1 GRAPHS

Objectives

After studying this chapter you should

- be able to use the language of graph theory;
- understand the concept of isomorphism;
- be able to search and count systematically;
- be able to apply graph methods to simple problems.

1.0 Introduction

This chapter introduces the language and basic theory of graphs. These are not graphs drawn on squared paper, such as you met during your GCSE course, but merely sets of points joined by lines. You do not need any previous mathematical knowledge to study this chapter, other than an ability to count and to do very simple arithmetic.

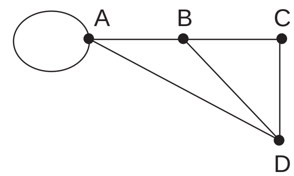
Although graph theory was first explored more than two hundred years ago, it was thought of as little more than a game for mathematicians and was not really taken seriously until the late twentieth century. The growth in computer power, however, led to the realisation that graph theory can be applied to a wide range of industrial and commercial management problems of considerable economic importance.

Some of the applications of graph theory are studied in later chapters of this book. Chapter 2, for example, looks at several different problems involving the planning of 'best' networks or routes, while Chapter 6 considers the question of planarity (very important in designing microchips and other electronic circuits). Chapter 7 deals with problems to do with the flow of vehicles through a road system or oil through a pipe, and Chapters 12 and 13 show how to analyse a complex task and determine the most efficient way in which it can be done. All these applications, however, depend on an understanding of the basic principles of graph theory.

1.1 The language of graphs

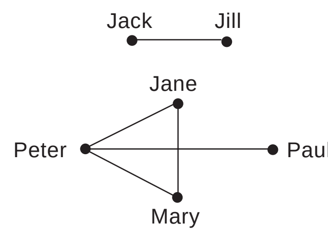
A **graph** is defined as consisting of a set of **vertices** (points) and a set of **edges** (straight or curved lines; alternatively called arcs): each edge joins one vertex to another, or starts and ends at the same vertex.

The diagrams show three different graphs, representing respectively the major roads between four towns, the friendships among a group of students, and the molecular structure of acetic acid - the theory of graphs can be applied in many different ways.



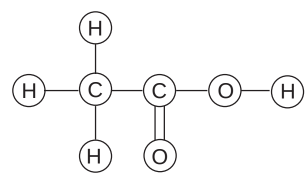
A road system

There are several things to note. One is that although nearly all the edges in these graphs have been drawn as straight lines, this is purely a matter of convenience. Curved lines would have done just as well, because what matters is which vertices are joined, not the shape of the line joining them. Second, each edge joins only two vertices, so that ABC in the first graph is two edges (AB and BC) rather than one long one. Third, the crossing in the middle of the second diagram is not a vertex of the graph; the only points counted as vertices are the ones identified as such at the start.



Friendships

The **degree** of a vertex is defined as the number of edges which start or finish at that vertex - an edge which starts and finishes at the same vertex (in other words, a **loop** such as the one at A in the first graph) is counted twice. So, for example, the degree of the vertex A in the first graph is 4, and the degree of the vertex 'Peter' in the second graph is 3. In the third case, the degree of each vertex corresponds to the valency of the atom.

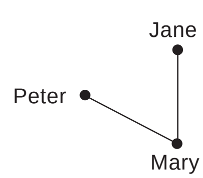


Acetic Acid CH_3COOH

There is actually something a little unusual in the third graph - two edges joining the same two vertices. A multiple edge of this kind can be of great importance in some situations: the difference between saturated and unsaturated fats in a healthy diet, for example, is largely a matter of multiple edges in their molecular structure. In other cases, however, such as the second graph here, a double or triple edge would be meaningless. A graph with no loops and no multiple edges is called a **simple** graph.

There is an oddity in the second graph too. Jack and Jill are friends with one another but with no one else, so that the graph 'falls apart' into two quite separate pieces. Such a graph is said to be **disconnected**. A **connected** graph is one in which every vertex is linked (by a single edge or a sequence of edges) to every other. If every vertex is linked to every other by a single edge, a simple graph is said to be **complete**.

A **subgraph** of a graph is another graph that can be seen within it; i.e. another graph consisting of some of the original vertices and edges. For example, the graph consisting of vertices 'Jane', 'Mary' and 'Peter' and edges from 'Jane' to 'Mary' and from 'Mary' to 'Peter' is a subgraph of the friendship graph above.

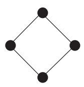
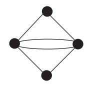
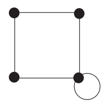
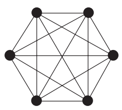
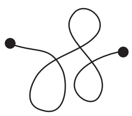
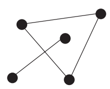


Exercise 1A

Note The answers to these questions will be used in later sections, and should be kept safely until then.

1. For each of the graphs shown below, write down

   (i) its number of vertices,
   (ii) its number of edges,
   (iii) the degree of each vertex.

(a)

(b)

(c)

(d)

(e)

(f)


2. Say which (if any) of the graphs in Question 1 are

   (i) simple (ii) connected and/or (iii) complete.

3. Draw graphs to fit the following descriptions:

   (a) The vertices are A, B, C and D; the edges join AB, BC, CD, AD and BD.

   (b) The vertices are P, Q, R, S and T, and there are edges joining PQ, PR, PS and PT.

   (c) The graph has vertices W, X, Y and Z and edges XY, YZ , YZ, ZX and XX.

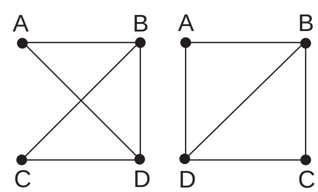
   (d) The graph has five vertices, each joined by a single edge to every other vertex.

   (e) The graph is a simple connected graph with four vertices and three edges.

1.2 Isomorphism

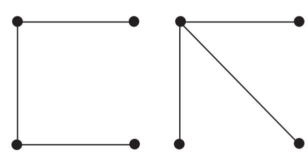
Look at your answers to Question 3 from Exercise 1A, and compare them with those of other students. You will probably find that some of the drawings look different from others and yet fit the descriptions equally well.

Two graphs which look different, but both of which are correctly drawn from a full description are said to be **isomorphic** - the word comes from Greek words meaning 'the same shape'. Isomorphism is a very important and powerful idea in advanced mathematics - it crops up in many different places - but at heart it is really very simple.



Two possible answers to Question 3 (a)

For example, the two graphs shown in the upper diagram each match the full description in Question 3(a) and so are isomorphic to one another. The graphs in the lower diagram each match the description in Question 3(e), but these are not isomorphic. The description did not say which vertices were to be joined by the edges, and the two graphs have joined the vertices differently.



Two possible answers to Question 3 (e)

If you are to say that two graphs are isomorphic, there must be a way of labelling or relabelling one or both of them so that the number of edges joining A to B in the first is equal to the number of edges joining A to B in the second, and so on through all possible pairs of vertices. In the upper diagram this is clearly possible: the labelling already on the graphs satisfies this condition, and indeed many people would say that the two graphs are more than isomorphic - they are identical. In the lower diagram, however, no such labelling can ever be found. The second graph has one vertex which is joined to three others, and no labelling of the first graph can ever match this.

Testing for isomorphism

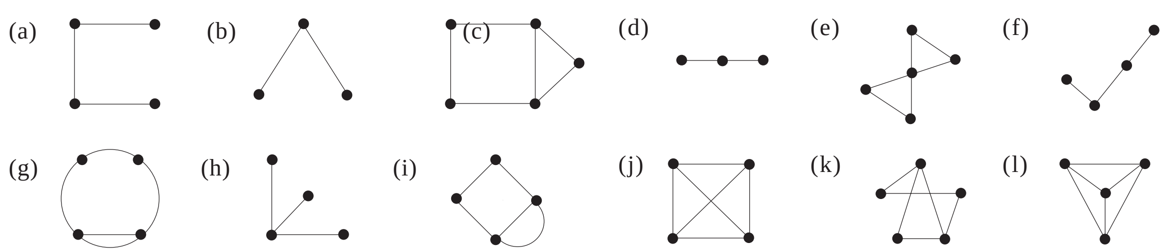
If you can match the labels it certainly shows that two graphs are isomorphic, but suppose you cannot. What does that show? It might mean that the graphs are not isomorphic, or it might simply be that you have not yet tried the right labelling combination.

How can you know if the two graphs are isomorphic?

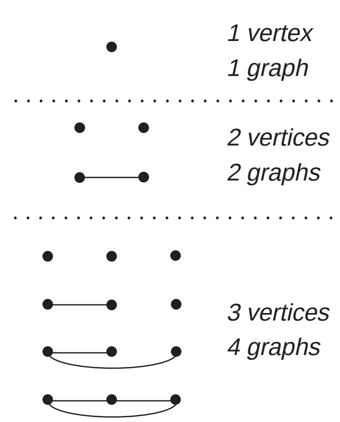
The clue is in the argument that has already been given. If one graph has a vertex of degree three, and the other does not, then no matching can ever be found and the graphs are not isomorphic. This idea can be extended to provide a partial test: a **necessary** condition for two graphs to be isomorphic is that the two graphs have the same number of vertices of degree 0, the same number of vertices of degree 1, and so on. If this condition is not satisfied the graphs are certainly not isomorphic. But it is not a **sufficient** condition; in other words, if the condition is satisfied you still do not know whether or not the graphs are isomorphic and you must go on looking for a match.

Exercise 1B

Look at the graphs below, and say which of them are isomorphic to which others.



Counting graphs



You may be wondering by now how many different simple graphs can be drawn with just a few vertices. With only one vertex (and no loops allowed) there is clearly only one such graph - the one with no edges.

With two vertices there are two possibilities: there is one graph with no edges and one with one edge, making two possible graphs altogether. You are counting simple graphs, remember, so multiple edges are excluded.

With three vertices there are four possibilities: one each with no edges, one edge, two edges and three edges respectively. Any other simple graph on three vertices must be isomorphic to one of these.

*Activity 1 Simple graphs

These results can be put into a table:

Vertices	1	2	3	4	5	6	...
Graphs	1	2	4	...	...	...	...

Try to predict the number of different simple graphs that can be drawn on four vertices, and then check your prediction by drawing them. It is worth doing this in discussion with another student to ensure that you do not leave out any possible graphs nor include two that are actually isomorphic.

When you have got a firm result for four vertices (and corrected your prediction if necessary), try to extend your prediction to five and/or six vertices.

Activity 2 Handshakes

At the beginning of the lesson, greet some of the other members of your group by shaking hands with them. You don't have to shake hands with everyone, and you can shake hands with the same person more than once if you like, but you must keep count of how many handshakes you take part in.

At the end, some members of the group will have been involved in an odd number of handshakes, and others in an even number, so consider this bet: if the number of people involved in an odd number of handshakes is odd, your teacher lets you off homework for a week, but if it is even you get a double dose - does that seem fair?

You may guess that this is not a good bet at all from your point of view - not unless you like doing maths homework, that is! In fact you can never win, because the number of people who shake hands an odd number of times is always even.

Look at your answers to Question 1 in Exercise 1A. Any handshaking situation can be represented by a graph, with people as vertices and handshakes as edges; it may have multiple edges, but not loops. For each graph, find the sum of the degrees of the vertices, and compare it with your other data.

The handshake lemma

You can see at once that the sum of the degrees of the vertices is always twice the number of edges. This is known as the handshake lemma - a lemma is a mini-theorem - and is easy to prove. The degree of each vertex is the number of edge-ends at that vertex, and since each edge has two ends, the number of edge-ends (and hence the total of the vertex degrees) must be twice the number of edges.

This lemma leads quite easily to the unwinnable bet. If the total of individual handshakes is twice the number of handshakes, as the lemma requires, it is certainly an even number. Some members of the class shook hands an even number of times, and the total of any number of even numbers is even. So the total for the rest must be even as well, and since they each shook hands an odd number of times, this can happen only if there are an even number of people. So the number of people involved an an odd number of handshakes must always be even.

The handshake lemma may seem trivial, but it has some quite important consequences and comes up again in Chapter 6.

Activity 3

Try the handshaking exercise again, and this time keep count not of the number of handshakes, but of the number of people with whom you shake hands (once or more times makes no difference). What are the chances that at the end there will be two people who have shaken hands with the same number of others?

The pigeonhole principle is considered in more depth in Chapter 3.

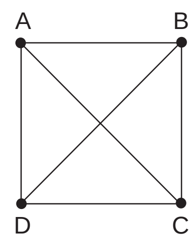
It will perhaps not surprise you to learn that such a coincidence is certain to happen. The proof of this depends on a simple but important principle known as the pigeonhole principle. If n objects have to be put into m pigeonholes, where $n > m$, it says that there must be at least one pigeonhole with more than one object in it. Like the handshake lemma, the pigeonhole principle seems obvious but has a number of uses.

For example, suppose there are nine people in the room: each must have shaken hands with 0, 1, 2, 3, 4, 5, 6, 7 or 8 others. Of course, if anyone has shaken hands with 8 others - that is, with everyone else - then there cannot be anyone who has shaken hands with 0 others, and vice versa. So among the nine people there are at most eight different scores and the pigeonhole principle says that at least two people must therefore have the same score. You can apply the same argument to any number of people more than one.

1.3 Walks, trails and paths

If you have read any other books about graph theory, you may find this next section rather confusing. Graph theory is a relatively new branch of mathematics, and as yet there is no universal agreement as to the meanings to be given to certain terms. The consequence is that what is called a trail here might be called a walk in another book and a path in a third - the ideas are common but the words are different. The definitions to be used in this book are as follows:

A **walk** is a sequence of edges of a graph such that the second vertex of each edge (except for the last edge) is the first vertex of the next edge. For example, the sequence CD, DA, AB, BD, DA defines a walk (which might be called a walk from C to A) in the graph shown in the diagram. A walk can be the trivial one with no edges at all!

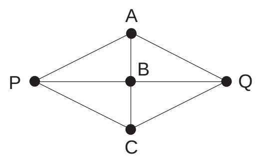


A **trail** is a walk such that no edge is included (in either direction) more than once in the sequence. The walk above is not a trail because the edge DA occurs twice, but CD, DA, AB, BC, CA is a trail from C to A.

A **path** is a trail such that no vertex is visited more than once (except that the first vertex may also be the last); the trail above is not a path because both A and C are visited more than once, but CD, DA, AB is a path from C to B.

Exercise 1C

1. Referring to the graph in the diagram below, list



   (a) all the paths from P to Q;

   (b) at least three trails from P to Q which are not paths;

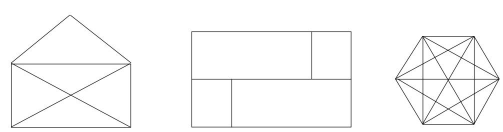
   (c) at least three walks from P to Q which are not trails;

   (d) all the paths which start and finish at P.

2. Which (if any) of the shapes below can you draw completely without lifting your pencil from the paper or going over any line twice?

   (If you invent appropriate vertices and imagine them as graphs, then you are looking for a trail which includes all the edges)

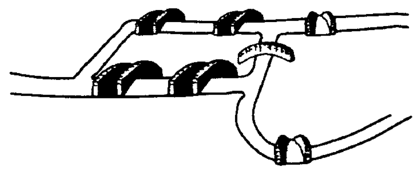
(a) (b) (c)



1.4 Cycles and Eulerian trails

Cycles

Puzzles involving trails and paths have been popular for many years, and you may well have seen some or all of the graphs above in books of recreational mathematics. Of particular interest are walks, etc which start and finish at the same place; a walk, trail or path which finishes at its starting point is said to be **closed**, and a closed path with one or more edges is called a **cycle**.

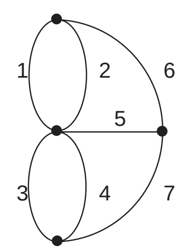


Konigsberg bridges

Modern graph theory effectively began with a problem concerning a closed trail. In the 18th century the citizens of the Prussian city of Königsberg (now called Kaliningrad) used to occupy their Sunday afternoons in going for walks. The city stood on the River Pregel and had seven bridges, arranged as shown in the diagram. The citizens' aim was to find a route that would take them just once over each bridge and home again.

Activity 4 Königsberg bridges

Try to find a route crossing each bridge just once and returning to the starting point.



Königsberg bridges in graph form

If you failed, don't worry - so did the people of Königsberg! They began to realise that such a route was impossible, but it was some years before the great Swiss mathematician *Leonhard Euler* (1707-83) proved that this was indeed so. The modern proof, developed from Euler's, is very simple once the bridges are represented by edges of a graph.

How can you be sure that there is no closed trail using all the edges of this graph?

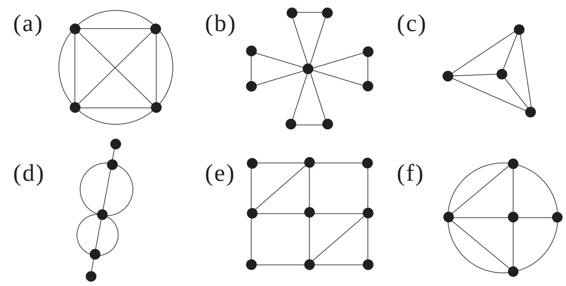
Eulerian trails

The four vertices of the graph have degrees 3, 3, 3 and 5 respectively - all odd numbers. Any closed trail, on the other hand, goes into a vertex and out of it again, thus adding 2 to its degree on each visit. A closed trail using all the edges cannot exist, therefore, unless every vertex has even degree. (If there are just two vertices with odd degree, they could be the start and finish of a non-closed trail using all the edges.) In fact the opposite is also known: if a connected graph has every vertex of even degree then there does exist a closed trail using all the edges (and if there are just two vertices of odd degree then there is a non-closed trail using all the edges).

As a mark of respect for Euler's work in this area, a trail which includes every edge of a graph is called an **Eulerian trail**. If the trail is closed, the graph itself is said to be **Eulerian**; a semi-Eulerian graph is one that has a non-closed trail including every edge.

Exercise 1D

By considering the degree of each vertex, determine whether each of the graphs shown opposite is Eulerian, semi-Eulerian, or neither. In the case of Eulerian and semi-Eulerian graphs, find an Eulerian trail.

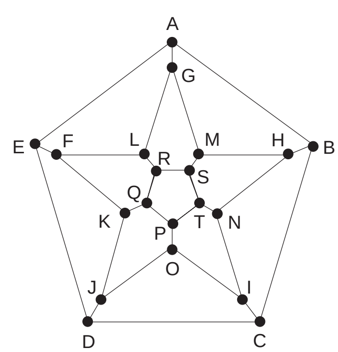


1.5 Hamiltonian cycles

Sir William Rowan Hamilton (1805-65) was a nineteenth-century Irish mathematician who invented in his spare time a game called the **Icosian Game**, based on the vertices of an icosahedron. The idea was essentially simple: given the first five vertices, the player had to find a route that would pass through the remaining fifteen and return to the start without using any vertex twice.

Activity 5 Icosian game

The diagram shows a graph representing a dodecahedron. Try to find such a route - a closed path, to use the modern phrase - beginning with ABCIN in that order.



A closed path that passes through every vertex of a graph is called a **Hamiltonian cycle**, and a graph in which a Hamiltonian cycle exists is said to be **Hamiltonian**. The dodecahedron is a Hamiltonian graph, and there are actually two Hamiltonian cycles beginning with the five vertices given:

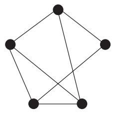
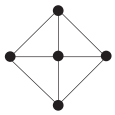
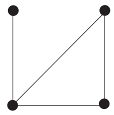
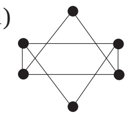
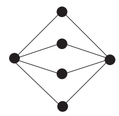
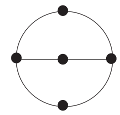
ABCINHMSTPOJDEFKQRLGA

and ABCINHMGLFKQRSTPOJDEA.

Distinguishing Hamiltonian from non-Hamiltonian graphs is not easy, and there is no simple test corresponding to the even-degree test for Eulerian graphs.

* Exercise 1E

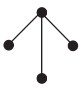
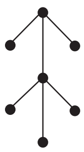
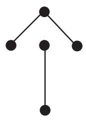
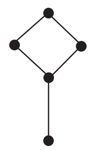
Decide by trial and error whether or not each of the graphs shown below is Hamiltonian.

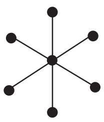
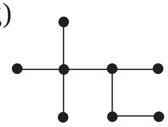
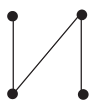
(a)  (b)  (c)  (d)  (e)  (f) 

1.6 Trees

A connected graph in which there are no cycles is called a **tree**.

Look at the graphs below and decide which of them are trees.

(a)  (b)  (c)  (d) 

(e)  (f) (g)  (h) 

Activity 6

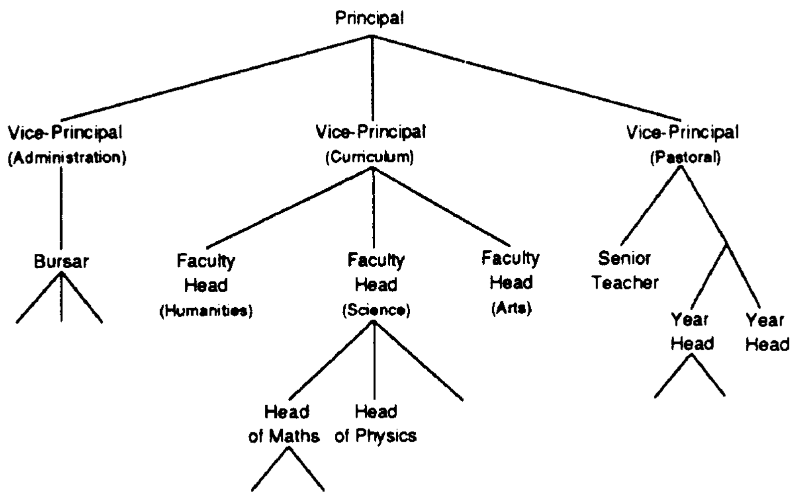
Look again at the graphs you have identified as trees, and count their vertices and their edges. Can you state a general theorem connecting the numbers of vertices and edges for trees? If so, can you prove it?

It is fairly easy to guess from the examples that the number of edges of a tree is always one less than the number of vertices. The proof too is straightforward: if the tree is built up one vertex at a time, starting with one vertex and no edges, each new vertex needs exactly one edge to join it to the body of the tree.

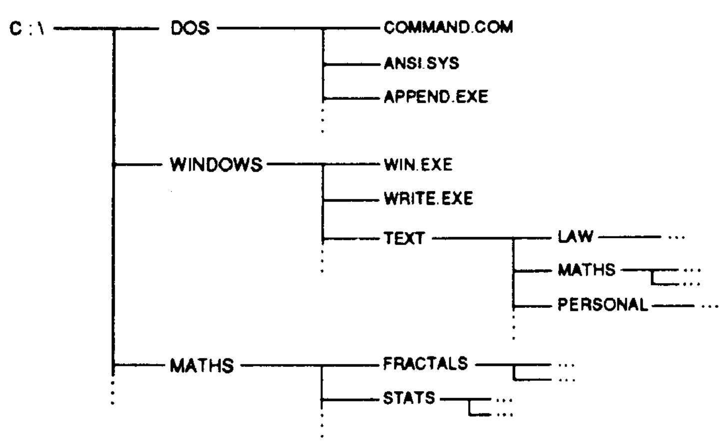
Trees of this kind occur quite often in real life - a biology book may include a 'tree' showing how all living creatures are ultimately descended from the same primitive life forms; a geography text may include a diagram of the entire Amazon river system; and you may find in a history book a diagram of the Kings and Queens of England, although a certain amount of from being truly a tree as defined above.

Hierarchies

Trees are also commonly used to represent hierarchical organisations. The first diagram below shows part of the management structure of a college, for example, while the second is an extract from a computer's hard disk directory.



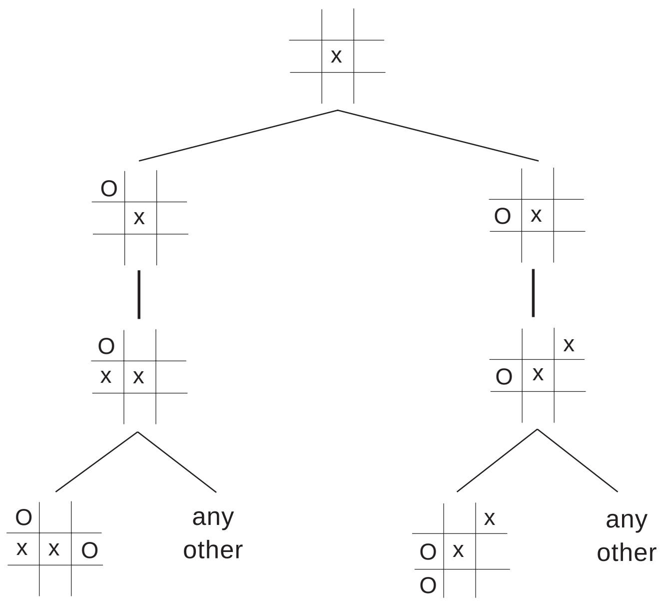
A management Tree



A computer directory tree

Game strategies

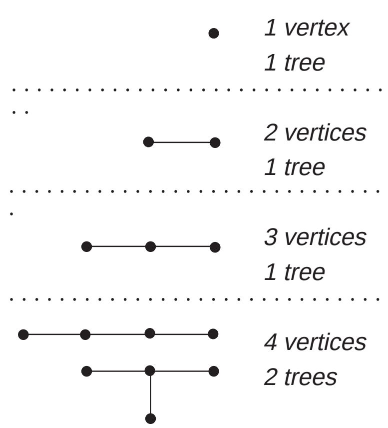
Another application of trees is in setting out strategies for playing certain games. For example, the diagram shows the first few stages of a strategy tree for the first player in the game "Noughts and Crosses". You may like to try to complete it.



A strategy tree for 'Noughts and Crosses'

Counting trees

There is clearly only one tree with one vertex, one with two, and one with three, as shown in the diagram - any other is isomorphic to one of these. There are two non-isomorphic trees with four vertices, however, and these figures can be set out in a table:



Vertices	1	2	3	4	5	6	7	...
Trees	1	1	1	2	...	...	...	...

*Activity 7 Counting trees

Draw all the different trees with five vertices, and all those with six, and try to predict from your results the number of seven-vertex trees. Check your prediction by drawing them.

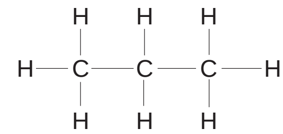
*Organic chemistry

In fact there is no simple formula for unlabelled trees - it turns out to be much easier to count trees if their vertices are labelled

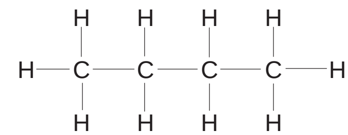
- but even the few results in your table can be of use in identifying chemical compounds.

The group of chemicals known as alkanes have molecules made up of a carbon 'tree' surrounded by hydrogen: each hydrogen atom is bonded to one carbon, and each carbon atom is bonded to four other atoms, of either kind. The diagram shows a molecule of propane (the fuel used in some camping gas stoves), which has three carbon and eight hydrogen atoms and so has molecular formula C_3H_8 .

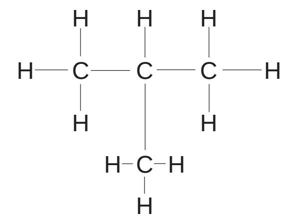
The molecule can actually be represented completely by its carbon tree, because once all the carbon atoms have been bonded in some formation the hydrogen atoms must go wherever there is a free bond. Now according to the table above there is only one possible carbon tree for CH_4 (methane), only one for C_2H_6 (ethane), and only one for C_3H_8 (propane), so each of these molecular formulae represents only one compound. But there are two distinct trees with four vertices, so the formula C_4H_{10} can represent either butane or isobutane, two different compounds with different properties.



Propane



Butane



Isobutane
(or 2-methyl-propane)

* Exercise 1F

1. How many different compounds have the molecular formula C_5H_{12}? (If you are studying A Level Chemistry, what are their names?)

2. How many different compounds have molecular formula C_5H_{14}? Think carefully before you answer.

* 1.7 Coloured cubes

You may have seen in the shops a puzzle consisting of four cubes with different colours or other designs on their sides. The aim of the puzzle is to stack the cubes in a tower so that each of the long faces shows four different colours or designs. A trial-and-error approach is very difficult, but the application of a little graph theory can lead directly to a solution.

Example

Suppose that the four cubes are coloured as shown.

Cube 1

red	*opposite*	yellow
green	*opposite*	yellow
blue	*opposite*	red

Cube 2

red	*opposite*	red
green	*opposite*	blue
blue	*opposite*	yellow

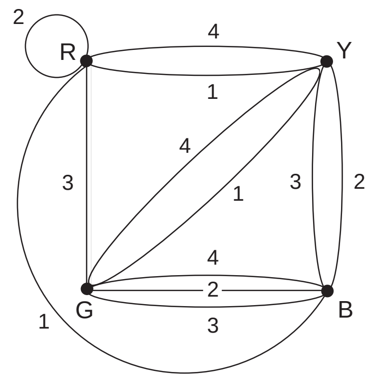
Cube 3

red	*opposite*	green
blue	*opposite*	green
blue	*opposite*	yellow

Cube 4

red	*opposite*	yellow
green	*opposite*	blue
green	*opposite*	yellow

Transfer all this information to a graph as shown in the diagram, joining the vertices representing opposite colours by an edge numbered to show the cube to which it belongs.



A graphical representation of the four cubes

From this graph, extract two disjoint subgraphs - that is, two subgraphs with no edges in common. Each subgraph must consist of four edges of the original graph, chosen in such a way that

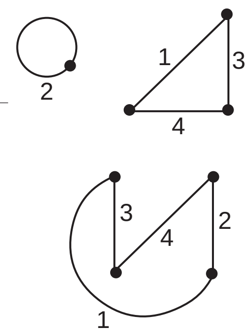
(i) the edges include one of each number, in any order,

(ii) each of the vertices R, Y, G, B has degree 2 in each subgraph.

Two subgraphs satisfying these conditions are shown in the lower diagram opposite.

The subgraphs now tell you how to stack the cubes:

	From first subgraph		From second subgraph	
	Front	**Back**	**Left**	**Right**
Cube 1	green	yellow	red	blue
Cube 2	red	red	blue	yellow
Cube 3	yellow	blue	green	red
Cube 4	blue	green	yellow	green



Two disjoint subgraphs

In this particular case there are two other solutions - there is a third subgraph that could have been chosen with either of the two above.

Activity 8

Find the third subgraph and interpret it in the same way as in the example.

Some patterns of cubes have only one solution, however, and others have no solution at all.

Activity 9

Try to get hold of a commercially-made puzzle of this kind, or make your own, and then amaze your family and friends (and perhaps yourself!) by using mathematics to solve it in just a few minutes.

1.8 Miscellaneous Exercises

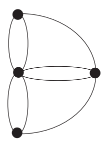
1. Draw two simple connected graphs, each with four vertices and four edges, which are not isomorphic.

2. If the vertices of a graph have degree 1, 2, 2, 2 and 3 respectively, how many edges has the graph? Draw two simple connected graphs, each with this vertex set, which are not isomorphic.

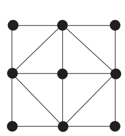
3. If P, Q, R, S and T are the vertices of a complete graph, list all the paths from S to T.

4. Determine whether each of the following graphs is Eulerian, semi-Eulerian or neither, and find an Eulerian trail if one exists.

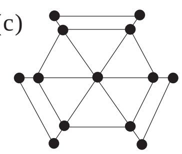
(a)



(b)



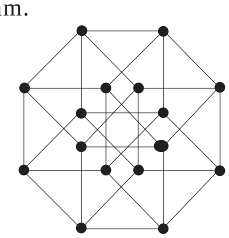
(c)



5. You are given nine apparently identical coins, eight of which are genuine, the other being counterfeit and different in weight from the rest - either heavier or lighter, but you do not know which. You are also given a two-sided balance on which to compare the weights of coins or groups of coins. Draw a tree to show a strategy for identifying the counterfeit coin in no more than three weighings.

*6. How many different compounds have molecular formula C_7H_{16}?

*7. Find a Hamiltonian cycle on the graph shown in the diagram.



A four-dimensional cube in graph form

*8. Prove that among any group of six people, there are either three who all know one another or three who are mutual strangers.

*9. Given that there are 23 different unlabelled trees with eight vertices, draw as many of them as you can.

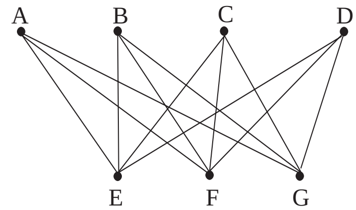
*10. A set of four coloured cubes has opposite faces coloured as follows:

Cube 1	R-B, R-Y, B-G;
Cube 2	R-B, Y-Y, Y-G;
Cube 3	R-Y, R-B, B-G;
Cube 4	R-G, G-G, B-Y;

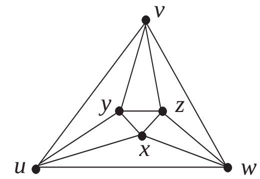
Either find a solution to the four-cube problem or explain why such a solution is impossible.

11. Consider the following graphs G_1 and G_2:

G_1



G_2



(a) Determine whether each of these graphs is Eulerian. In each case, either give an Eulerian trail or state why such a trail cannot exist.

(b) Determine whether each of these graphs is Hamiltonian. In each case, either give a Hamiltonian cycle or state why such a cycle cannot exist.

12. A simple graph G has five vertices, and each of those vertices has the same degree d.

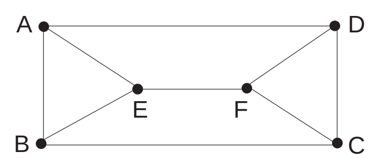
(a) State the possible values of d.

(b) If G is connected, what are the possible values of d?

(c) If Eulerian, what are the possible values of d?

(AEB)

13. Consider the following graph of G.



(a) Is G Eulerian? If so, write down an Eulerian trail.

(b) Is G Hamiltonian? If so, write down a Hamiltonian cycle.