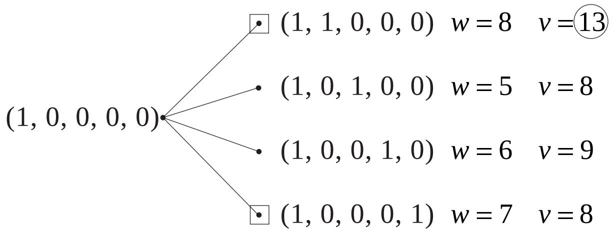
So, for example,

$$\mathrm{x} = (0, 0, 1, 0, 1)$$

means buying books C and E, which have total cost £8 and total value $4 + 4 = 8$ hours of reading time.

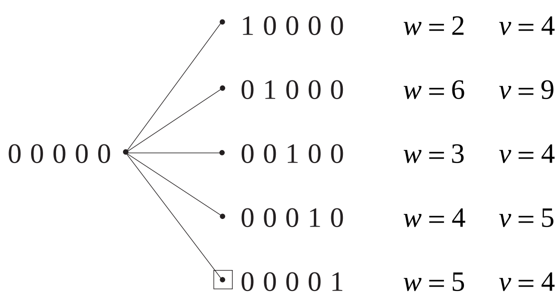
The method uses a branching method to search for the optimal solution.

For example let us look at the possible branches from (1, 0, 0, 0, 0).

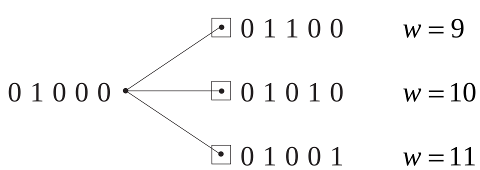


The 'square' shows that there can be no further branching from this point. For example, there is a square by (1, 1, 0, 0, 0) because the total allowed cost (weight) of 8 has been reached. Also we always add additional 1s to the right of the last 1so that, for example, you could branch from (1, 0, 0, 1, 0) to (1, 0, 0, 1, 1) but you would never consider branching from (1, 0, 0, 1, 0) to (1, 0, 1, 1, 0). So when, in the example above, x_5 is 1, no further branching is possible : that accounts for the square by (1, 0, 0, 0, 1). Note that the best solution (i.e. maximum *v*) at this stage is $v = 13$.

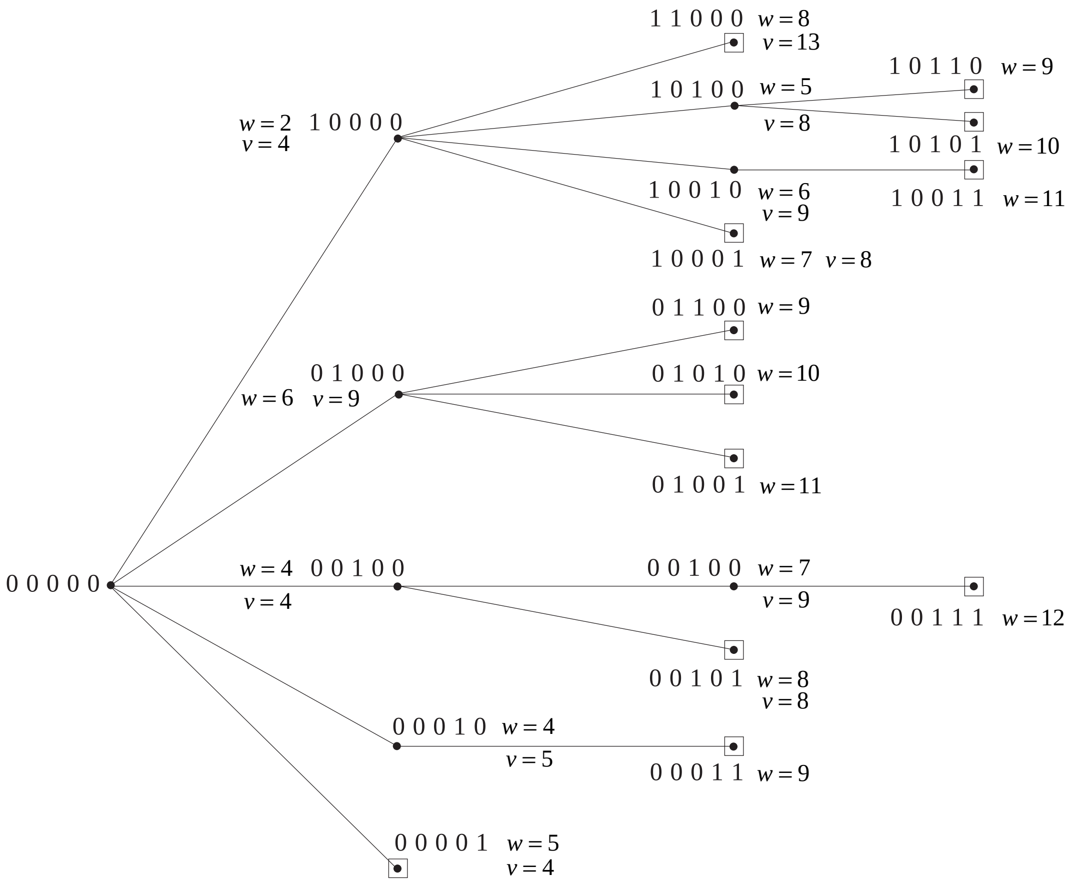
You start the full process at the **null** solution (0, 0, 0, 0, 0), now written more simply as 0 0 0 0 0, and then keep repeating the process as outlined below.



You can now branch from any of the four vertices which are not squared: for example, the branches from 1 0 0 0 0 are as shown above and the branches from 0 1 0 0 0 are as shown below :



Continue in this way to give one single diagram, as shown on the next page:



Then from amongst the squared vertices with $w \le 8$, you find the one with the highest *v*, namely 1 1 0 0 0 in the this case. That means that the traveller should take books D and B, giving a total reading time of 13 hours.

Exercise 13C

1. Suppose that a hiker can pack up to 9 kg of items and that the following items are available to take. The value of each item is also specified.

Item	A	B	C	D	E
Weight (kg)	3	8	6	4	2
Value	2	12	9	3	5

Use the branch and bound method to find the items that can be taken which give a maximum total value.

2. The manager of a firm which has installed a small computer system has received requests from four potential users. The computer will be run for up to 24 hours per day but can cope with only one user at a time. The manager estimates that the number of hours of computer time required by each user per day, and the consequent likely income for the firm, are as follows:

User	A	B	C	D
Use (hours/day)	8	12	13	4
Income (£1000/year)	72	102	143	38

Use the branch and bound method to determine which users should be allocated time on the computer system so as to maximise the total income.

3. A machine in a factory can be used to make any one of five items, A, B, C, D and E. The time taken to produce each item, and the value of each item, are shown in the table opposite.

   If the machine is available for only 10 days, use the branch and bound method to determine which of the items should be produced so that the total value is as large as possible.

Item	A	B	C	D	E
Production time (in days)	3	7	2	4	4
Value	3	14	3	7	8

13.4 Miscellaneous Exercises

1. A project consists of ten activities A-J with the following durations (in hours). There are no precedence relations.

Activity	A	B	C	D	E	F	G	H	I	J
Duration	2	3	4	5	6	7	8	9	10	11

   (a) Find the minimum number of workers needed to complete this project in 16 hours.

   (b) Use (i) the first-fit packing method,

   (ii) the first-fit decreasing method.

   Does either of these methods produce an optimum solution?

2. A hiker wishes to take with her a number of items. Their weights and values are given in the table below.

Item	A	B	C	D	E
Weight (kg)	5	4	7	3	6
Value	3	3	4	2	4

   If the maximum weight she can carry is 12 kg, find by trial and error the best combination of items to carry. Use the branch and bound method to confirm your solution as optimal.

3. A small firm orders planks of wood of length 20 m. Each week the firm orders a certain number of planks and then has to meet the orders for that week. Use a bin-filling method to find the minimum number of planks required to meet the following weekly orders.

(a)

Length	Number Required
3 m	5
4 m	6
5 m	2
7 m	2
8 m	1
9 m	1

(b)

Length	Number Required
11 m	1
9 m	1
7 m	2
5 m	2
3 m	12

(c)

Length	Number Required
15 m	1
12 m	2
11 m	1
7 m	3
3 m	2

   Does the method always produce the optimum solution?

4. The durations of nine activities A to I are given below, in days:

Activity	A	B	C	D	E	F	G	H	I
Duration	5	9	1	7	3	2	8	4	6

   There are no precedence relations, but each activity must be completed by just one worker. We wish to find the minimum number of workers needed to complete all nine activities in 12 days.

   (a) What answer is given by the first-fit packing method?

   (b) If instead the first-fit decreasing method were used, what answer would then be found?

5. A project consists of eight activities A - H with the following durations (in days). There are no precedence relations.

Activity	A	B	C	D	E	F	G	H
Duration	8	3	9	1	2	6	7	4

   It is required to find the minimum number of workers needed to complete this project in 10 days. Each activity is to be completed by a single worker.

   (a) What answer is given to this problem by the first-fit packing method?

   (b) What answer is given by the first-fit decreasing method?

   (In each part you should draw a diagram to show which tasks are allocated to which workers.)